Passivity Preserving Model Reduction

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Projection Methods for Model Reduction

- LTI Model Reduction via Projection
- Passivity and Positive Real
- Invariant Subspace Approach
- Positive Real Lemma Sufficiency
- Riccati Equations
- Minimal Solutions and Balancing
- Performance

LTI Systems and Model Reduction

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du}\end{aligned}$$

$\mathbf{A} \quad \in \mathbb{R}^{n \times n}, \quad \mathbf{B} \in \mathbb{R}^{n \times m}, \quad \mathbf{C} \in \mathbb{R}^{p \times n}, \mathbf{D} \in \mathbb{R}^{p \times m} \quad n >> m, p$

Construct LOW dimensional system

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\ \hat{\mathbf{y}} &= \hat{\mathbf{C}}\hat{\mathbf{x}} + \mathbf{Du}\end{aligned}$$

Goal: $\underline{\hat{\mathbf{y}} \text{ should approximate } \mathbf{y}}$

Want the small system response to be the same as the original system response to the same input

Model Reduction by Projection

Assume $\mathbf{x} \in \mathcal{S}_V = Range(\mathbf{V})$ k-diml. subspace

i.e. $\mathbf{x} = \mathbf{V}\hat{\mathbf{x}}$

Then

$$\begin{aligned} \mathbf{V}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\mathbf{V}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{V}\hat{\mathbf{x}} + \mathbf{D}\mathbf{u} \end{aligned}$$

If we have $\mathbf{W}^T\mathbf{V} = \mathbf{I}_k$,

Then a k dimensional reduced model is

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\ \hat{\mathbf{y}} &= \hat{\mathbf{C}}\hat{\mathbf{x}} + \mathbf{D}\mathbf{u} \end{aligned}$$

where $\hat{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\hat{\mathbf{B}} = \mathbf{W}^T\mathbf{B}$, $\hat{\mathbf{C}} = \mathbf{C}\mathbf{V}$.

Preserving Passivity

Rohrer

Ober

Feldman, Freund, Bai

Gugercin, Antoulas

Positive Real Lemma

Yakubovich-Kalman-Popov-Anderson Lemma

Passive and Positive Real

Passive Systems:

$$Re \int_{-\infty}^{t} \mathbf{u}(\tau)^T \mathbf{y}(\tau) \mathrm{d}\tau \geq 0$$
for all $t \in \mathbb{R}$ and all $\mathbf{u} \in \mathcal{L}_2(\mathbb{R})$,

Positive Real: $\mathbf{G}(s) \equiv \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$**.**

(1) $\mathbf{G}(s)$ is analytic for $Re(s) > 0$,
(2) $\mathbf{G}(\bar{s}) = \overline{\mathbf{G}(s)}$ for all $s \in \mathbb{C}$,
(3) $\mathbf{G}(s) + (\mathbf{G}(s))^* \succeq \mathbf{0}$ for $Re(s) > 0$.

Spectral Zeros:

Property (3) $\Rightarrow$ $\mathbf{W}(s)$ (with stable inverse) s.t.
$\mathbf{G}(s) + \mathbf{G}^T(-s) = \mathbf{W}(s)\mathbf{W}^T(-s)$.

$\mathcal{S}_{\mathbf{G}} := \{s \in \mathbb{C} : \det[\mathbf{G}(s) + \mathbf{G}^T(-s)] = 0\}$

Assumptions:

- Transfer function: $\mathbf{G}(s) := \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$.
- $m = p$, The system is square.
- System is stable, controllable, observable and **passive**.
- $\mathcal{D} := \mathbf{D} + \mathbf{D}^T$ is positive definite
  $\mathcal{D} = \mathbf{W}_o^T \mathbf{W}_o$ with $\mathbf{W}_o$ nonsingular.

Spectral Zeros $\mathcal{S}_{\mathbf{G}}$

Finite generalized eigenvalues $\lambda \in \sigma(\mathcal{A}, \mathcal{E})$:

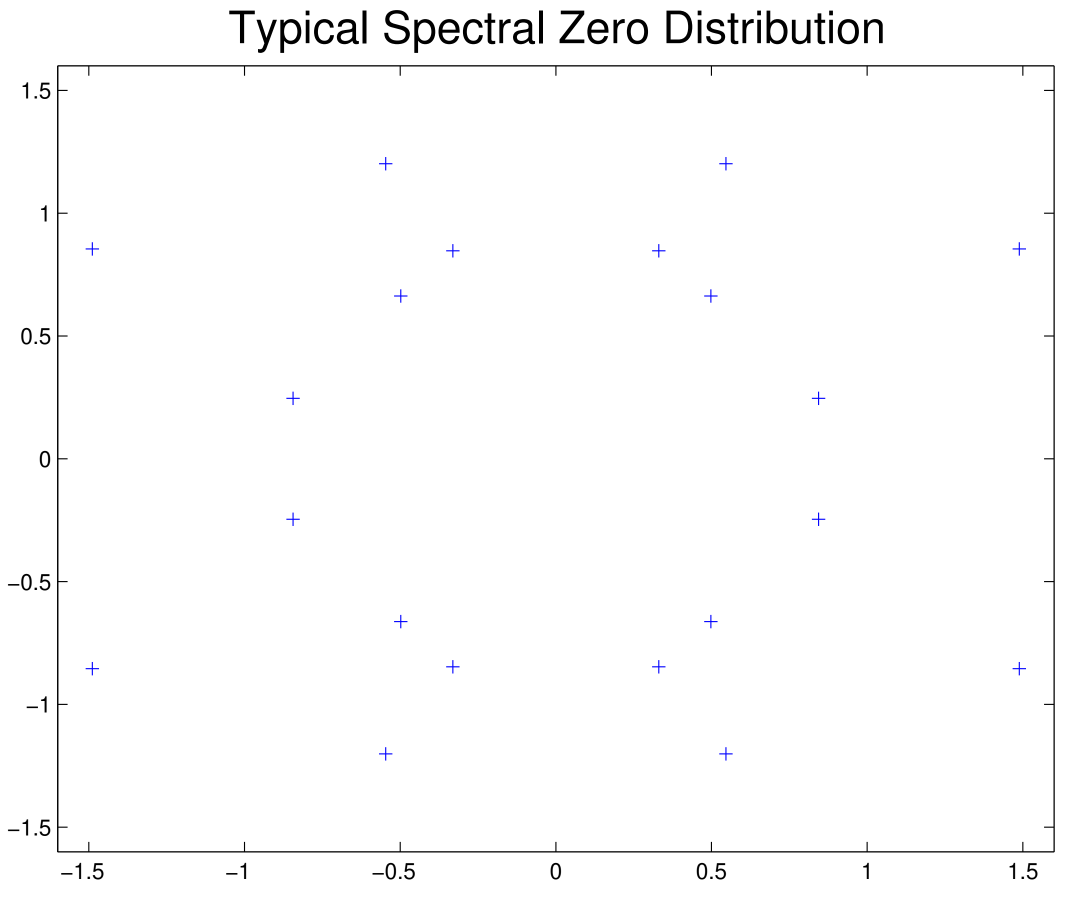
$$Rank(\mathcal{A} - \lambda\mathcal{E}) < 2n + p,$$

where

$$\mathcal{A} := \begin{bmatrix} \mathbf{A} & & \mathbf{B} \\ & -\mathbf{A}^T & -\mathbf{C}^T \\ \mathbf{C} & \mathbf{B}^T & \mathbf{D}+\mathbf{D}^T \end{bmatrix} \text{ and } \mathcal{E} := \begin{bmatrix} \mathbf{I} & & \\ & \mathbf{I} & \\ & & \mathbf{0} \end{bmatrix}$$

Reflection Properties: $\lambda \in \mathcal{S}_{\mathbf{G}} \iff -\bar{\lambda} \in \mathcal{S}_{\mathbf{G}}$.

