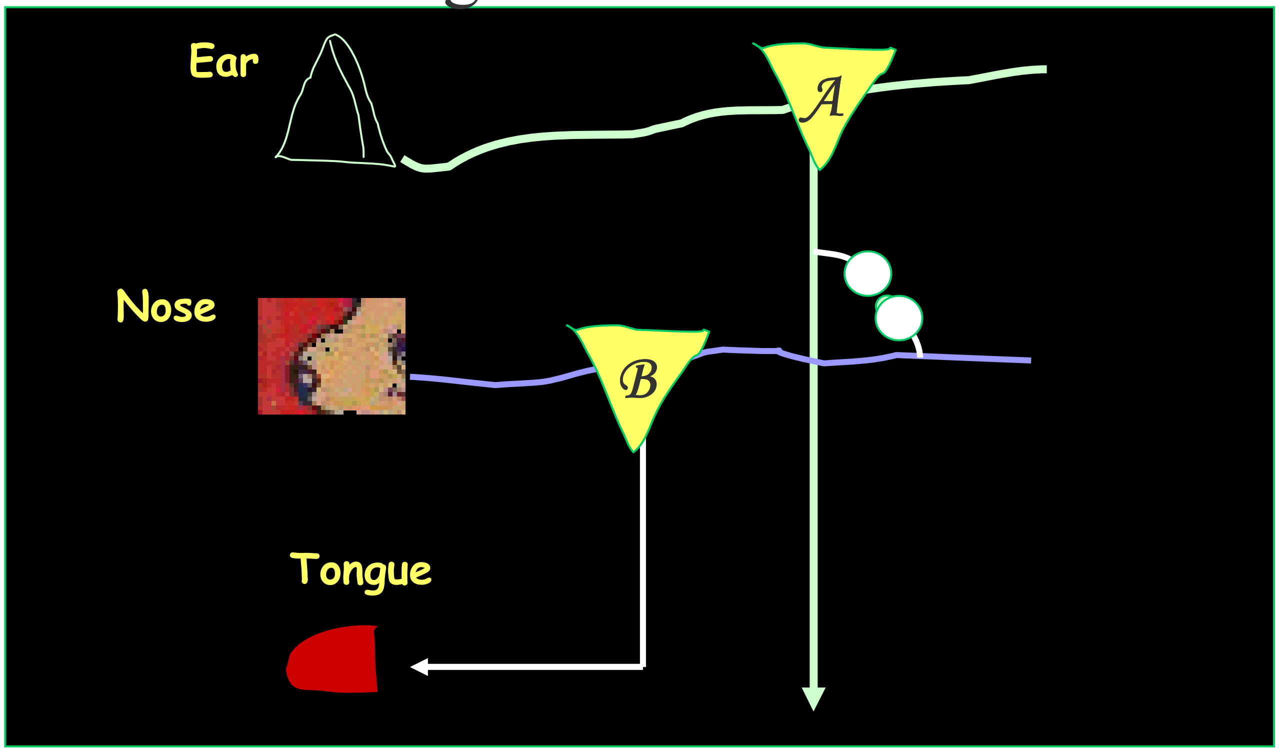
Ordinary Differential Equations in Neuroscience with Matlab examples.

- Aim 1- Gain understanding of how to set up and solve ODE's

- Aim 2 – Understand how to set up an solve a simple example of the Hebb rule in 2D

Our goal at end of class – understand Hebbian plasticity mathematically

Classical Conditioning and Hebb's rule



"When an axon in cell $\mathcal{A}$ is near enough to excite cell $\mathcal{B}$ and repeatedly and persistently takes part in firing it, some growth process or metabolic change takes place in one or both cells such that $\mathcal{A}$'s efficacy in firing $\mathcal{B}$ is increased"

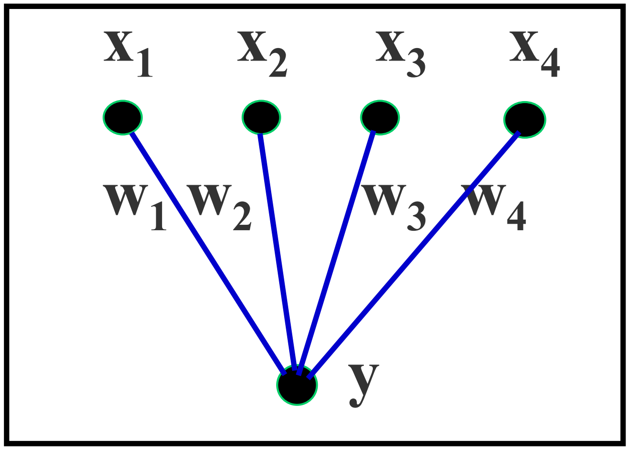
The generalized Hebb rule:

