Overview of Grammian Based Model Reduction

D.C. Sorensen

joint work with **A.C. Antoulas**
also **S. Gugercin and Y.K. Zhou**

Rice University
Houston, TX

Second Sandia Workshop on PDE-constrained Optimization
The Bishop's Lodge, Santa Fe, NM 2004

Grammian Based Reduction - LTI Systems

Motivation:

- Recent Workshops
  - Oberwolfach (Oct 03)
  - Banff (Apr 04)
- Overview
  - Krylov Methods
  - Relation POD – Grammian Based Reduction
  - Balanced Reduction
  - Solving Large Lyapunov Equations
  - Preserving Passivity
  - Second Order Systems

LTI Systems and Model Reduction

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{x}
\end{aligned}
$$

$$\mathbf{A} \quad \in \mathbf{R}^{n\times n}, \quad \mathbf{B} \in \mathbf{R}^{n\times m}, \quad \mathbf{C} \in \mathbf{R}^{p\times n}, \quad n >> m, p$$

Model Reduction: Construct $\{\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}\}$

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\
\hat{\mathbf{y}} &= \hat{\mathbf{C}}\hat{\mathbf{x}}
\end{aligned}
$$

with LOW dimension $k << n$

Goal: $\underline{\hat{\mathbf{y}} \text{ should approximate } \mathbf{y}}$

Model Reduction by Projection

Approximate $\mathbf{x} \in \mathcal{S}_V = Range(\mathbf{V})$ k-diml. subspace

i.e. Put $\mathbf{x} = \mathbf{V}\hat{\mathbf{x}}$, and then force

$$\begin{aligned} \mathbf{W}^T[\mathbf{V}\dot{\hat{\mathbf{x}}} &- (\mathbf{A}\mathbf{V}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u})] = 0 \\ \hat{\mathbf{y}} &= \mathbf{C}\mathbf{V}\hat{\mathbf{x}} \end{aligned}$$

If $\mathbf{W}^T\mathbf{V} = \mathbf{I}_k$,

Then a k dimensional reduced model is

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\ \hat{\mathbf{y}} &= \hat{\mathbf{C}}\hat{\mathbf{x}} \end{aligned}$$

where $\hat{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\hat{\mathbf{B}} = \mathbf{W}^T\mathbf{B}$, $\hat{\mathbf{C}} = \mathbf{C}\mathbf{V}$.

Transfer Function – Approximation

$$\mathcal{G}(s) \equiv \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} \quad \text{and} \quad \hat{\mathcal{G}}(s) \equiv \hat{\mathbf{C}}(s\mathbf{I}-\hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}$$

In frequency domain,

$$\mathbf{y}(s) = \mathcal{G}(s)\mathbf{u}(s) \quad \text{and} \quad \hat{\mathbf{y}}(s) = \hat{\mathcal{G}}(s)\mathbf{u}(s)$$

Rational Approximation $\qquad \hat{\mathcal{G}}(s) \approx \mathcal{G}(s)$

Local: Moment Matching = Krylov Projection

Global : Grammian Based = SVD ,
(*Balanced Truncation*)

Moments $\leftrightarrow$ Krylov Subspace Projection

- Padé via Lanczos (PVL)
  - Freund, Feldmann
  - Bai, Freund, Feldmann
- Multipoint Rational Interpolation
  - Grimme
  - Gallivan, Grimme, Van Dooren

Moment Matching by Krylov Projection

Arnoldi – Markov Parameters – Match at ∞

$$\mathbf{AV} = \mathbf{VH} + \mathbf{f}\mathbf{e}_k^T$$

Shift Invert Arnoldi – Match at σ_o

$$(\mathbf{A} - \sigma_o \mathbf{I})^{-1}\mathbf{V} = \mathbf{VH} + \mathbf{f}\mathbf{e}_k^T$$

Rational Krylov (Ruhe) – Multipoint Rational Interpolation

$$\mathbf{AV}_k\mathbf{H}_k = \mathbf{V}_k\mathbf{K}_k + \mathbf{f}_k\mathbf{e}_k^T$$

$$(\mathbf{A} - \sigma_j \mathbf{I})^{-\ell_j}\mathbf{b} \in Range(\mathbf{V}), \quad \text{at several } \sigma_j$$