Spectral Zero Symmetry



Passivity via Interpolation at Spectral Zeros

Theorem: (Antoulas) *If* $\mathcal{S}_{\hat{\mathbf{G}}} \subset \mathcal{S}_{\mathbf{G}}$ *and* $\hat{\mathbf{G}}$ *interpolates* $\mathbf{G}$ *on* $\mathcal{S}_{\hat{\mathbf{G}}}$*, then the reduced system* $(\hat{A}, \hat{B}, \hat{C}, D)$ *is both passive and stable.*

Interpolation via Invariant Subspace

$$\begin{bmatrix} \mathbf{A} & & \mathbf{B} \\ & -\mathbf{A}^T & -\mathbf{C}^T \\ \mathbf{C} & \mathbf{B}^T & \mathbf{D}+\mathbf{D}^T \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{0} \end{bmatrix} \mathbf{R}.$$

Construct $\mathbf{X} = \mathbf{V}\hat{\mathbf{X}}, \quad \mathbf{Y} = \mathbf{W}\hat{\mathbf{Y}}, \quad \hat{\mathbf{A}} := \mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \mathbf{W}^T\mathbf{V} = \mathbf{I}$

$$\begin{bmatrix} \hat{\mathbf{A}} & & \hat{\mathbf{B}} \\ & -\hat{\mathbf{A}}^T & -\hat{\mathbf{C}}^T \\ \hat{\mathbf{C}} & \hat{\mathbf{B}}^T & \mathbf{D}+\mathbf{D}^T \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}} \\ \hat{\mathbf{Y}} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{X}} \\ \hat{\mathbf{Y}} \\ \mathbf{0} \end{bmatrix} \mathbf{R}.$$

Positive Real Lemma – Sufficiency

Positive Real Lemma

Sufficiency: If $\hat{\mathbf{A}}$ *is stable and there exists* $\hat{P} \succeq \mathbf{0}$, $\mathbf{C}_o$, $\mathbf{W}_o$ such that

$$\begin{aligned} \hat{\mathbf{A}}\hat{P} + \hat{P}\hat{\mathbf{A}}^T &= -\mathbf{C}_o^T\mathbf{C}_o, \\ \hat{\mathbf{C}}\hat{P} - \hat{\mathbf{B}}^T &= -\mathbf{W}_o^T\mathbf{C}_o, \\ \mathbf{D} + \mathbf{D}^T &= \mathbf{W}_o^T\mathbf{W}_o, \end{aligned}$$

then $\mathbf{G}$ *is positive real.*

Prelim to Positive Real Sufficiency

Lemma 1 :

If $\sigma(\mathbf{R}) \subset \mathbb{C}_+$ *or* $\sigma(\mathbf{R}) \subset \mathbb{C}_-$*, then* $\mathbf{X}$ *and* $\mathbf{Y}$ *are both full rank with* $\mathbf{X}^T\mathbf{Y} = \mathbf{Y}^T\mathbf{X}$*.*

Lemma 2 :

$$\begin{aligned} \mathbf{Y}^T\mathbf{A}\mathbf{X} + \mathbf{X}^T\mathbf{A}^T\mathbf{Y} &= \mathbf{Z}^T\mathcal{D}\mathbf{Z} \\ \mathbf{C}\mathbf{X} + \mathbf{B}^T\mathbf{Y} &= -\mathcal{D}\mathbf{Z} \end{aligned}$$

Riccati Equations

Laub

Byers

Bunse-Gerstner and Merhmann

Ammar, Benner, and Merhmann

Riccati Eqns., Minimal Solutions, and Balancing

Interpolate n spectral zeros

$$\begin{aligned} \mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T &= -(\mathbf{B} - \mathcal{P}\mathbf{C}^T)\mathcal{D}^{-1}(\mathbf{B}^T - \mathbf{C}\mathcal{P}) \\ \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} &= -(\mathbf{C}^T - \mathcal{Q}\mathbf{B})\mathcal{D}^{-1}(\mathbf{C} - \mathbf{B}^T\mathcal{Q}) \end{aligned}$$

with

$$\mathcal{P} = -\mathbf{X}\mathbf{Y}^{-1} \succ \mathbf{0}, \quad \mathcal{Q} = -\mathbf{Y}\mathbf{X}^{-1} \succ \mathbf{0}.$$

For $k << n$ (low dimensional invariant subspace),

$$\mathcal{P}_k = -\mathbf{X}(\mathbf{X}^T\mathbf{Y})^{-1}\mathbf{X}^T \quad \text{and} \quad \mathcal{Q}_k = -\mathbf{Y}(\mathbf{Y}^T\mathbf{X})^{-1}\mathbf{Y}^T$$

give low rank approximate solutions.

Minimal and Maximal Solutions

Interpolate n spectral zeros in right half plane

$$\mathcal{P} \text{ is } \underline{\text{minimal}} \text{ and } \mathcal{Q} = \mathcal{P}^{-1} \text{ is maximal}$$

Interpolate n spectral zeros in left half plane

$$\mathcal{Q} \text{ is } \underline{\text{minimal}} \text{ and } \mathcal{P} = \mathcal{Q}^{-1} \text{ is maximal}$$

Definition:

$\mathcal{Q}$ is maximal if $\mathcal{Q} \succeq \hat{\mathcal{Q}}$ for any other solution $\hat{\mathcal{Q}}$

$\mathcal{Q}$ is minimal if $\mathcal{Q} \preceq \hat{\mathcal{Q}}$ for any other solution $\hat{\mathcal{Q}}$

Key: (anti) stability of $\tilde{\mathbf{A}} := \mathbf{A} - \mathbf{B}\mathcal{D}^{-1}(\mathbf{C} - \mathbf{B}^T\mathcal{Q})$

Minimal Solutions in Factored Form – The GSVD

Useful tool: **GSVD**

There is a computable pair of factorizations

$$\mathbf{X} = \mathbf{Q}_x \mathbf{C}_x \mathbf{G} \quad \text{and} \quad \mathbf{Y} = \mathbf{Q}_y \mathbf{S}_y \mathbf{G},$$