$$\frac{dw_i}{dt} = \eta x_i y$$

where x_i are the inputs

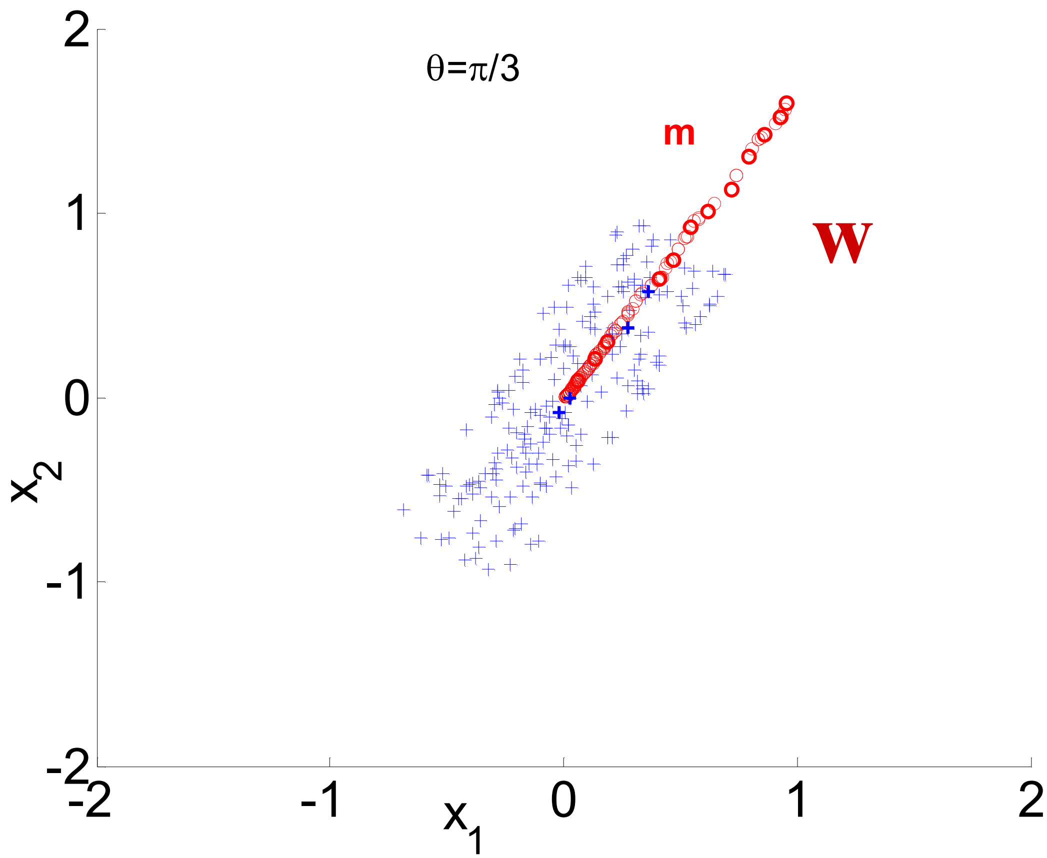
and y the output is assumed linear:

$$y = \sum_j w_j x_j$$



Results in 2D

Example of Hebb in 2D



The + symbols represent randomly chosen input pairs (x_1,x_2) and the red circles the evolution of the synaptic weights (w_1=w_2).

(Note: here inputs have a mean of zero)

Part I: examples of ODE's how to set them up and how to solve them.

Example 1: Radioactive decay:

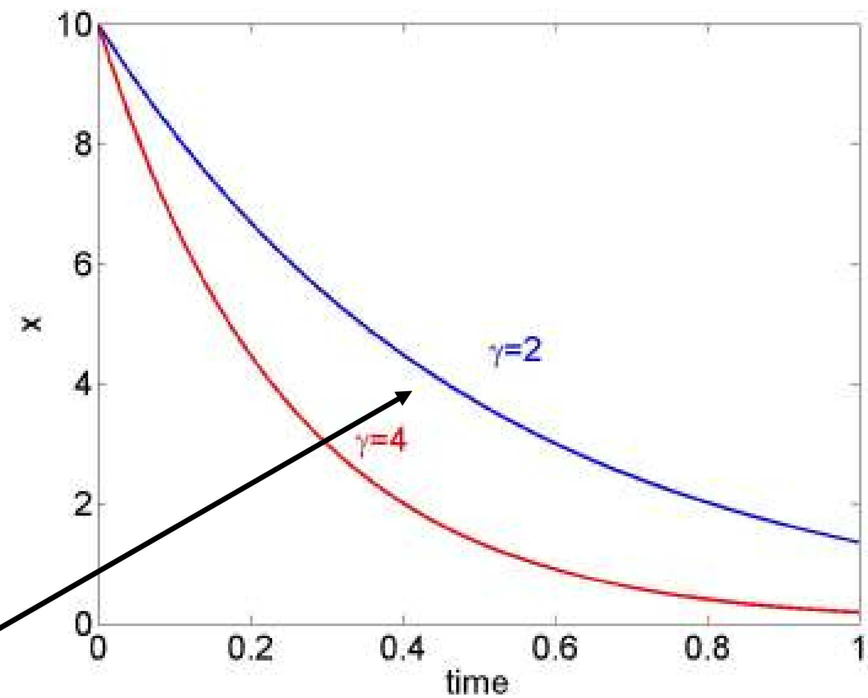
This is the canonical example for a simple 1D ODE, it is also a good example for a random statistical processes, like you saw yesterday

Assume there is a given amount or radioactive material defined by the variable X. X(t) is the amount at time t.
The probability that each atom will decay in a small time period Δt is independent of what the other molecules do, and only radioactive molecules can decay to a non radioactive state. The probability of decay over a small period Δt is defined as $\gamma \cdot \Delta t$.The amount remaining at time $t+\Delta t$ is:

$$x(t+\Delta t) = x(t) - x(t)\gamma\Delta t$$

Lets run the following simple matlab program to see what will happen to x(t).

```
>> x=zeros(1,1001);
>> dt=1/1000;
>> t=0:dt:1000*dt;
>>
>> x(1)=10;
>> gamma=2;
>>for jj=1:(length(t)-1)
x(jj+1)=x(jj)-dt*gamma*x(jj);
end
>> plot(t,x)
```



What is the shape of these curves?

How do they depend on the parameter γ?

This is a difference equation: $x(t+\Delta t) = x(t) - x(t)\gamma\Delta t$

A little simple math:

$$x(t+\Delta t) - x(t) = -x(t)\gamma\Delta t$$

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = -x(t)\gamma$$

Now assume that the time step Δt approached 0 (is very small)

$$\lim_{\Delta t \to 0}\left(\frac{x(t+\Delta t) - x(t)}{\Delta t}\right) = \frac{dx}{dt} = -x\gamma$$

This is now a differential equation: $\dfrac{dx}{dt} = -\gamma x$

$$\frac{dx}{dt} = -\gamma x$$

How do we solve this ODE?

Make a guess, assume that:

$$x(t) = A \cdot \exp(B \cdot t)$$

Note for this choice of x(t): $\frac{dx}{dt} = B \cdot A \cdot \exp(Bt) = B \cdot x(t)$

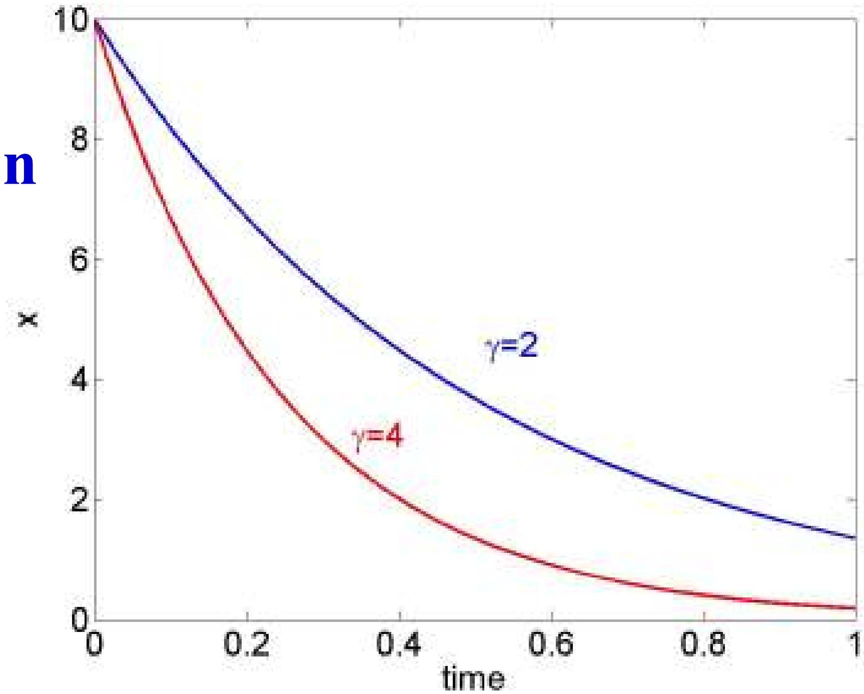
Insert this back into the ODE above, get: $B \cdot x(t) = -\gamma \cdot x(t)$