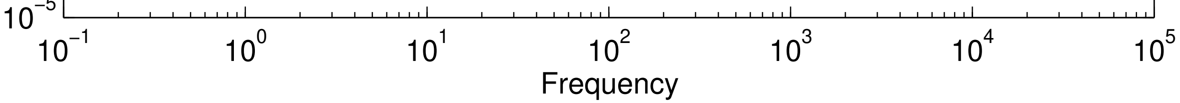


Figure 2.1: Partial Realization and Padé Approximation of Example 2.1

From Eric Grimme's Ph.D. Thesis

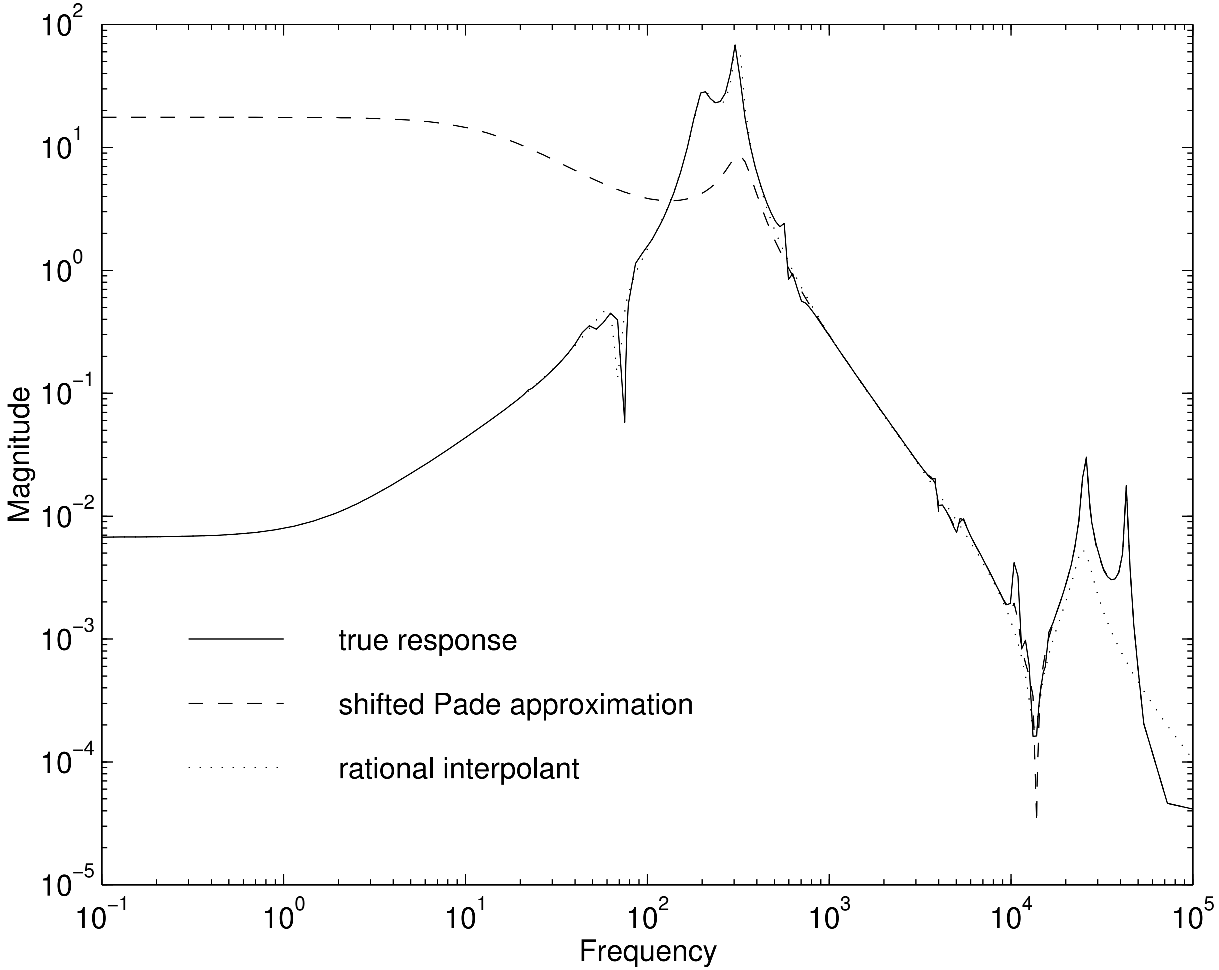


Figure 2.2: Shifted Padé Approximation and Rational Interpolation of Example 2.1

Summary – Krylov Methods

- Efficient Means for Moment Matching
- Tradeoffs – Arnoldi (Orthogonal) vs Lanczos (Oblique) Projection
- Multipoint Rational Interpolation – Very Effective
- Automatic Error Control, Bounds ?
- Aposteriori Error Bounds - S. Gugercin's Thesis
- Preservation of System Properties (Implicit Restarting)

Grammian Based Model Reduction

