Ordinary Differential Equations in Neuroscience with Matlab examples.

- **Aim 1- Gain understanding of how to set up and solve ODE's**

- **Aim 2 – Understand how to set up an solve a simple example of the Hebb rule in 2D**

Part I: examples of ODE's how to set them up and how to solve them.

Example 1: Radioactive decay:

This is the canonical example for a simple 1D ODE, it is also a good example for a random statistical processes, like you saw yesterday

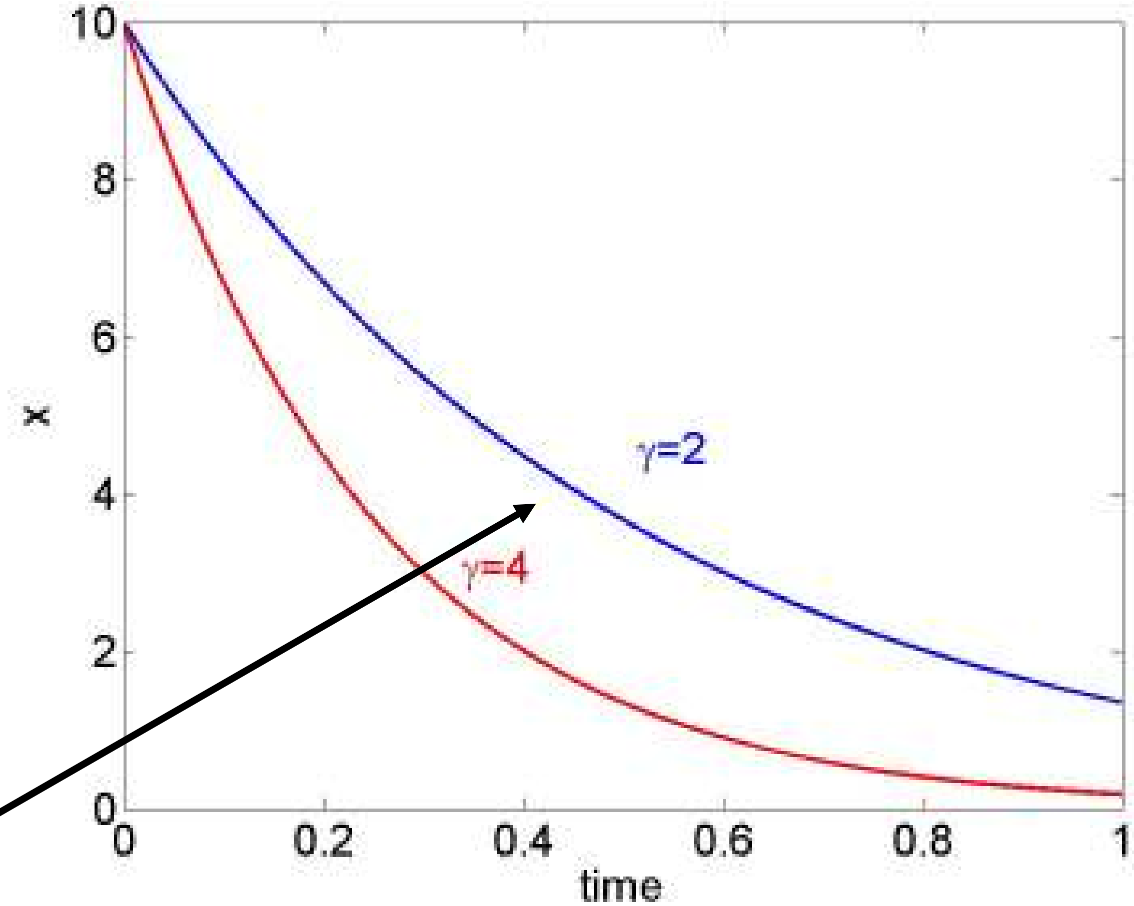
Assume there is a given amount or radioactive material defined by the variable X. X(t) is the amount at time t.

The probability that each atom will decay in a small time period Δt is independent of what the other molecules do, and only radioactive molecules can decay to a non radioactive state. The probability of decay over a small period Δt is defined as $\gamma \cdot \Delta t$.The amount remaining at time $t+\Delta t$ is:

$$x(t+\Delta t) = x(t) - x(t)\gamma\Delta t$$

Lets run the following simple matlab program to see what will happen to x(t).

```
>> x=zeros(1,1001);
>> dt=1/1000;
>> t=0:dt:1000*dt;
>>
>> x(1)=10;
>> gamma=2;
>>for jj=1:(length(t)-1)
x(jj+1)=x(jj)-dt*gamma*x(jj);
end
>> plot(t,x)
```



What is the shape of these curves?

How do they depend on the parameter γ?

This is a difference equation: $x(t+\Delta t) = x(t) - x(t)\gamma\Delta t$

A little simple math: $x(t+\Delta t) - x(t) = -x(t)\gamma\Delta t$

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = -x(t)\gamma$$

Now assume that the time step Δt approached 0 (is very small)

$$\lim_{\Delta t \to 0}\left(\frac{x(t+\Delta t) - x(t)}{\Delta t}\right) = \frac{dx}{dt} = -x\gamma$$

This is now a differential equation: $\dfrac{dx}{dt} = -\gamma x$

$$\frac{dx}{dt} = -\gamma x$$

How do we solve this ODE?