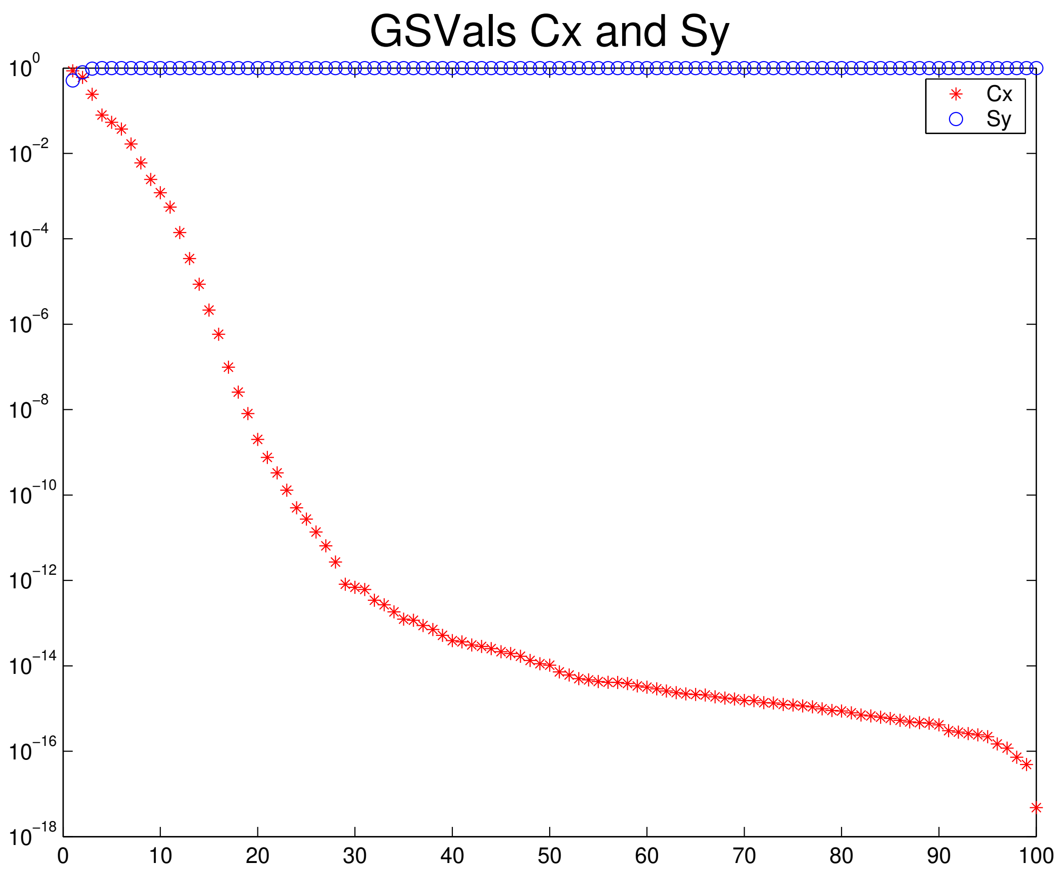
with Q_x, Q_y orthogonal, $\mathbf{C}_x, \mathbf{S}_y$ diagonal and non-negative with $\mathbf{C}_x^2 + \mathbf{S}_y^2 = \mathbf{I}$.

Key:

$$\mathbf{X}\mathbf{Y}^{-1} = \mathbf{Q}_x (\mathbf{C}_x / \mathbf{S}_y) \mathbf{Q}_y^T = -\mathbf{Q}_x (\mathbf{C}_x / \mathbf{S}_y) \mathbf{Q}_x^T$$

$$\mathbf{Y}\mathbf{X}^{-1} = \mathbf{Q}_y (\mathbf{S}_y / \mathbf{C}_x) \mathbf{Q}_x^T = -\mathbf{Q}_y (\mathbf{S}_y / \mathbf{C}_x) \mathbf{Q}_y^T \text{ !!}$$

Minimal Solutions are Low (est) Rank



Balanced Passive Truncation

Minimal Gramians:

$$\mathcal{P} = \mathbf{V}_\rho \mathbf{V}_\rho^T \quad \text{(RHP interp.)}, \qquad \mathcal{Q} = -\mathbf{W}_\lambda \mathbf{W}_\lambda^T \quad \text{(LHP interp.)}$$

$$\mathbf{Q}_v \mathbf{S}_\beta^2 \mathbf{Q}_w^T = \mathbf{V}_\rho^T \mathbf{W}_\lambda \text{ (SVD)}$$

Put

$$\mathbf{V}_\beta := \mathbf{V}_\rho \mathbf{Q}_v \mathbf{S}_\beta^{-1}, \qquad \mathbf{W}_\beta := \mathbf{W}_\lambda \mathbf{Q}_w \mathbf{S}_\beta^{-1},$$

Can Truncate in Both Places

Passive Balanced (Reduced) Model

$$\mathbf{A}_\beta := \mathbf{W}_\beta^T \mathbf{A} \mathbf{V}_\beta, \; \mathbf{B}_\beta = \mathbf{W}_\beta^T \mathbf{B}, \; \mathbf{C}_\beta = \mathbf{C} \mathbf{V}_\beta.$$

Riccati Equations in Balanced Form

$$\hat{\mathbf{A}}_\beta^T \mathcal{S} + \mathcal{S}\hat{\mathbf{A}}_\beta = -(\hat{\mathbf{C}}_\beta^T - \mathcal{S}\hat{\mathbf{B}}_\beta)\mathcal{D}^{-1}(\hat{\mathbf{C}}_\beta - \hat{\mathbf{B}}_\beta^T \mathcal{S}).$$

$$\hat{\mathbf{A}}_\beta \mathcal{S} + \mathcal{S}\hat{\mathbf{A}}_\beta^T = -(\hat{\mathbf{B}}_\beta - \mathcal{S}\hat{\mathbf{C}}_\beta^T)\mathcal{D}^{-1}(\hat{\mathbf{B}}_\beta^T - \hat{\mathbf{C}}_\beta \mathcal{S}).$$

Every Truncation gives

Balanced, Stable, Passive Reduced Model

Reduction with Low Rank ($k << n$) X,Y

With

$$\mathbf{X} = \mathbf{Q}_x \mathbf{C}_x \mathbf{G} \;\; \text{and} \;\; \mathbf{Y} = \mathbf{Q}_y \mathbf{S}_y \mathbf{G},$$

Low rank approximate gramian is

$$-\mathbf{X}(\mathbf{X}^T\mathbf{Y})^{-1}\mathbf{X}^T = \hat{\mathbf{Q}}_x \hat{\mathcal{P}} \hat{\mathbf{Q}}_x^T,$$

where $\hat{\mathcal{P}} := -(\mathbf{Q}_y^T \mathbf{Q}_x)(\mathbf{C}_x / \mathbf{S}_y)$ and $\hat{\mathbf{Q}}_x := \mathbf{Q}_x(\mathbf{Q}_y^T \mathbf{Q}_x)^{-1}$

Note: $\mathbf{Q}_y^T \hat{\mathbf{Q}}_x = \mathbf{I}_k$ and

$$(\mathbf{C}_x/\mathbf{S}_y)(\mathbf{Q}_x^T \mathbf{Q}_y) = \mathbf{S}_y^{-1}\mathbf{G}^{-T}\mathbf{X}^T\mathbf{Y}\mathbf{G}^{-1}\mathbf{S}_y^{-1} \prec \mathbf{0}.$$

(symm. neg. def)

Reduction with Low Rank ($k << n$) X,Y

Recall:

$$\begin{aligned}\mathbf{Y}^T\mathbf{A}\mathbf{X}+\mathbf{X}^T\mathbf{A}^T\mathbf{Y} &= \mathbf{Z}^T\mathcal{D}\mathbf{Z}\\ \mathbf{C}\mathbf{X}+\mathbf{B}^T\mathbf{Y} &= -\mathcal{D}\mathbf{Z}\end{aligned}$$

This transforms to

$$\begin{aligned}\hat{\mathbf{A}}\hat{\mathcal{P}}+\hat{\mathcal{P}}\hat{\mathbf{A}}^T &= -\hat{\mathbf{Z}}^T\mathcal{D}\hat{\mathbf{Z}}\\ \hat{\mathbf{C}}\hat{\mathcal{P}}-\hat{\mathbf{B}}^T &= \mathcal{D}\hat{\mathbf{Z}}\end{aligned}$$

with $\hat{\mathbf{A}}=\mathbf{Q}_y^T\mathbf{A}\hat{\mathbf{Q}}_x$, $\hat{\mathbf{C}}=\mathbf{C}\hat{\mathbf{Q}}_x$ and $\hat{\mathbf{B}}=\mathbf{Q}_y^T\mathbf{B}$.