Proper Orthogonal Decomposition (POD)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{y} = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$$

The Grammian

$$\mathcal{P} = \int_o^\infty \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau$$

Eigenvectors of $\mathcal{P}$

$$\mathcal{P} = \mathbf{V}\mathbf{S}^2\mathbf{V}^T$$

Orthogonal Basis

$$\mathbf{x}(t) = \mathbf{V}\mathbf{S}\mathbf{w}(t)$$

POD Reduced Basis

Low Rank Approximation

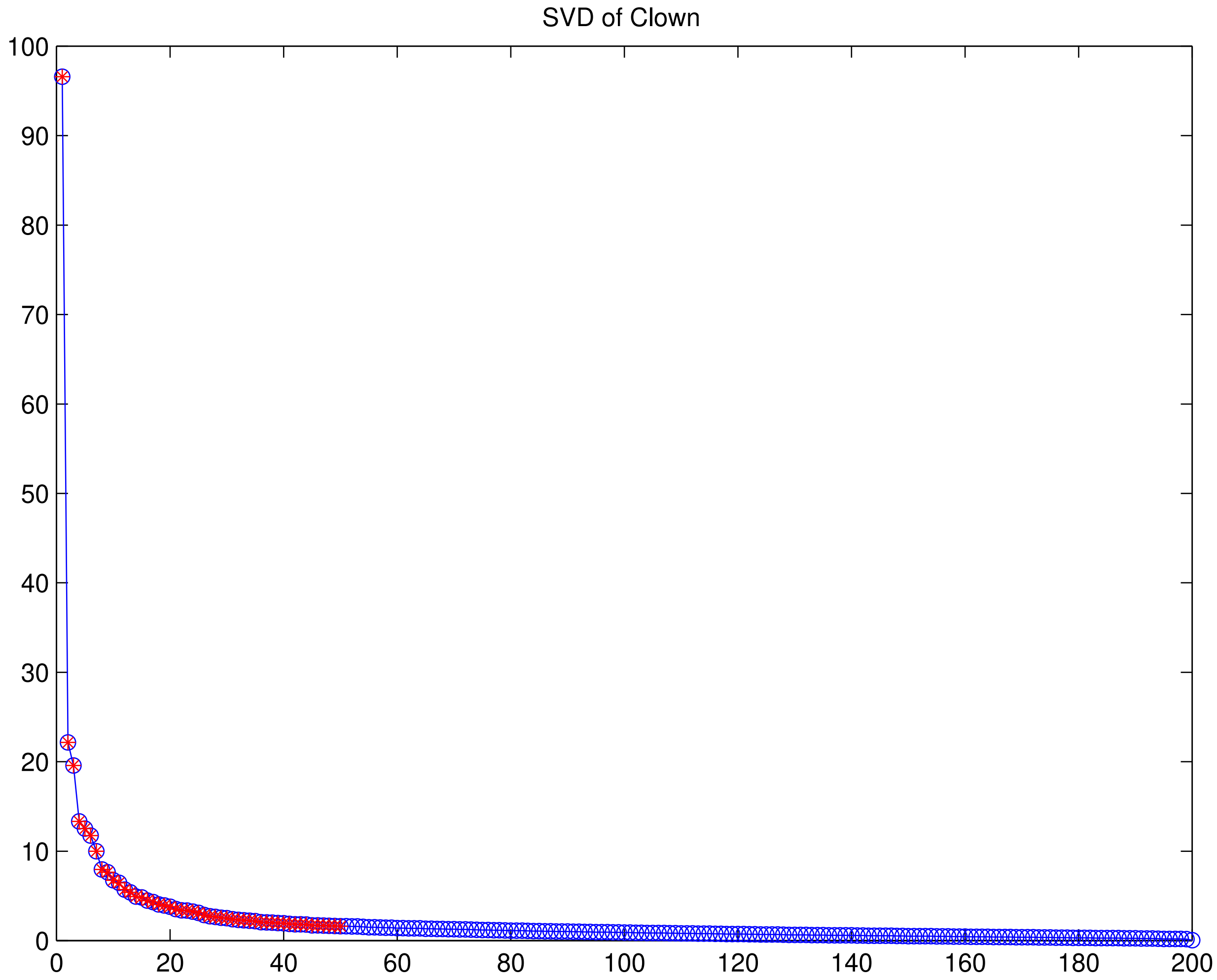
$$\mathbf{x} \approx \mathbf{V}_k \hat{\mathbf{x}}_k(t), \quad \text{where} \quad \hat{\mathbf{x}}_k(t) = \mathbf{S}_k \mathbf{w}_k(t)$$