Make a guess, assume that:

$$x(t) = A \cdot \exp(B \cdot t)$$

Note for this choice of x(t): $\frac{dx}{dt} = B \cdot A \cdot \exp(Bt) = B \cdot x(t)$

Insert this back into the ODE above, get: $B \cdot x(t) = -\gamma \cdot x(t)$

Which is a solution if B=-γ.

So: $x(t) = A \cdot \exp(-\gamma \cdot t) = x(0) \exp(-1 / \tau)$

Example 2: Chemical reactions. Assume that when a molecule
Of type A binds to a molecule of type B they can form a product of type C. Denote as A+B $\longrightarrow$ C.

A represents the concentration of type A etc.

Assume now that the probability that type A will bind with be depends on their concentration. Then:

$$\left.\begin{aligned} &(1)\quad \frac{dA}{dt} = -\gamma A \cdot B \\ &(2)\quad \frac{dA}{dt} = -\gamma A \cdot B \\ &(3)\quad \frac{dC}{dt} = +\gamma A \cdot B \end{aligned}\right\}$$

These are three coupled ordinary differential equations

Where γ is a rate constant.

Matlab program

```
len=1000
dt=1/len;
A=zeros(1,1000);
B=zeros(1,1000);
C=zeros(1,1000);
gamma=2;

A(1)=2;
B(1)=10;
timeline=0:dt:dt*len;
for ii=1:len
    A(ii+1)=A(ii)-dt*gamma*A(ii)*B(ii);
    B(ii+1)=B(ii)-dt*gamma*A(ii)*B(ii);
    C(ii+1)=C(ii)+dt*gamma*A(ii)*B(ii);
end
plot(timeline,A,'b'); hold on;
plot(timeline,B,'r-.');plot(timeline,C,'k');
```

$$(1)\quad \frac{dA}{dt} = -\gamma A \cdot B$$

$$(2)\quad \frac{dA}{dt} = -\gamma A \cdot B$$

$$(3)\quad \frac{dC}{dt} = +\gamma A \cdot B$$

Note that there are conservation equations: $A+C=A^{tot}$ and $B+C=B^{tot}$

$$(1)\quad \frac{dA}{dt} = -\gamma A \cdot B$$
$$(2)\quad \frac{dA}{dt} = -\gamma A \cdot B$$
$$(3)\quad \frac{dC}{dt} = +\gamma A \cdot B$$

Simplification.
Lets assume a case where B>>A.

The smallest value on B possible is B(0)-A(0) which is close to B(0). Replace then the dynamical variable B with the parameter B=B0=B(0). Use conservation A+C=A(0), get:

$$(1)\quad \frac{dA}{dt} = -\gamma B_0 A$$
$$(2)\quad C = A(0) - A$$

Obtain solution:

$$A = A(0)\exp(-\gamma B_0 t)$$
$$C = A(0)(1 - \exp(-\gamma B_0 t))$$

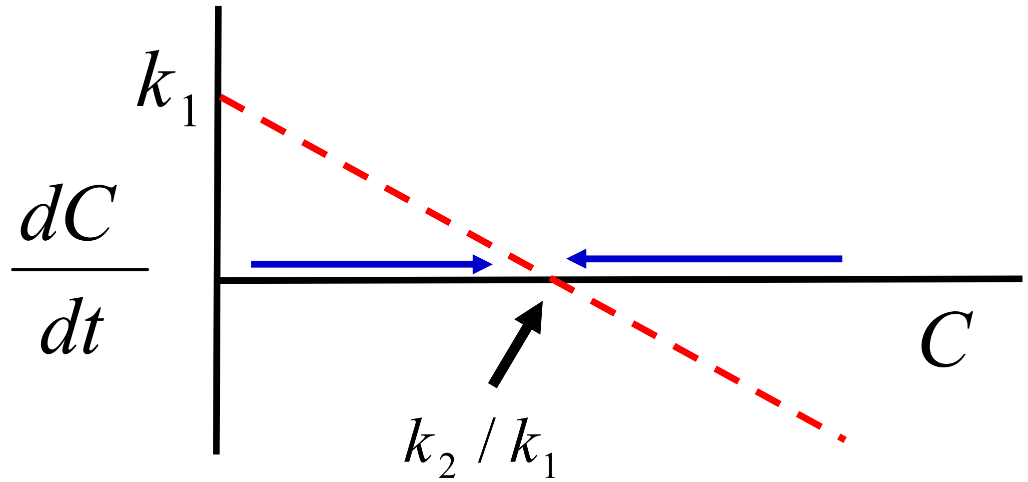
Fixed points and their stability:

The problems until now are very simple and exactly, most problems are not. Lets take a simple problem and pretend it is not to see what we would do in such a case. Same problem rewritten:

$$\frac{dC}{dt} = \underbrace{\gamma B_0 A_0}_{k_1} - \underbrace{\gamma B_0}_{k_2} C = k_1 - k_2 C$$

Fixed points when: $\frac{dC}{dt} = 0 \Rightarrow C = \frac{k_2}{k_1} = A_0$

Is this fixed point stable, that is when we slightly move away from the FP will the dynamics return us to the FP or take us away from it?