Follows from

$$\mathbf{S}_y^{-1}\mathbf{G}^{-T}[\mathbf{Y}^T\mathbf{A}\mathbf{X}]\mathbf{G}^{-1}\mathbf{S}_y^{-1}=\mathbf{Q}_y^T\mathbf{A}\hat{\mathbf{Q}}_x\hat{\mathcal{P}}$$

The Reduced Model is Passive !

We have

$$\begin{aligned} \hat{\mathbf{A}}\hat{\mathcal{P}} + \hat{\mathcal{P}}\hat{\mathbf{A}}^T &= -\hat{\mathbf{Z}}^T\mathcal{D}\hat{\mathbf{Z}} \\ \hat{\mathbf{C}}\hat{\mathcal{P}} - \hat{\mathbf{B}}^T &= -\mathcal{D}\hat{\mathbf{Z}}, \end{aligned}$$

which gives

$$\begin{aligned} \hat{\mathbf{A}}\hat{\mathcal{P}} + \hat{\mathcal{P}}\hat{\mathbf{A}}^T &= -\mathbf{C}_o^T\mathbf{C}_o, \\ \hat{\mathbf{C}}\hat{P} - \hat{\mathbf{B}}^T &= -\mathbf{W}_o^T\mathbf{C}_o, \\ \mathbf{D} + \mathbf{D}^T &= \mathbf{W}_o^T\mathbf{W}_o, \end{aligned}$$

where $\mathbf{C}_o := \mathbf{W}_o\hat{\mathbf{Z}}$.

This is PRL sufficiency.

Passive Reduction Algorithm

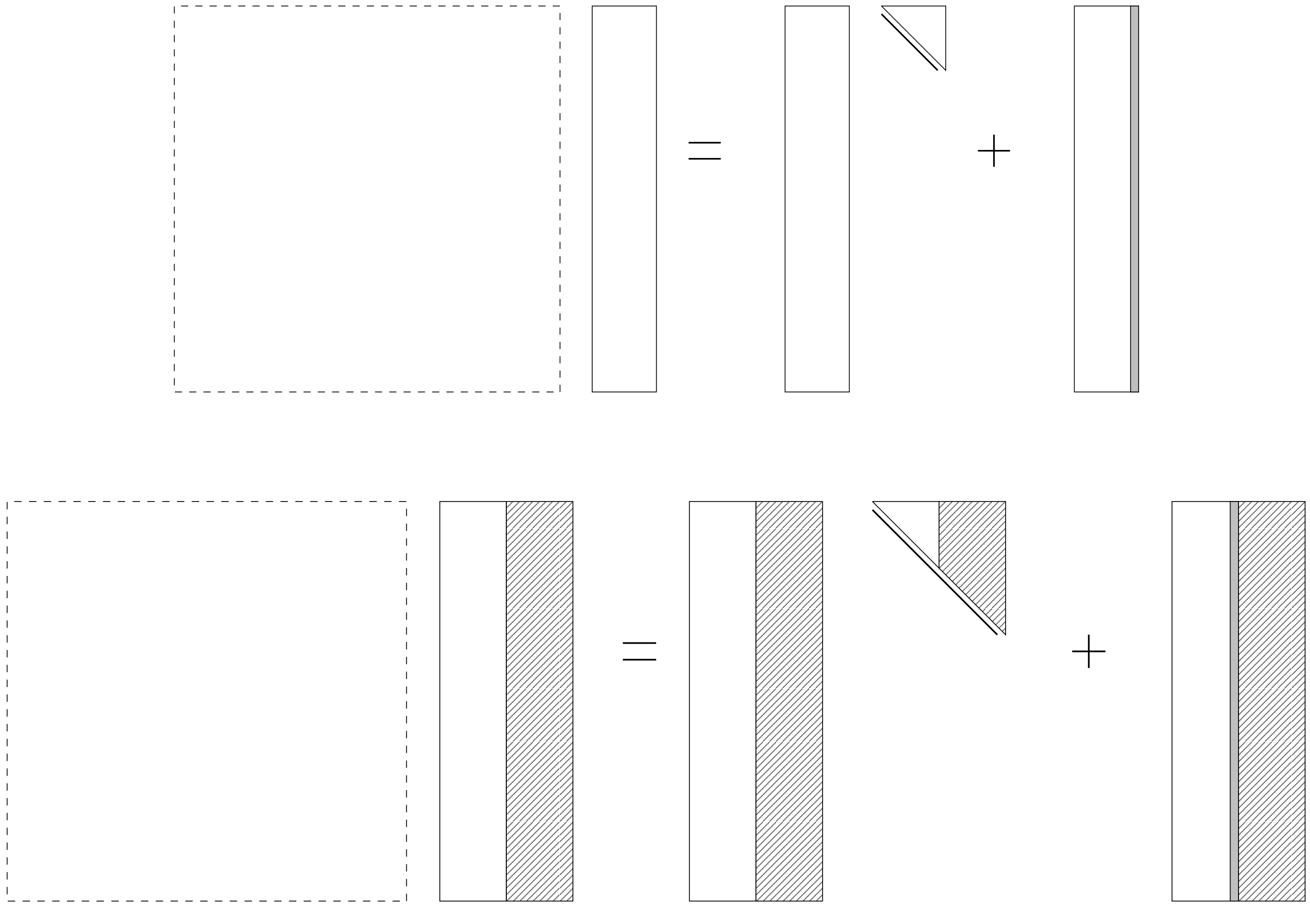
function $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}] = \mathrm{Posreal}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, k)$;

Compute a k-th order partial real Schur decomposition
$\mathcal{A}\mathbf{Q} = \mathcal{E}\mathbf{Q}\mathbf{R}$;

$\mathbf{X} = \mathbf{Q}(1\!:\!n, :); \quad \mathbf{Y} = \mathbf{Q}(n\!+\!1\!:\!2n, :);$
$[\mathbf{Q}_x, \mathbf{Q}_y, \mathbf{G}, \mathbf{C}_x, \mathbf{S}_y] = \mathrm{gsvd}(\mathbf{X}, \mathbf{Y}); \quad \mathbf{S} \leftarrow (\mathbf{C}_x/\mathbf{S}_y)^{1/2};$
$\mathbf{V} = \mathbf{Q}_x/(\mathbf{Q}_y^T\mathbf{Q}_x); \quad \mathbf{W} = \mathbf{Q}_y;$

$\hat{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}; \quad \hat{\mathbf{B}} = \mathbf{W}^T\mathbf{B}; \quad \hat{\mathbf{C}} = \mathbf{C}\mathbf{V};$