Snapshot Approximation to $\mathcal{P}$

$$\mathcal{P} \approx \frac{1}{m} \sum_{j=1}^{m} \mathbf{x}(t_j)\mathbf{x}(t_j)^T$$

Get $\mathbf{V}_k$ and $\mathbf{S}_k$ from SVD of Snapshot Trajectory

$$\mathbf{X} = [\mathbf{x}(t_1), \mathbf{x}(t_2), \ldots, \mathbf{x}(t_m)]$$


SVD of Clown
100
90
80
70
60
50
40
30
20
10
0
0
20
40
60
80
100
120
140
160
180
200

POD for LTI systems

Impulse Response – Input to State Map: $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{B}$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{u} = \delta(t) \quad \text{– unit impuse}$$

Controllability Grammian:

$$\mathcal{P} = \int_o^\infty \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau = \int_o^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau$$

State to Output Map: $\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$

Observability Grammian:

$$\mathcal{Q} = \int_o^\infty e^{\mathbf{A}^T\tau}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}\tau} d\tau$$

Balanced Reduction

Control Energy

Min. Input Energy: $0 \rightarrow \mathbf{x}$ $\qquad$ $\mathcal{E}_r^2 \equiv \mathbf{x}^T \mathcal{P}^{-1} \mathbf{x}$

Max. Observation Energy: $\mathbf{x}_o \rightarrow 0$ $\qquad$ $\mathcal{E}_o^2 \equiv \mathbf{x}_o^T \mathcal{Q} \mathbf{x}_o$

Large $\mathcal{E}_r$: State difficult to reach

Small $\mathcal{E}_o$: State difficult to observe

Balanced Reduction: Construct basis $\mathbf{V}$ so that

Difficult to reach = Difficult to observe

Balanced Reduction (Moore 81)

Find $\mathbf{V}, \mathbf{W}$ such that $\mathbf{W}^T \mathbf{V} = \mathbf{I}$

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u}, \quad \mathbf{y} = \hat{\mathbf{C}}\hat{\mathbf{x}}$$

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0$$

With $\mathcal{P} = \mathcal{Q} = \mathbf{S}$: Want Grammians Diagonal and Equal

Reduced Model $\mathbf{A}_k = \mathbf{W}_k^T \mathbf{A} \mathbf{V}_k$, $\mathbf{B}_k = \mathbf{W}_k^T \mathbf{B}$, $\mathbf{C}_k = \mathbf{C}_k \mathbf{V}_k$

- Reduced Model Grammians $\mathcal{P}_k = \mathbf{S}_k$ and $\mathcal{Q}_k = \mathbf{S}_k$.