Which is a solution if B=-γ.

So: $x(t) = A \cdot \exp(-\gamma \cdot t) = x(0)\exp(-1/\tau)$

Solution: $x(t) = x(0)\exp(-1/\tau)$

Where τ the time constant is
$\tau = 1/\gamma$, and the initial condition determines the other
Free constant x(0)=A



Terminology, this is a first order (only first derivatives) linear differential equation. The dynamical variable x depends on only one parameter, t. If it depends on additional paramerters we might obtain partial differential equations, which we will not discuss here.

Example 2: Chemical reactions. Assume that when a molecule
Of type A binds to a molecule of type B they can form a product of type C. Denote as A+B $\longrightarrow$ C.

A represents the concentration of type A etc.

Assume now that the probability that type A will bind with be depends on their concentration. Then:

$$\left.\begin{aligned} &(1)\quad \frac{dA}{dt} = -\gamma A \cdot B \\ &(2)\quad \frac{dA}{dt} = -\gamma A \cdot B \\ &(3)\quad \frac{dC}{dt} = +\gamma A \cdot B \end{aligned}\right\}$$

These are three coupled ordinary differential equations

Where γ is a rate constant.

Matlab program

```
len=1000
dt=1/len;
A=zeros(1,1000);
B=zeros(1,1000);
C=zeros(1,1000);
gamma=2;

A(1)=2;
B(1)=10;
timeline=0:dt:dt*len;
for ii=1:len
    A(ii+1)=A(ii)-dt*gamma*A(ii)*B(ii);
    B(ii+1)=B(ii)-dt*gamma*A(ii)*B(ii);
    C(ii+1)=C(ii)+dt*gamma*A(ii)*B(ii);
end
plot(timeline,A,'b'); hold on;
plot(timeline,B,'r-.');plot(timeline,C,'k');
```

$$(1)\quad \frac{dA}{dt} = -\gamma A \cdot B$$

$$(2)\quad \frac{dA}{dt} = -\gamma A \cdot B$$

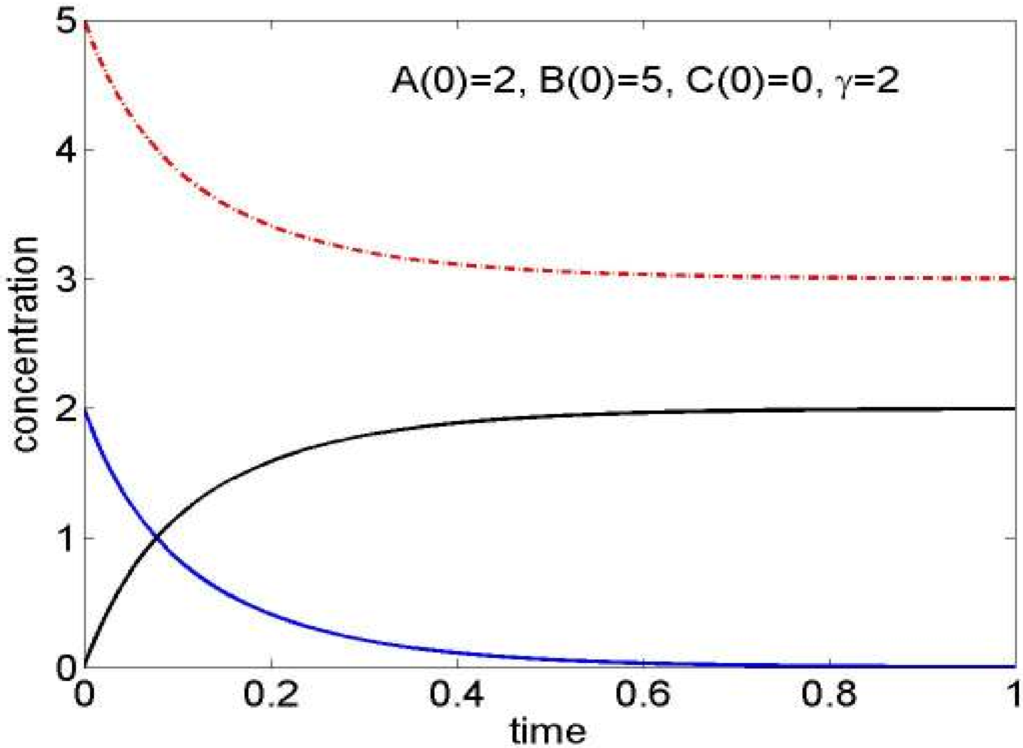
$$(3)\quad \frac{dC}{dt} = +\gamma A \cdot B$$