Here, the flow is toward the FP, and the FP is stable. If the line had a positive slope, at the FP it would be unstable.

Isoclines

For a system:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = F(x_1, x_2)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = G(x_1, x_2)$$

isoclines are defined by

$$F(x_1, x_2) = 0$$
$$G(x_1, x_2) = 0$$

Negative feedback in visual cortex

The following is a model of divisive gain control
originally developed in visual cortex, and applied to the response of a bipolar cell interacting with an amacrine cell in the retina.

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{1}{\tau_{\mathrm{B}}}\left(-B + \frac{L}{1+A}\right)$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{\tau_{\mathrm{A}}}(-A + 2B)$$

This equation is potentially undefined at A=-1. Why does this not matter?

isoclines: $A = 2B; \quad B = \frac{-1+\sqrt{1+8L}}{4}$

Stability of fixed point

We can find the fixed point by computing where the nulclines intersect. How do we determine its stability?

Theorem 8: Given the nonlinear system described by the equation:

$$\frac{\mathrm{d}\vec{X}}{\mathrm{d}t} = \vec{F}(\vec{X})$$

and an equilibrium point at X_{Eq}, which is a solution to:

$$\vec{F}(\vec{X}_{\mathrm{Eq}}) = \vec{0}$$

calculate the Jacobian to produce an associated linear equation:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{A}\vec{x}$$

where

$$\overleftrightarrow{A} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots \\ \vdots & \vdots & \frac{\partial F_N}{\partial x_N} \end{pmatrix}$$

where all partial derivatives are evaluated at X_{Eq}. Then sufficiently near X_{Eq}: (a) if all eigenvalues of the linear system have negative real parts, the nonlinear system is asymptotically stable; and (b) if the linear system has at least one eigenvalue with a positive real part, the nonlinear system is unstable. In addition, the type of equilibrium point for the nonlinear system, i.e. spiral point, node, or saddle point, will be the same as that for the associated linear equation.

When L = 10 we find

$$\overleftrightarrow{A} = \begin{pmatrix} -\frac{1}{10} & -\frac{1}{(1+A)^2} \\ \frac{1}{5} & -\frac{1}{10} \end{pmatrix} = \begin{pmatrix} -\frac{1}{10} & -\frac{1}{25} \\ \frac{1}{5} & -\frac{1}{10} \end{pmatrix}$$