Implicitly Restarted Arnoldi



Example: RLC Ladder Circuit

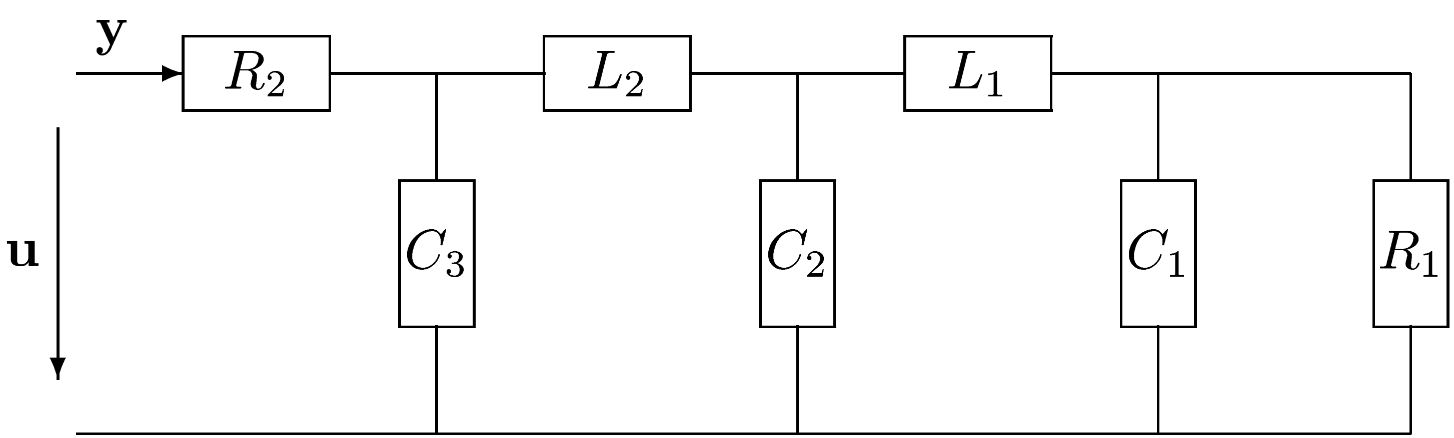


Figure 1: RLC circuit of order 5

n is odd (201)

x_{2i-1} is voltage across capacitor C_i for $i = 1, 2, \ldots \frac{n+1}{2}$

x_{2i} is the current through inductor L_i for $i = 1, 2, \ldots \frac{n-1}{2}$

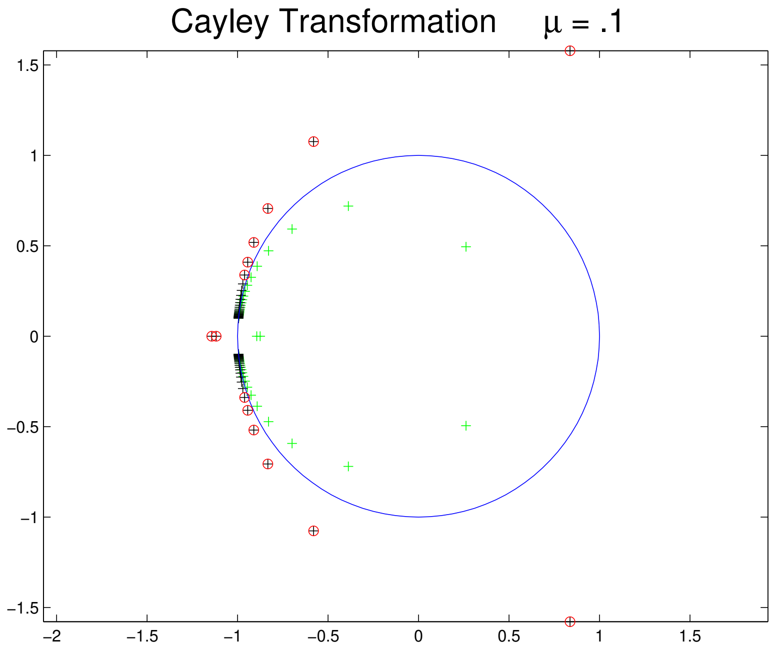
Two resistors R_1, R_2 at either end

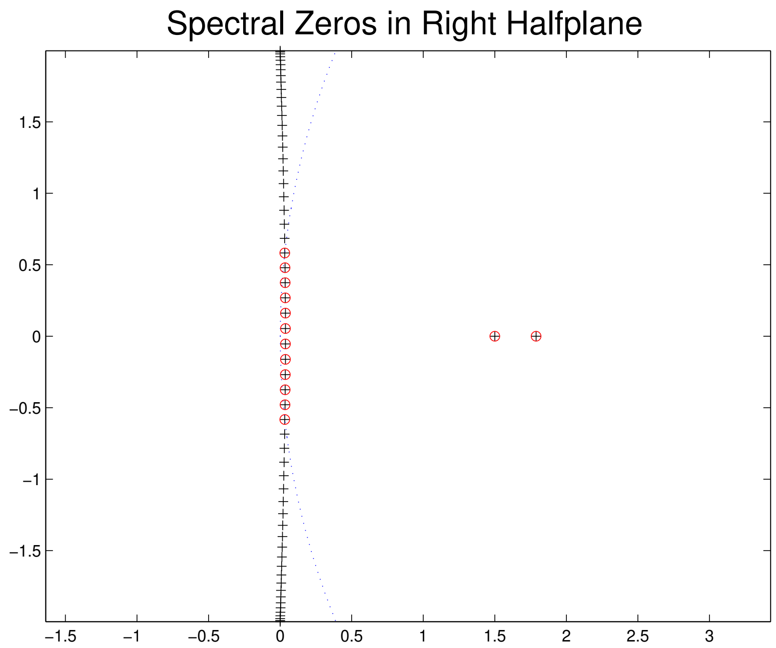
RLC Circuit of Order 5

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -5 \end{bmatrix}, \; \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \; \mathbf{C} = [0\ 0\ 0\ 0\ -2],$$