Note that there are conservation equations: $A+C=A^{tot}$ and $B+C=B^{tot}$

$$(1)\quad \frac{dA}{dt} = -\gamma A \cdot B$$

$$(2)\quad \frac{dA}{dt} = -\gamma A \cdot B$$

$$(3)\quad \frac{dC}{dt} = +\gamma A \cdot B$$



$$(1)\quad \frac{dA}{dt} = -\gamma A \cdot B$$
$$(2)\quad \frac{dA}{dt} = -\gamma A \cdot B$$
$$(3)\quad \frac{dC}{dt} = +\gamma A \cdot B$$

Simplification.
Lets assume a case where B>>A.

The smallest value on B possible is B(0)-A(0) which is close to B(0). Replace then the dynamical variable B with the parameter B=B0=B(0). Use conservation A+C=A(0), get:

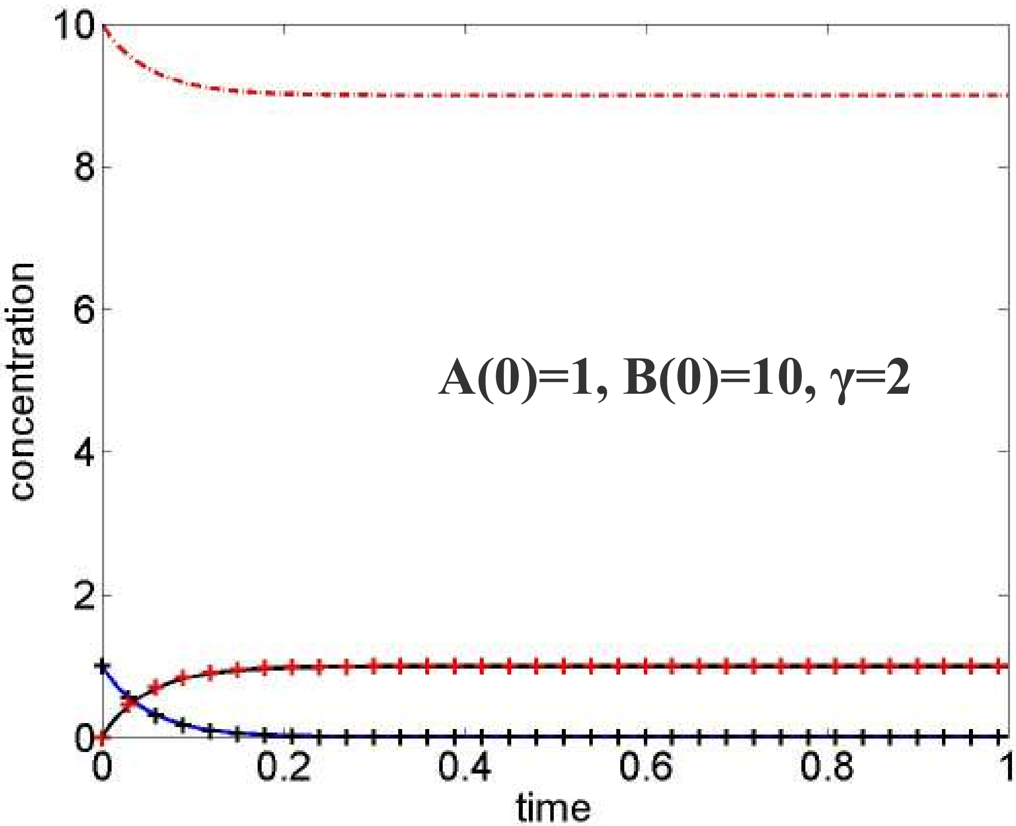
$$(1)\quad \frac{dA}{dt} = -\gamma B_0 A$$
$$(2)\quad C = A(0) - A$$