with eigenvalues

$$\lambda = -0.1 \pm 0.089\mathrm{i}$$

A short term memory circuit

Short term memory is frequently examined by delayed matching tasks

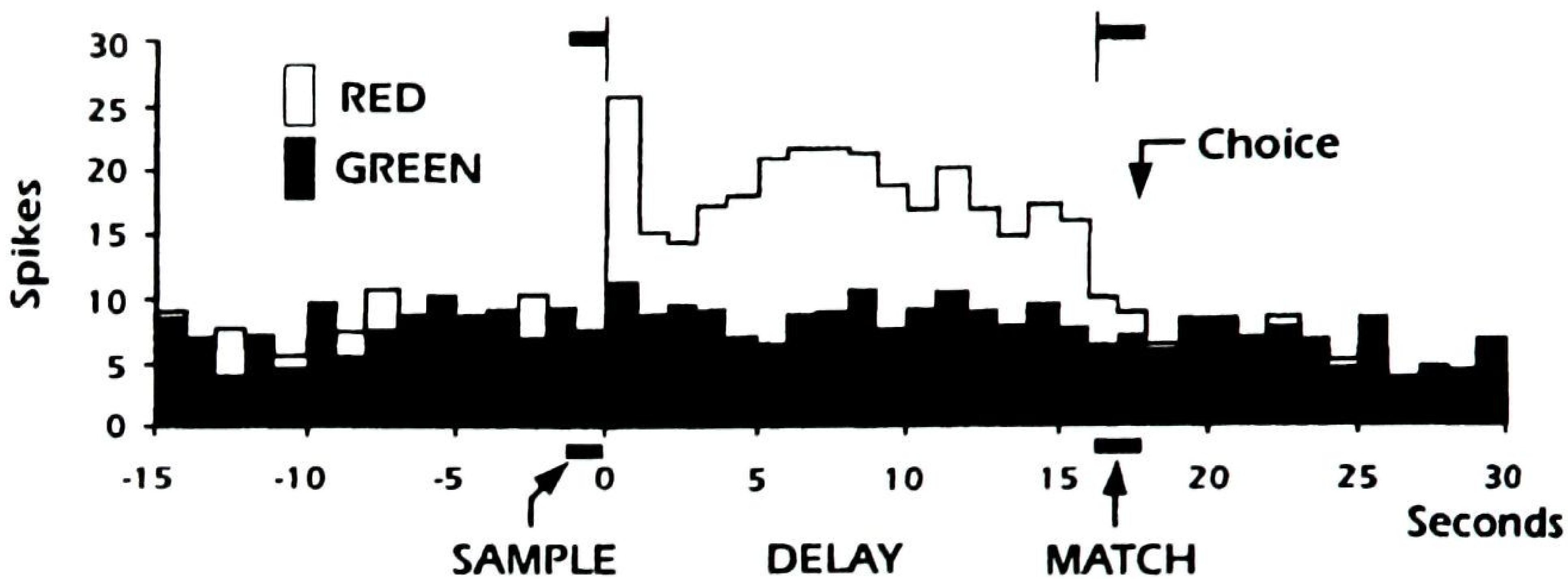


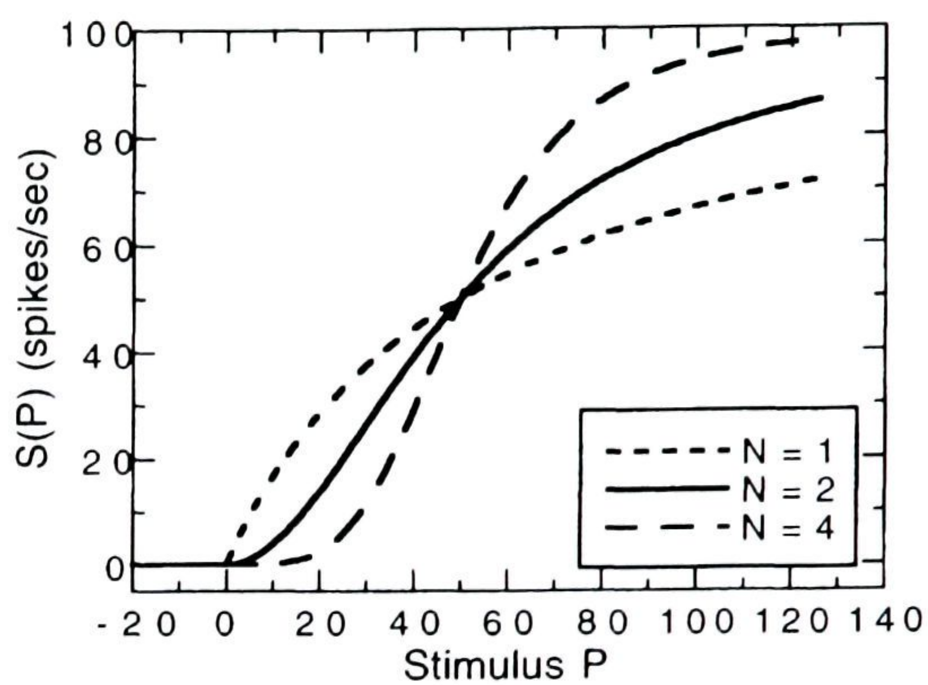
Fig. 6.3 Responses of a neuron in monkey inferiortemporal cortex during a short-term memory task (reproduced with permission, Fuster, 1995). Following a 1.0 s presentation of a red sample, this neuron fires at more than twice its resting level for 16 s until the signal to make a match appears and the monkey makes a choice to receive a reward. The same neuron did not increase its response when the sample was green.

A simple model

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = \frac{1}{\tau}\left(-E_1 + \frac{100(3E_2)^2}{120^2 + (3E_2)^2}\right)$$