$$\mathbf{A}_k\mathcal{P}_k + \mathcal{P}_k\mathbf{A}_k^T + \mathbf{B}_k\mathbf{B}_k^T = 0 \quad \mathbf{A}_k^T\mathcal{Q}_k + \mathcal{Q}_k\mathbf{A}_k + \mathbf{C}_k^T\mathbf{C}_k = 0$$

Hankel Norm Error estimate (Glover 84)

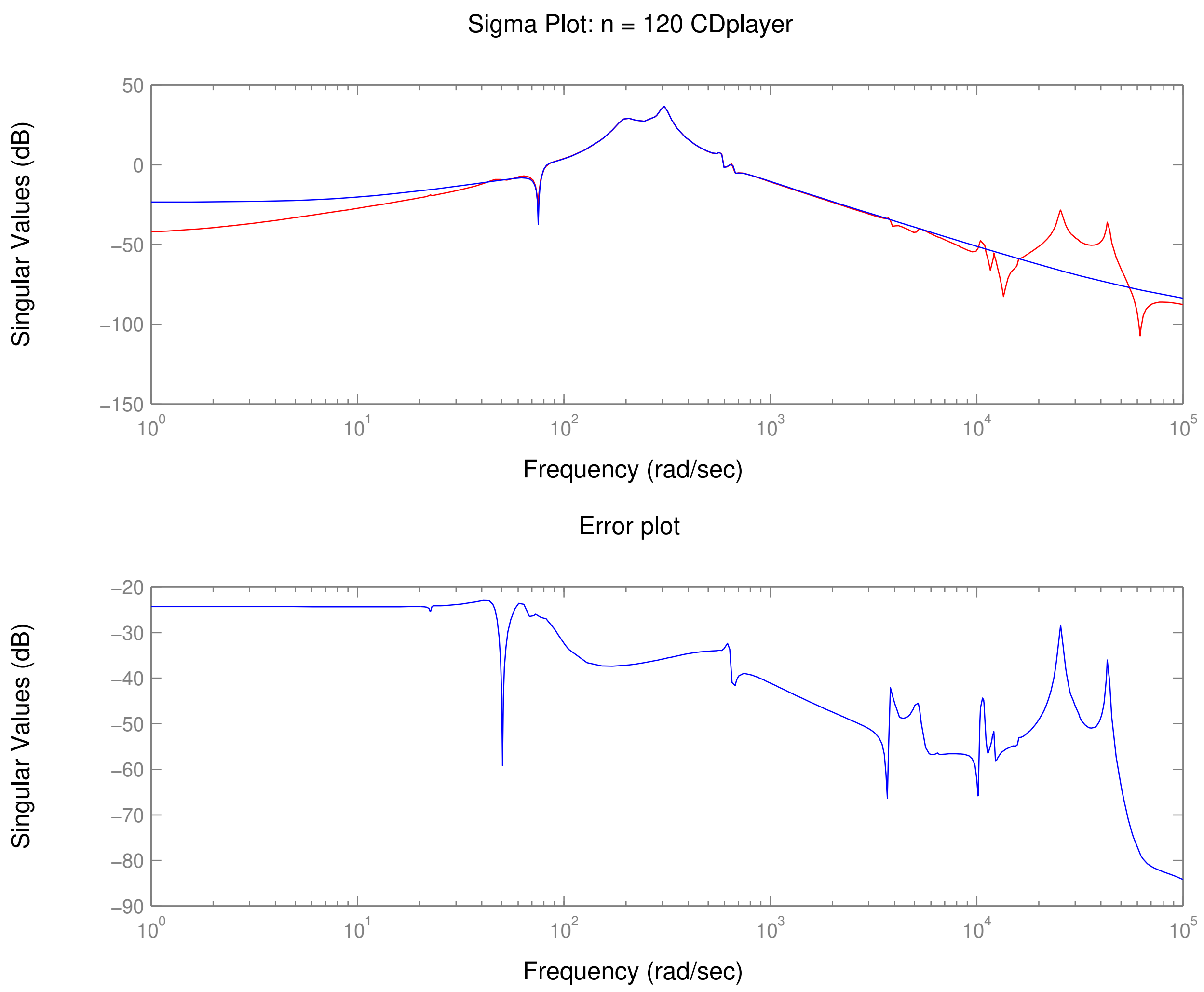
Why Balanced Realization?