$$\mathbf{D} = 1$$

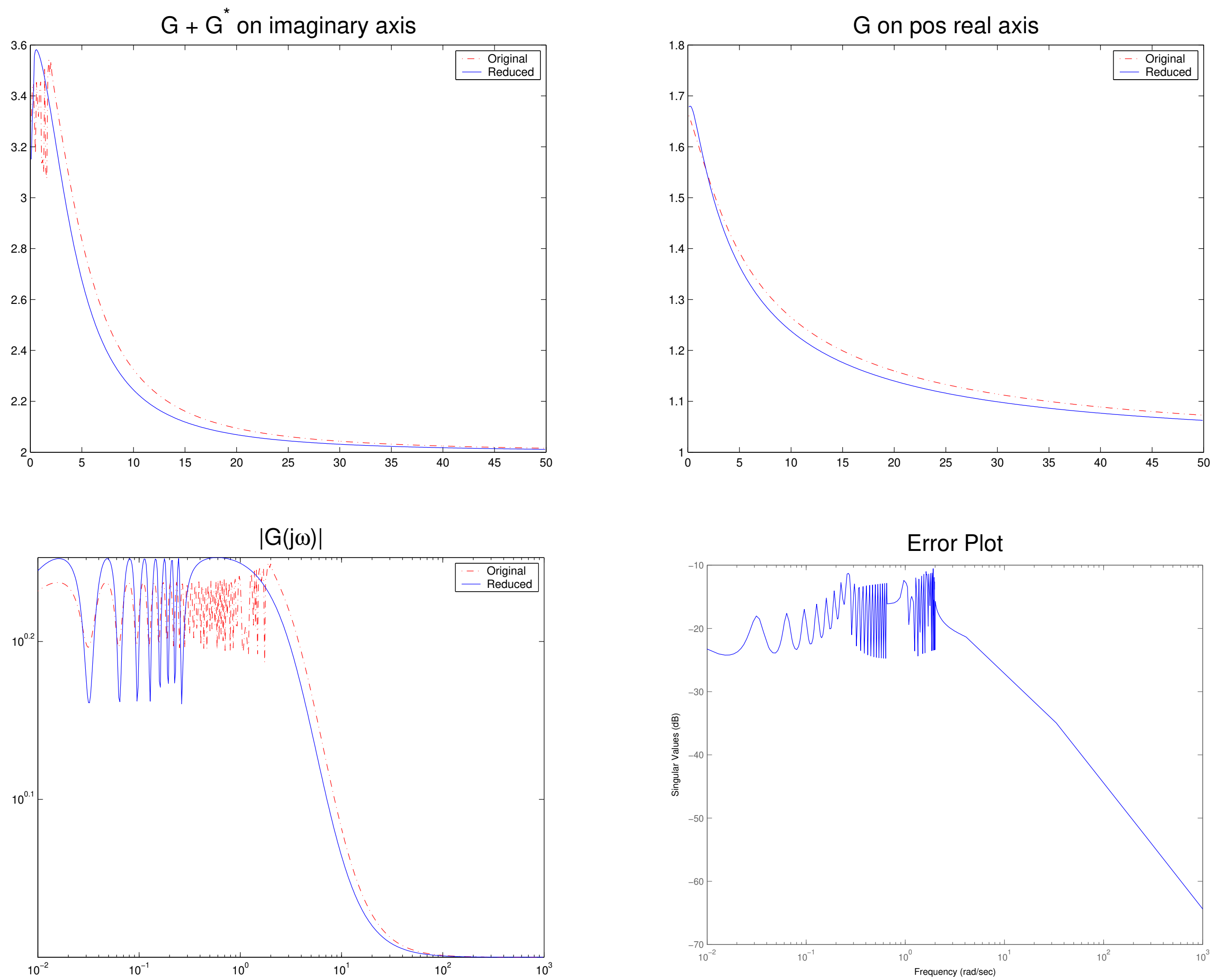
Cayley Transformation and Spectral Zeros