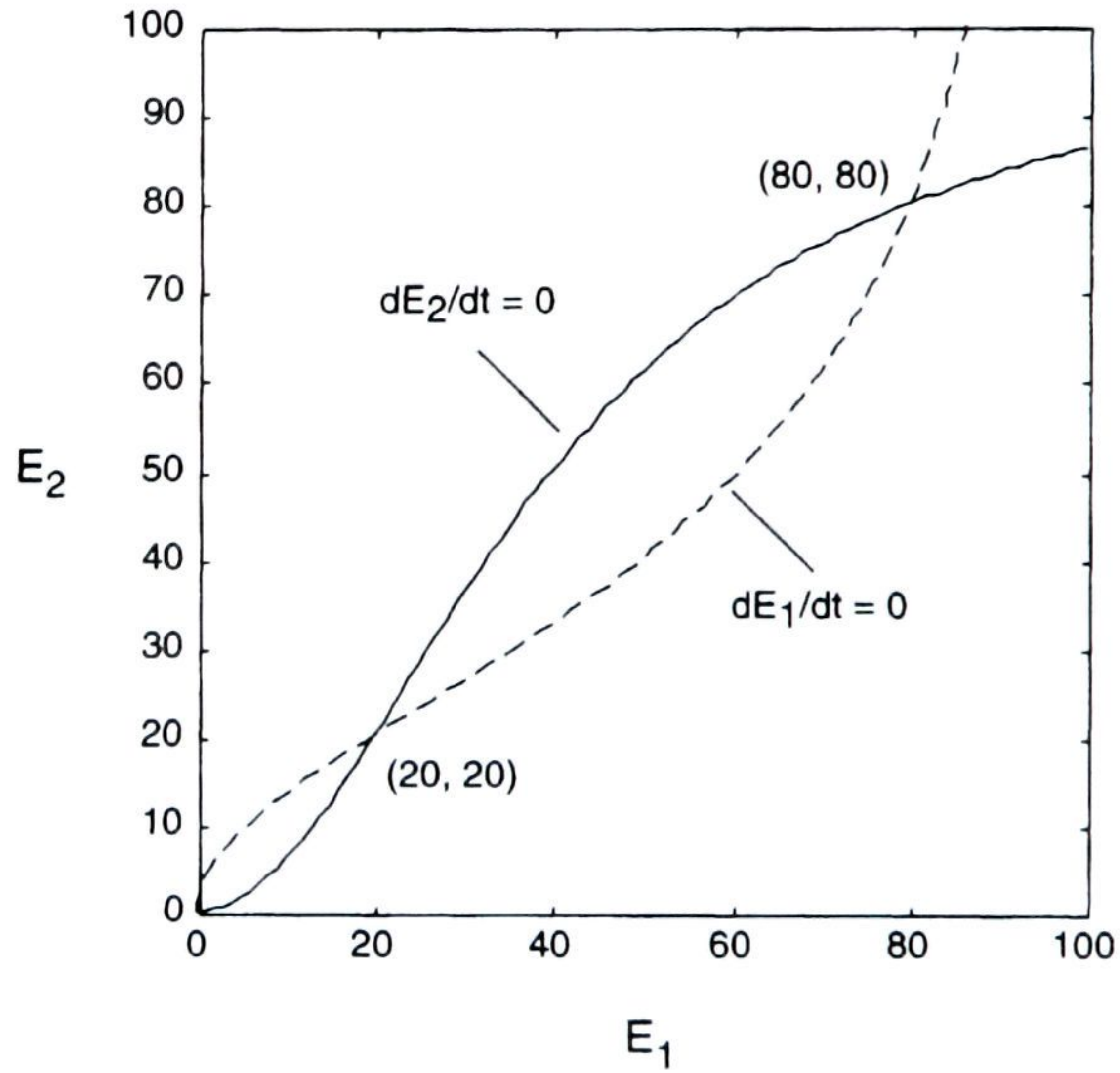
$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = \frac{1}{\tau}\left(-E_2 + \frac{100(3E_1)^2}{120^2 + (3E_1)^2}\right)$$

The interaction between the cells is given by the Naka Rushton function



$$S(P) = \begin{cases} \dfrac{MP^N}{\sigma^N + P^N} & \text{for } P \geq 0 \\ 0 & \text{for } P < 0 \end{cases}$$

Nulclines



$$\begin{pmatrix}0\\0\end{pmatrix}:\ \overleftrightarrow{A} = \begin{pmatrix}-0.05 & 0\\ 0 & -0.05\end{pmatrix}, \lambda = -0.05,\ -0.05 \quad \text{Asymptotically stable node.}$$

$$\begin{pmatrix}20\\20\end{pmatrix}:\ \overleftrightarrow{A} = \begin{pmatrix}-0.05 & 0.08\\ 0.08 & -0.05\end{pmatrix}, \lambda = +0.03,\ -0.13 \quad \text{Unstable saddle point.}$$

$$\begin{pmatrix}80\\80\end{pmatrix}:\ \overleftrightarrow{A} = \begin{pmatrix}-0.05 & 0.02\\ 0.02 & -0.05\end{pmatrix}, \lambda = -0.07,\ -0.03 \quad \text{Asymptotically stable node.}$$

Bifurcations

Let's add an external input, K, to both cells, so that the fixed points are determined by