- Hankel singular values $= \sqrt{\lambda(\mathcal{P}\mathcal{Q})}$

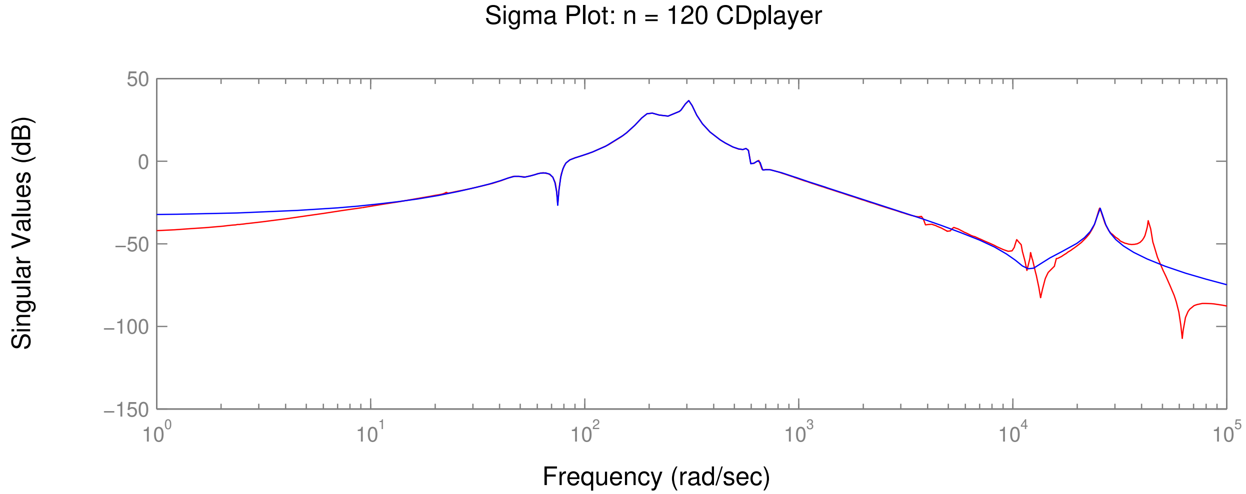
- Model reduction error (Glover)

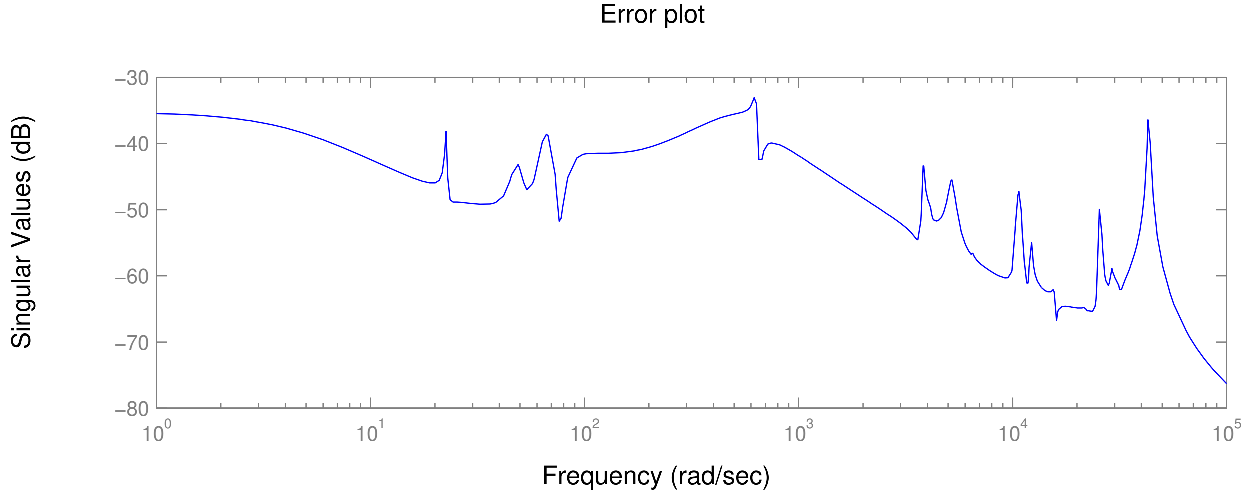
  $\|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq 2 \times (\text{sum neglected singular values})\|u\|_2$