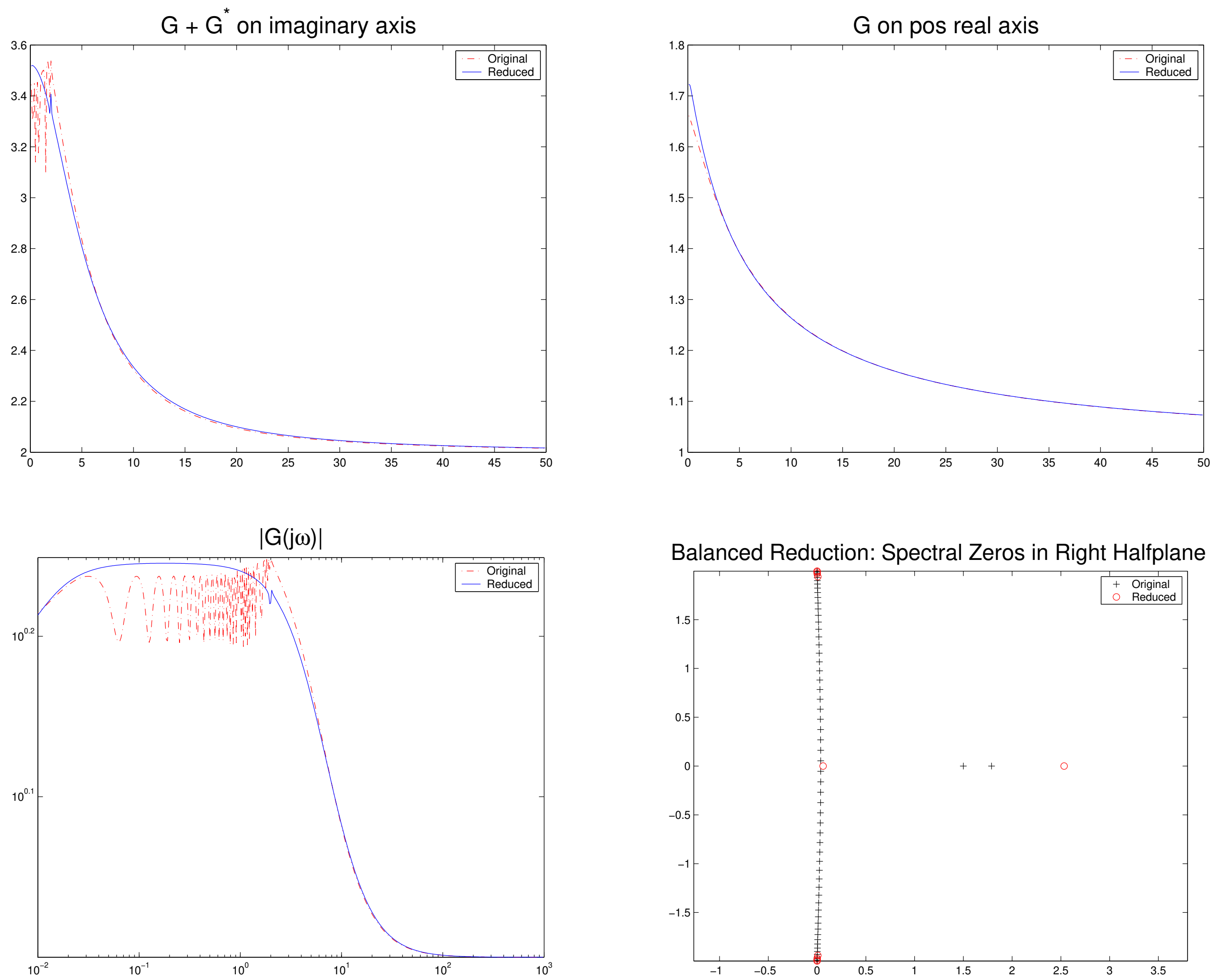
Passive Reduction

$n = 201,\ k = 20$

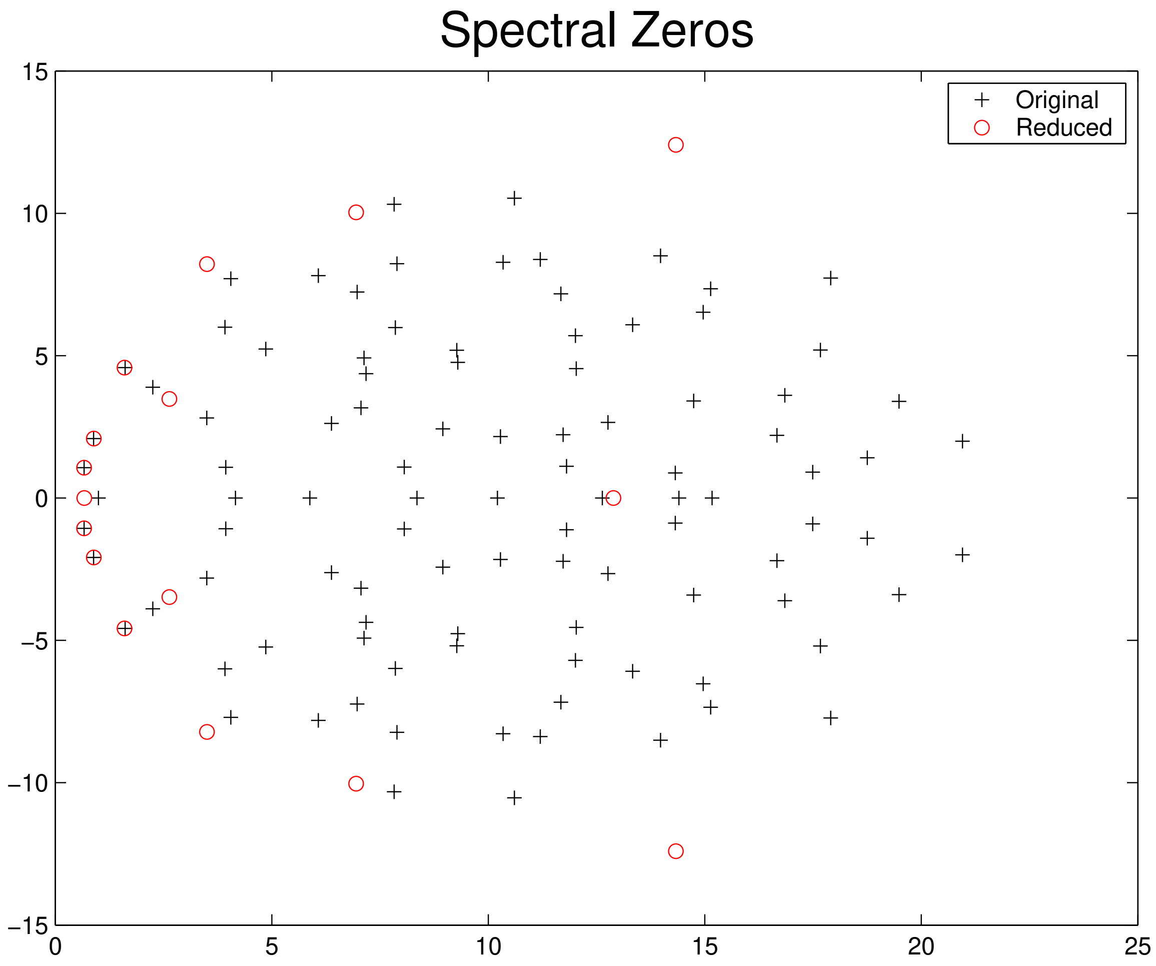


Balanced Passive Reduction

$n = 201,\ k = 10$



Random Problem Spectral Zeros.



Symmetric Spiral Passive Model Reduction

From: J.-R. Li and J. White (via P. Benner)