$$E_1 = \frac{100(3E_1 + K)_+^2}{120^2 + (3E_1 + K)_+^2}$$

A stimulus can cause a switch between two stable states

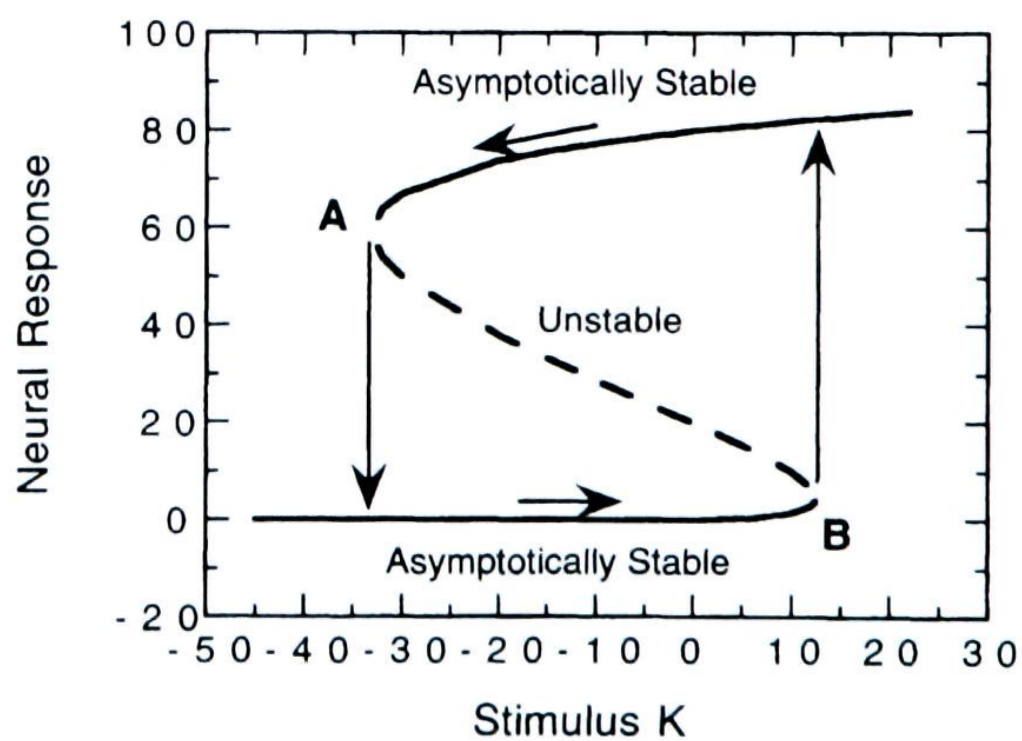


Fig. 6.5 Hysteresis loop and bifurcations generated by (6.8) in the presence of stimulus K in (6.13). Between A and B two steady states are asymptotically stable nodes, while the intervening one is an unstable saddle point. If K is swept back and forth across range AB, the system will trace out the hysteresis loop shown by the arrows.

Adaptation

Over time, neurons that are firing will tend to adapt.

$$\frac{dE_1}{dt} = \frac{1}{\tau}\left(-E_1 + \frac{100(3E_2)_+^2}{(120 + A_1)^2 + (3E_2)_+^2}\right)$$

$$\frac{dE_2}{dt} = \frac{1}{\tau}\left(-E_2 + \frac{100(3E_1)_+^2}{(120 + A_2)^2 + (3E_1)_+^2}\right)$$

$$\frac{dA_1}{dt} = \frac{1}{\tau_a}(-A_1 + 0.7E_1)$$

$$\frac{dA_2}{dt} = \frac{1}{\tau_a}(-A_2 + 0.7E_2)$$

Adaptation can cause loss of stability of active state

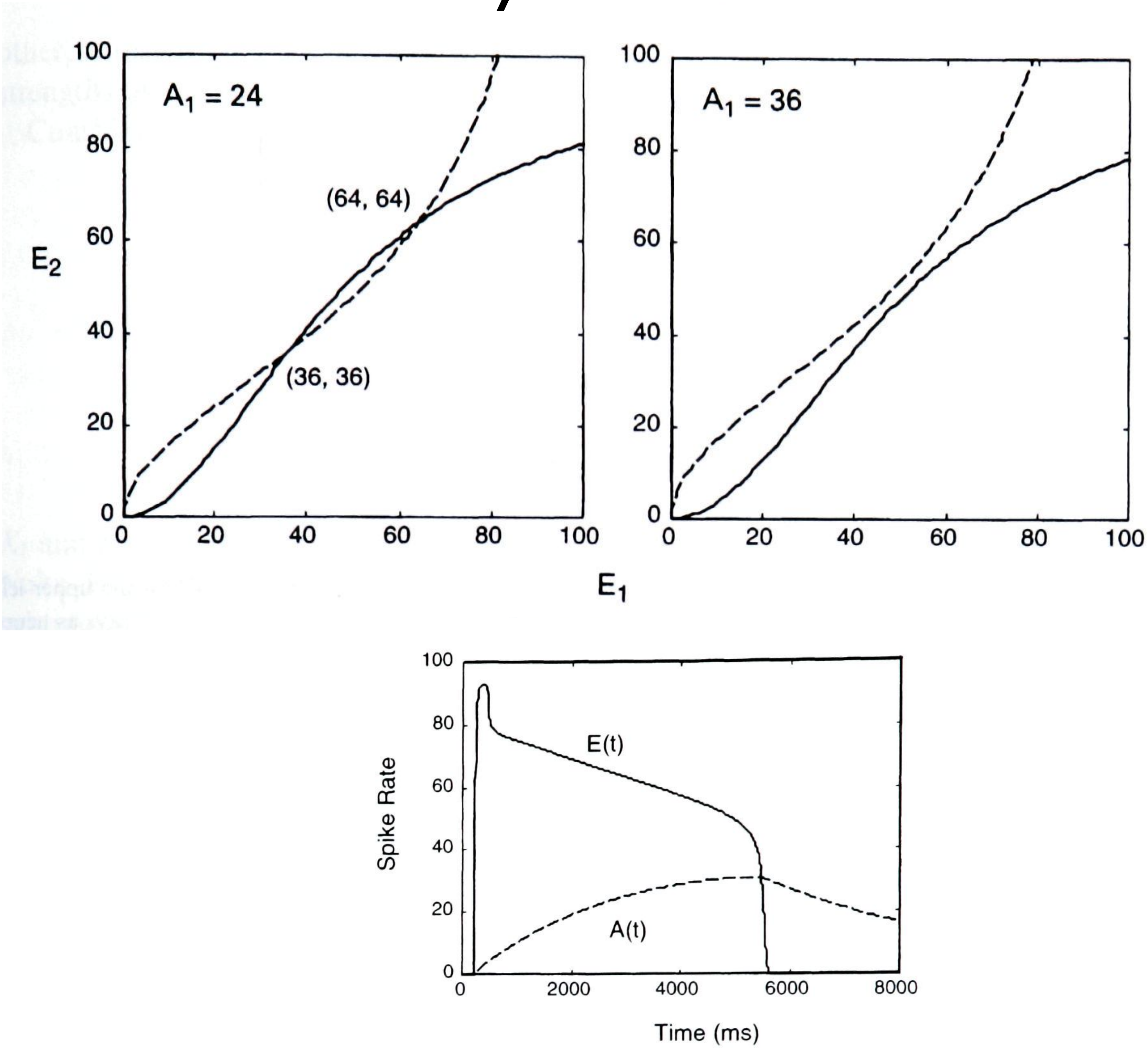
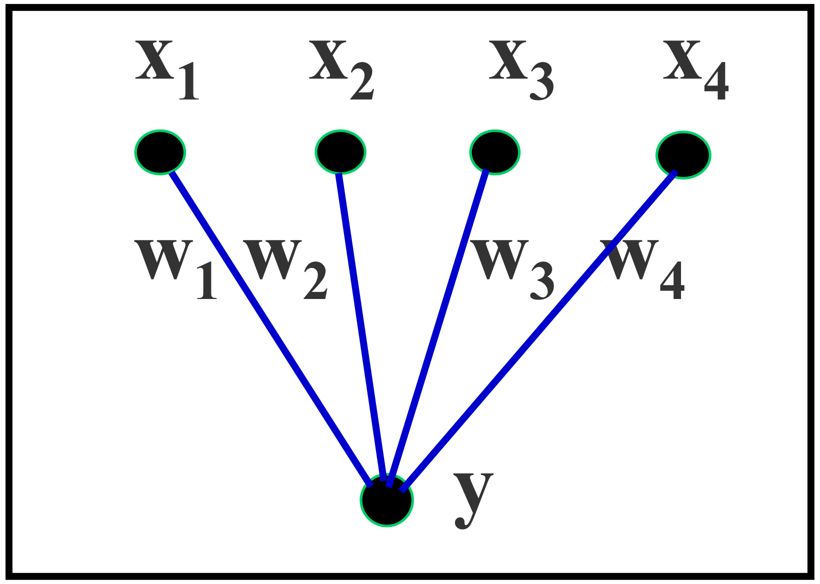