- Extends to MIMO
- Preserves Stability

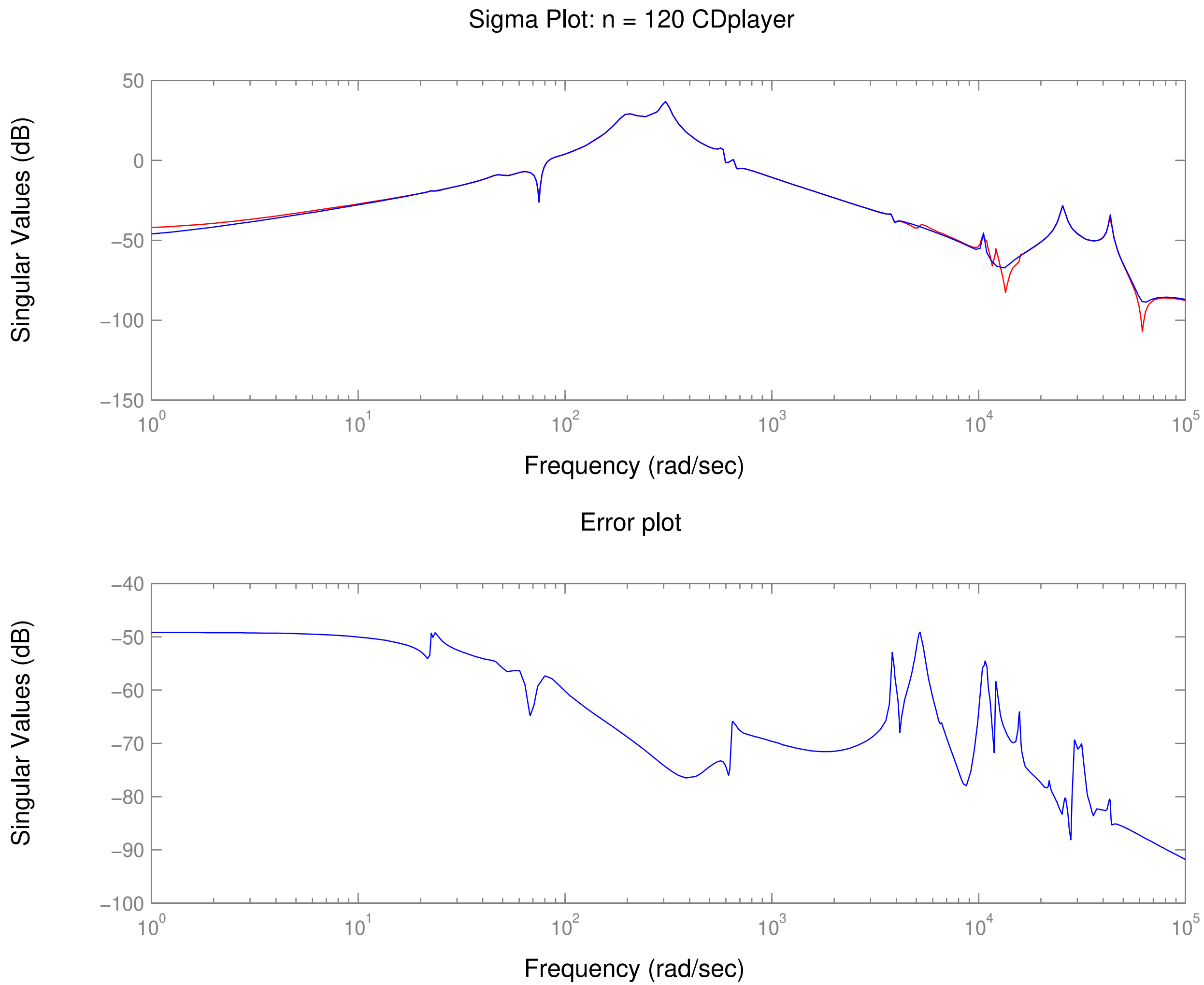