Symmetric Spiral Passive Model Reduction

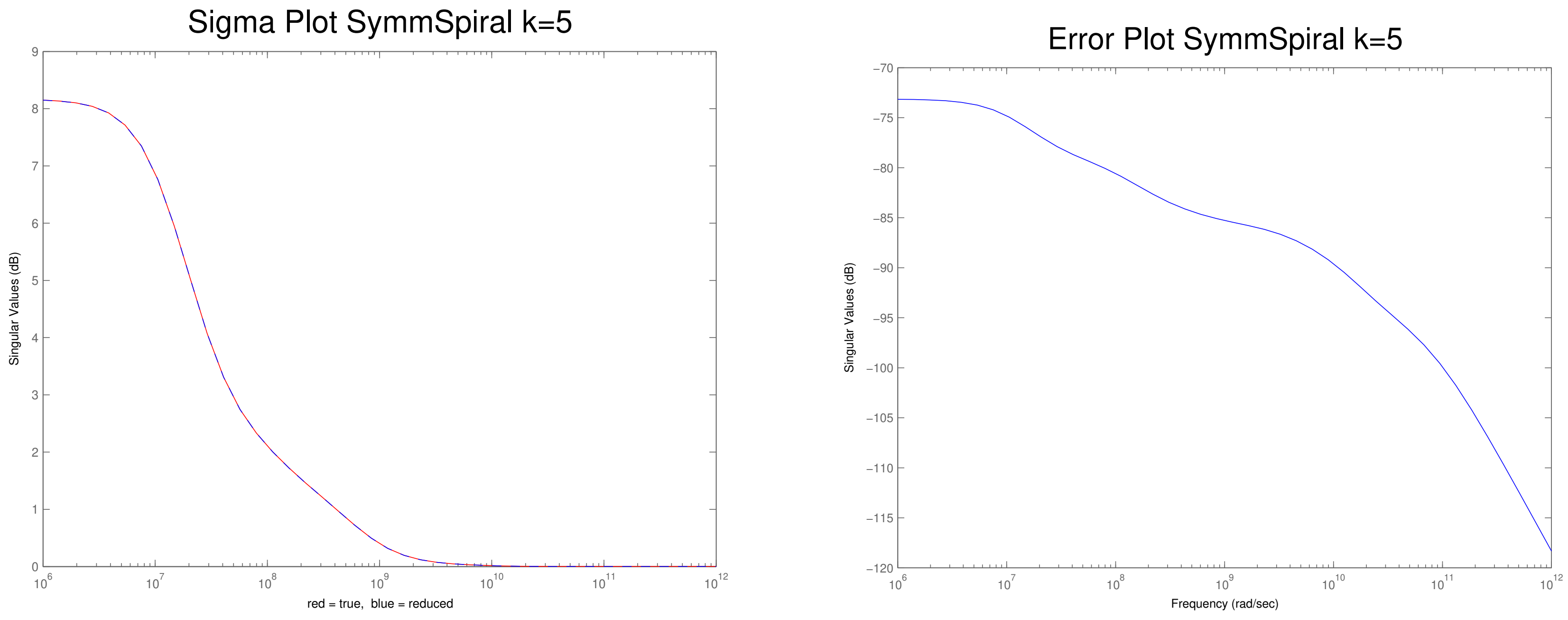
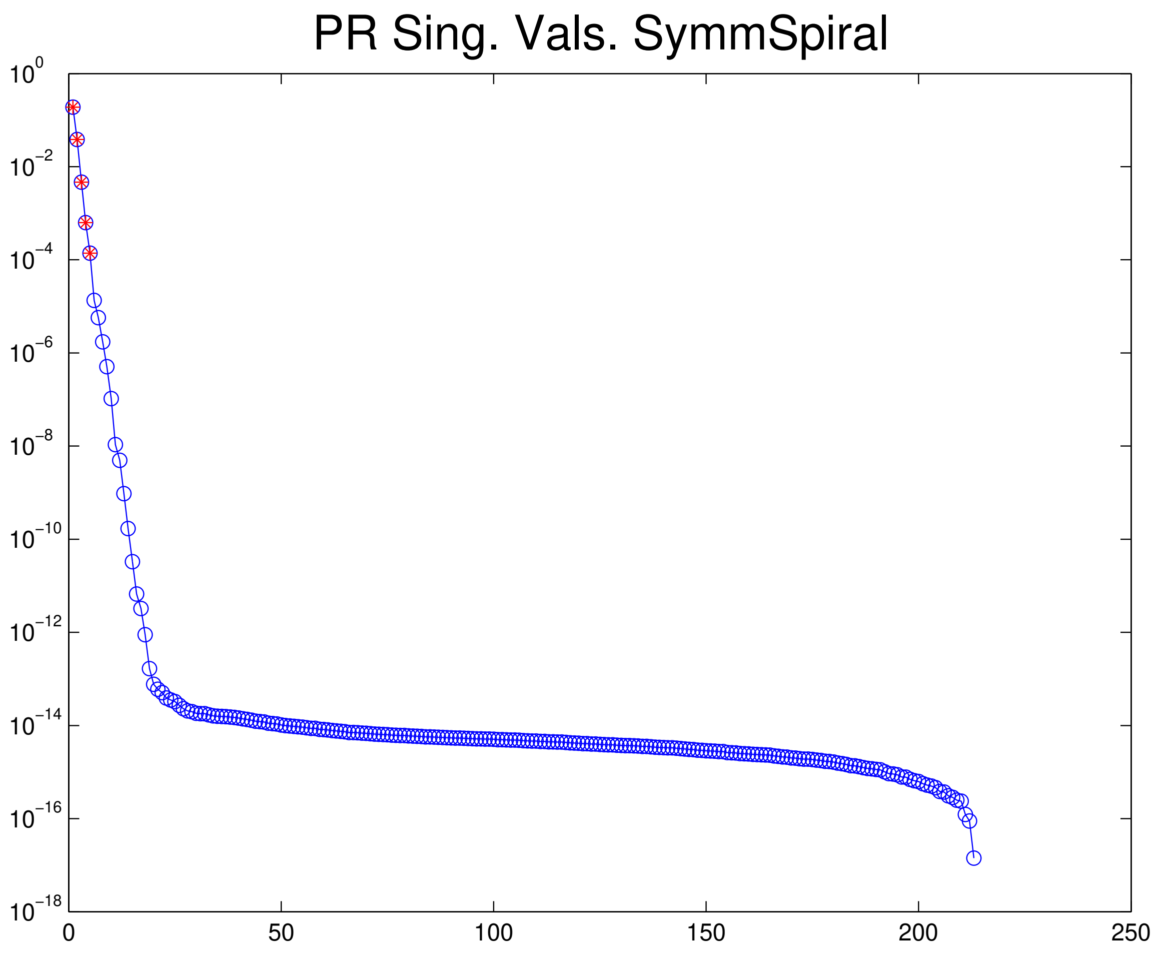
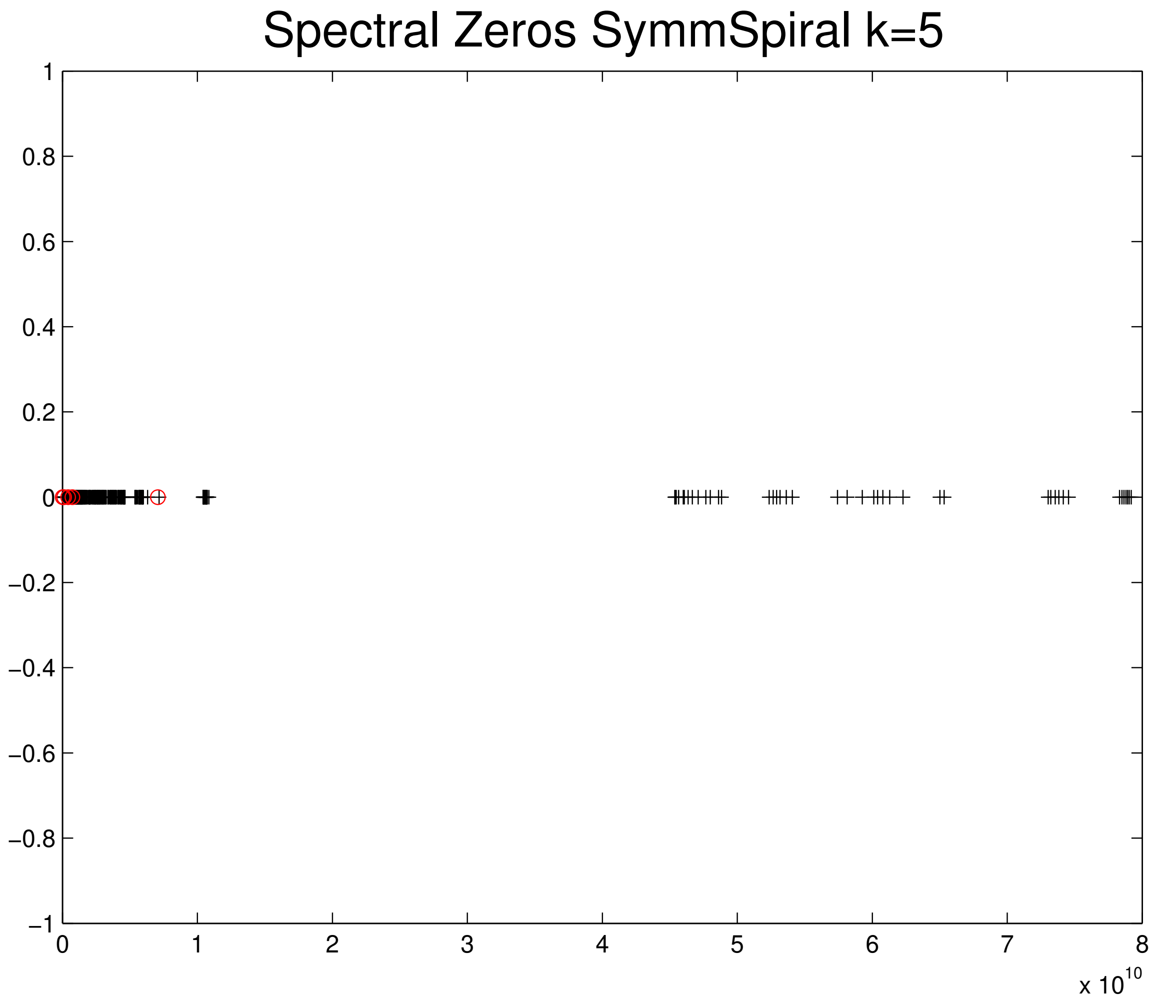


Figure 2: Left: Bode Plot True vs. Reduced, Right: Error Plot

Symm. Spiral Passive Singular Vals.



Symm. Spiral Spectral Zeros.



Future Work

1. Obtain experience with more (realistic) examples
2. Develop Riccati solver – Low Rank Stabilizing Approximate Solution
3. Learn how to choose "best" spectral zeros for interpolation minimize error
4. Approximate reduced order balancing – with error bounds

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- D.C. Sorensen, *Passivity Preserving Model Reduction via Interpolation of Special Zeros,* Tech. Rept. 02-15, Dept. Comp. and Appl. Math., Rice University, (2002).
- D.C. Sorensen, Y. Zhou, and A.C. Antoulas, *On the Decay Rate of Hankel Singular Values and Related issues, Systems and Control Lett.*, **46** , (5), pp. 323-342 (2002).
  (Caam Tech. Rept. TR01-09)
- S. Gugercin, A. C. Antoulas, and D. C. Sorensen, *A modified low-rank Smith method for large-scale Lyapunov equations,* **Numerical Algorithms**, **32**, Issue 1, pp. 27-55, Jan. 2003.
  (Caam Tech Rept TR01-10)