Obtain solution:

$$A = A(0)\exp(-\gamma B_0 t)$$
$$C = A(0)(1 - \exp(-\gamma B_0 t))$$

$$A(t) = A(0)\exp(-\gamma B_0 t)$$
$$C(t) = A(0)(1 - \exp(-\gamma B_0 t))$$

+ signs represent the approximate analytical solutions



Play with the program parameters and initial conditions and compare to the analytical solution, see when the approximate solutions are no longer a good approximation.

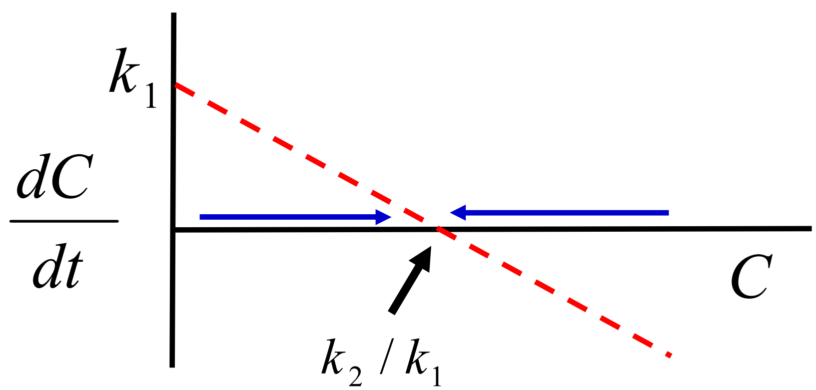
Fixed points and their stability:

The problems until now are very simple and exactly, most problems are not. Lets take a simple problem and pretend it is not to see what we would do in such a case. Same problem rewritten:

$$\frac{dC}{dt} = \underbrace{\gamma B_0 A_0}_{k_1} - \underbrace{\gamma B_0}_{k_2} C = k_1 - k_2 C$$

Fixed points when: $\frac{dC}{dt} = 0 \Rightarrow C = \frac{k_2}{k_1} = A_0$

Is this fixed point stable, that is when we slightly move away from the FP will the dynamics return us to the FP or take us away from it?



Here, the flow is toward the FP, and the FP is stable. If the line had a positive slope, at the FP it would be unstable.

Terms used:

1. First order, only first derivatives $\frac{dx}{dt}$

The equation $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + b = 0$ is second order. It too has a simple solution try inserting $x(t) = A \cdot \exp(\lambda t)$

2. This equation is linear because it does not have terms of the form $\left(\frac{dx}{dt}\right)^n$.

Non linear equation are usually hard to solve exactly, and we usually resort to finding their fixed points and the stability of these fixed points.

3. An equation of the form $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + b(x) = 0$

Is non-homogeneous we have methods for solving such equations.

Textbooks for differential equations:

1. Elementary differential equations and boundary value problems. Boyce and DiPrima. This has mostly analytical solutions and background on fixed points and their stability.

To see examples of equations for specific problems solved in terms of the fixed points and their stability you can look at:

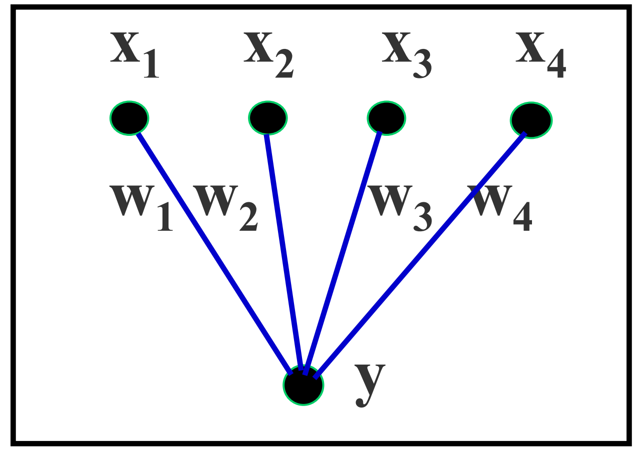
2. Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering. S. Stogatz.

This is not a differntial equations textbook, but the methods used and examples are very useful and it is an easy read.

Recap – what did we learn until now?

Part II: The Hebb rule, an example of learning dynamics and how we can solve a 2D example.