Fig. 6.7 Response of (6.14) to a brief, 200 ms stimulus coinciding with the narrow peak on the upper left. Recurrent excitation maintains activity of both $E(t)$ cells at a high level, but activity slowly decays as neural adaptation $A(t)$ builds up. After 5000 ms a sudden loss of neural activity occurs at a bifurcation.

Part II: The Hebb rule, an example of learning dynamics and how we can solve a 2D example.

The Hebb rule: $\Delta w_i = \Delta t \cdot \eta \, x_i y$



$$\frac{\Delta w_i}{\Delta t} \rightarrow \frac{dw_i}{dt} = \eta \, x_i y$$

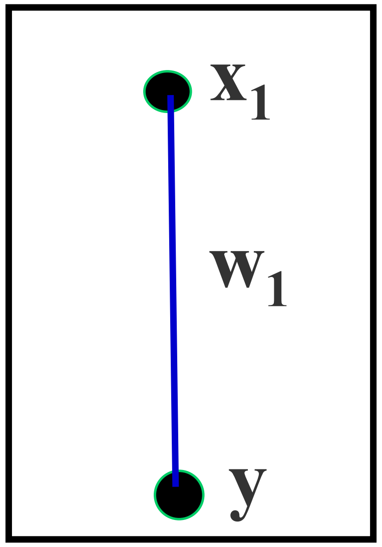
Where: $y = \sum_i w_i x_i$

If we insert into equation above we get:

$$\frac{dw_i}{dt} = \eta \sum_j w_j x_i x_j$$

Simple 1D example (x_1=x).

a. The input is constant over time: x=a



$$\frac{dw}{dt} = \eta\, wx^2 = \eta a^2 w$$

$$w(t) = w(0)\exp(\eta a^2 t)$$

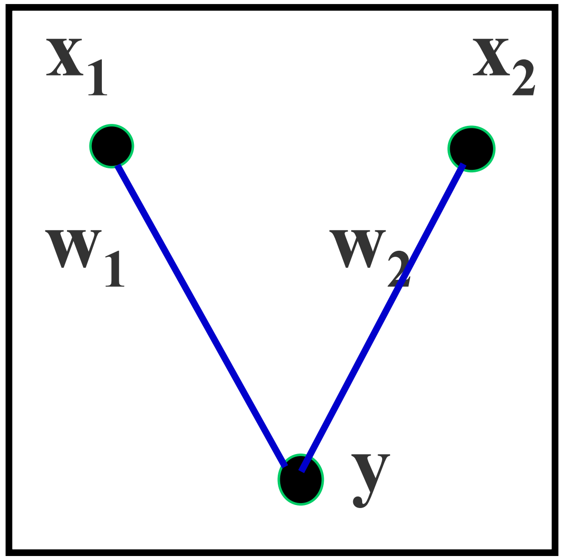
b. The input has a probability of ½ of being 1 and a probability of ½ of being -1. Assume learning is slow so that can take average over input distribution:

$$\left\langle \frac{dw}{dt} \right\rangle = \eta\, w\left\langle x^2 \right\rangle = \eta w \underbrace{\left(0.5\cdot(1^2) + 0.5\cdot(-1)^2\right)}_{1} = \eta w$$

$$w(t) = w(0)\exp(\eta t)$$

2D example.

The Hebb rule: $$\frac{dw_i}{dt} = \eta \sum_j w_j x_i x_j$$



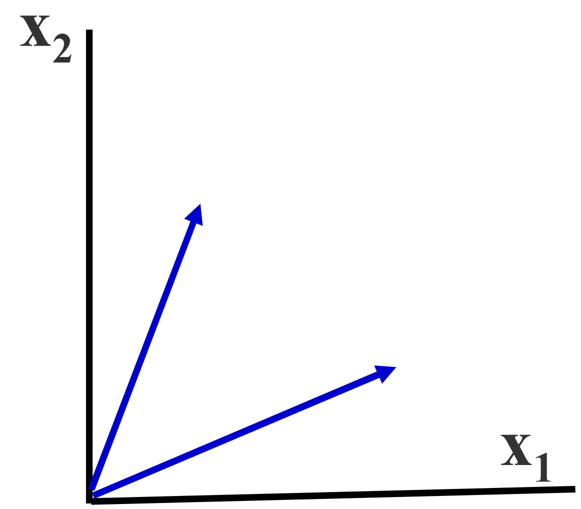
Average: $$\left\langle \frac{dw_i}{dt} \right\rangle = \eta \sum_j w_j \langle x_i x_j \rangle = \eta \sum_j w_j Q_{ij}$$

Matrix notation: $$\left\langle \frac{d\mathbf{w}}{dt} \right\rangle = \eta \mathbf{w} \mathbf{Q}$$

Assume: $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1.0 \\ 0.5 \end{pmatrix} & with \quad p = 0.5 \\ \begin{pmatrix} 0.5 \\ 1.0 \end{pmatrix} & with \quad p = 0.5 \end{cases}$$

Show simple matlab simulation of this example

$$\left\langle \frac{dw_i}{dt} \right\rangle = \sum_j w_j Q_{ij} \quad \text{where:} \quad Q_{ij} = \langle x_i x_j \rangle$$



$$\text{Assume} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1.0 \\ 0.5 \end{pmatrix} & \textit{with} \quad p = 0.5 \\ \begin{pmatrix} 0.5 \\ 1.0 \end{pmatrix} & \textit{with} \quad p = 0.5 \end{cases}$$

$$\mathbf{Q} = 0.5 \begin{pmatrix} 1^2 & 1 \cdot 0.5 \\ 0.5 \cdot 1 & 0.5^2 \end{pmatrix} + 0.5 \begin{pmatrix} 0.5^2 & 0.5 \cdot 1 \\ 1 \cdot 0.5 & 1^2 \end{pmatrix} = \begin{pmatrix} 0.625 & 0.5 \\ 0.5 & 0.625 \end{pmatrix}$$

$$\left\langle \frac{d\mathbf{w}}{dt} \right\rangle = \eta \mathbf{w} \mathbf{Q}$$

Lets pretend now that we can simply drop this average symbol

$$\frac{d\mathbf{w}}{dt} = \eta \mathbf{w} \mathbf{Q}$$

$$\frac{d\mathbf{w}}{dt} = \eta \mathbf{Q}\mathbf{w}$$

Find the eigen vectors of Q: $\mathbf{Q} \cdot \mathbf{u}_i = \lambda_i \mathbf{u}_i$

Matlab code

```
>> Q=[0.625, 0.5
        0.5, 0.625];

>> [U,lam]=eig(Q)

U =

   -0.7071    0.7071
    0.7071    0.7071

lam =

    0.1250         0
         0    1.1250
```

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \lambda_1 = 1.125$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \lambda_2 = 0.125$$

The averaged 'Hebbian' ODE:

$$\frac{d\mathbf{w}}{dt} = \eta \mathbf{Q}\mathbf{w}$$

Find the eigen-vectors of Q: $\mathbf{Q} \cdot \mathbf{u}_i = \lambda_i \mathbf{u}_i$

If Q has the common form: $\mathbf{Q} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$$\lambda_1 = a + b \quad u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ +1 \end{pmatrix}$$

Then:

$$\lambda_1 = a - b \quad u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ -1 \end{pmatrix}$$

$$\mathbf{w(t)} = a_1(0)\exp(\lambda_1 t)\mathbf{u_1} + a_2(0)\exp(\lambda_2 t)\mathbf{u_2}$$

What happens if $\lambda_1 >> \lambda_2$?

What would happen with the learning rule:

$$\frac{dw_i}{dt} = \eta \left(x_i y - w_i y^2 \right)$$

Oja (1982)

Where are the F.P, how does this relate to the eigen-vectors, and why