The Hebb rule: $\Delta w_i = \Delta t \cdot \eta \, x_i y$



$$\frac{\Delta w_i}{\Delta t} \rightarrow \frac{dw_i}{dt} = \eta \, x_i y$$

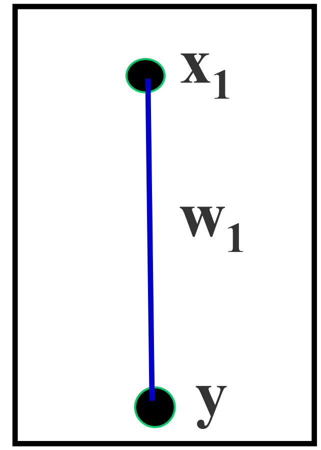
Where: $y = \sum_i w_i x_i$

If we insert into equation above we get:

$$\frac{dw_i}{dt} = \eta \sum_j w_j x_i x_j$$

Simple 1D example (x_1=x).

a. The input is constant over time: x=a



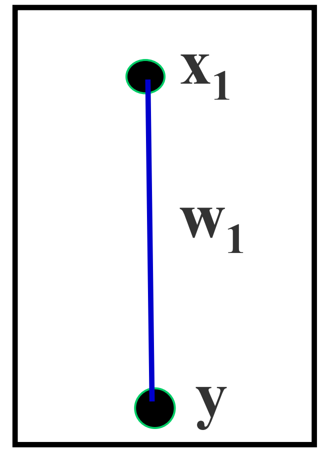
$$\frac{dw}{dt} = \eta\, wx^2 = \eta a^2 w$$

$$w(t) = w(0)\exp(\eta a^2 t)$$

b. The input has a probability of ½ of being 1 and a probability of ½ of being -1. Assume learning is slow so that can take average over input distribution:

$$\left\langle \frac{dw}{dt} \right\rangle = \eta\, w\langle x^2 \rangle = \eta w \underbrace{\left(0.5\cdot(1^2) + 0.5\cdot(-1)^2\right)}_{1} = \eta w$$

$$w(t) = w(0)\exp(\eta t)$$



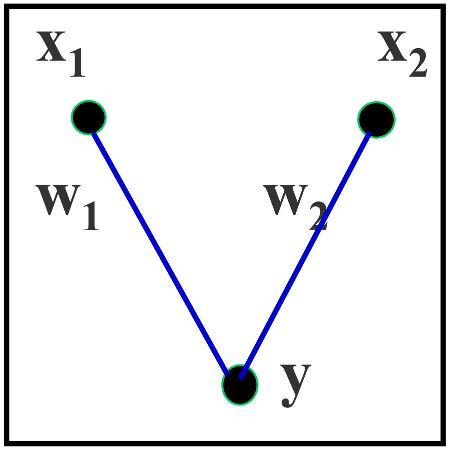
$$w(t) = w(0)\exp(\eta a^2 t)$$

$$\left\langle \frac{dw}{dt} \right\rangle = \eta\ w\left\langle x^2 \right\rangle = \eta w \underbrace{\left(0.5\cdot(1^2) + 0.5\cdot(-1)^2\right)}_{1} = \eta w$$

$$w(t) = w(0)\exp(\eta t)$$

2D example.

The Hebb rule: $$\frac{dw_i}{dt} = \eta \sum_j w_j x_i x_j$$



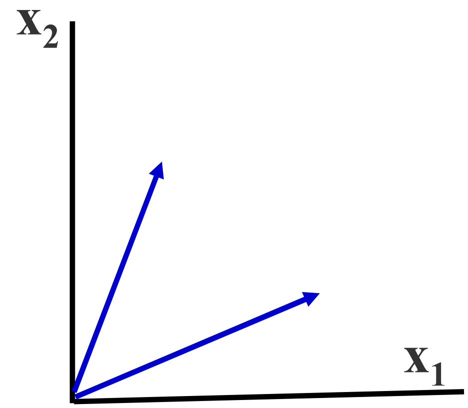
Average: $$\left\langle \frac{dw_i}{dt} \right\rangle = \eta \sum_j w_j \langle x_i x_j \rangle = \eta \sum_j w_j Q_{ij}$$

Matrix notation: $$\left\langle \frac{d\mathbf{w}}{dt} \right\rangle = \eta \mathbf{w} \mathbf{Q}$$

Assume: $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1.0 \\ 0.5 \end{pmatrix} & with \quad p = 0.5 \\ \begin{pmatrix} 0.5 \\ 1.0 \end{pmatrix} & with \quad p = 0.5 \end{cases}$$

Show simple matlab simulation of this example

$$\left\langle \frac{dw_i}{dt} \right\rangle = \sum_j w_j Q_{ij} \quad \text{where:} \quad Q_{ij} = \langle x_i x_j \rangle$$



Assume $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1.0 \\ 0.5 \end{pmatrix} & \textit{with} \quad p = 0.5 \\ \begin{pmatrix} 0.5 \\ 1.0 \end{pmatrix} & \textit{with} \quad p = 0.5 \end{cases}$

$$\mathbf{Q} = 0.5 \begin{pmatrix} 1^2 & 1 \cdot 0.5 \\ 0.5 \cdot 1 & 0.5^2 \end{pmatrix} + 0.5 \begin{pmatrix} 0.5^2 & 0.5 \cdot 1 \\ 1 \cdot 0.5 & 1^2 \end{pmatrix} = \begin{pmatrix} 0.625 & 0.5 \\ 0.5 & 0.625 \end{pmatrix}$$

$$\left\langle \frac{d\mathbf{w}}{dt} \right\rangle = \eta \mathbf{w} \mathbf{Q}$$

Lets pretend now that we can simply drop this average symbol

$$\frac{d\mathbf{w}}{dt} = \eta \mathbf{w} \mathbf{Q}$$

$$\frac{d\mathbf{w}}{dt} = \eta \mathbf{Q}\mathbf{w}$$

Find the eigen vectors of Q: $\mathbf{Q} \cdot \mathbf{u}_i = \lambda_i \mathbf{u}_i$

Matlab code

```
>> Q=[0.625, 0.5
      0.5, 0.625];

>> [U,lam]=eig(Q)

U =

  -0.7071    0.7071
   0.7071    0.7071

lam =

   0.1250         0
        0    1.1250
```

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \lambda_1 = 1.125$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \lambda_2 = 0.125$$

The averaged 'Hebbian' ODE:

$$\frac{d\mathbf{w}}{dt} = \eta \mathbf{Q}\mathbf{w}$$

Find the eigen-vectors of Q: $\mathbf{Q} \cdot \mathbf{u}_i = \lambda_i \mathbf{u}_i$

These form a complete orthonormal basis

Rewrite: $\mathbf{w(t)} = a_1(t)\mathbf{u_1} + a_2(t)\mathbf{u_2}$

$$\frac{d\mathbf{w}}{dt} = \eta\left(a_1(t)\lambda_1\mathbf{u}_1 + a_2(t)\lambda_2\mathbf{u}_2\right)$$

So $\dfrac{da_i}{dt} = \eta\lambda_i a_i$

$$\mathbf{w(t)} = a_1(0)\exp(\lambda_1 t)\mathbf{u_1} + a_2(0)\exp(\lambda_2 t)\mathbf{u_2}$$

The averaged 'Hebbian' ODE:

$$\frac{d\mathbf{w}}{dt} = \eta \mathbf{Q}\mathbf{w}$$

Find the eigen-vectors of Q: $\mathbf{Q} \cdot \mathbf{u}_i = \lambda_i \mathbf{u}_i$

If Q has the common form: $\mathbf{Q} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$$\lambda_1 = a + b \quad u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ +1 \end{pmatrix}$$

Then:

$$\lambda_1 = a - b \quad u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ -1 \end{pmatrix}$$

$$\mathbf{w(t)} = a_1(0)\exp(\lambda_1 t)\mathbf{u_1} + a_2(0)\exp(\lambda_2 t)\mathbf{u_2}$$

What happens if $\lambda_1 >> \lambda_2$?

General comments about the form of correlation matrixes.

• They are symmetric $Q_{ij}=Q_{ji}$., and therefore all of their eigen-values are real.

• All of their eigen-values are positive (Try to prove this yourself)

HW- For the general form of a correlation matrix in 2D:

$$\mathbf{Q} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

1. **Learn how to find analytically the eigen-vectors and eigen-values.**
2. **Show that in this case all eigen-values are positive**
3. **Show that in the higer dimnstional case all eigen-values are positive**

What did we learn today?

HW-

What would happen with the learning rule:

$$\frac{dw_i}{dt} = \eta \left(x_i y - w_i y^2 \right)$$

Oja (1982)

Where are the F.P, how does this relate to the eigen-vectors, and why