CD Player Sigma plot: $n = 120$ $k = 11$, tol $= 5e\text{-}3$

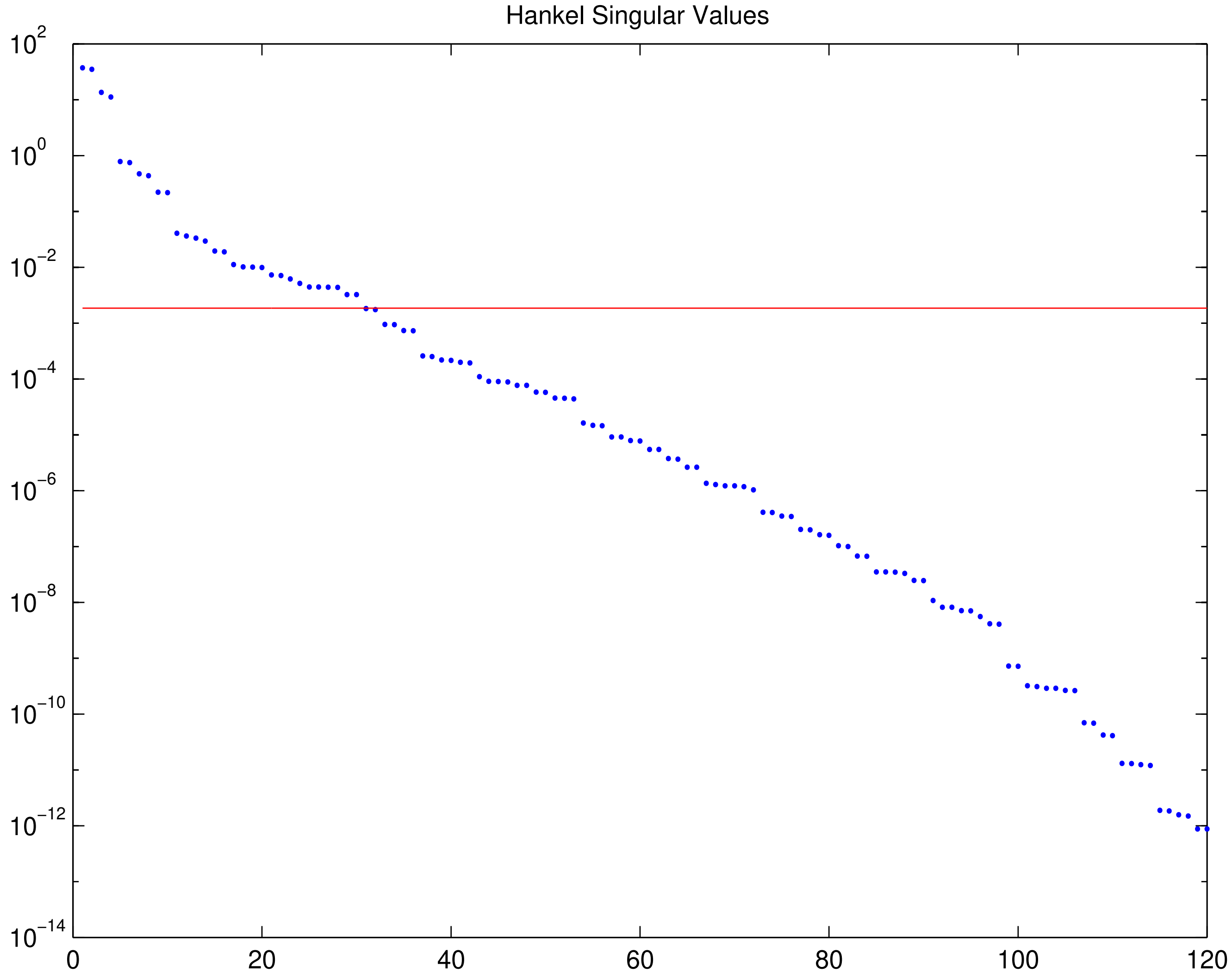


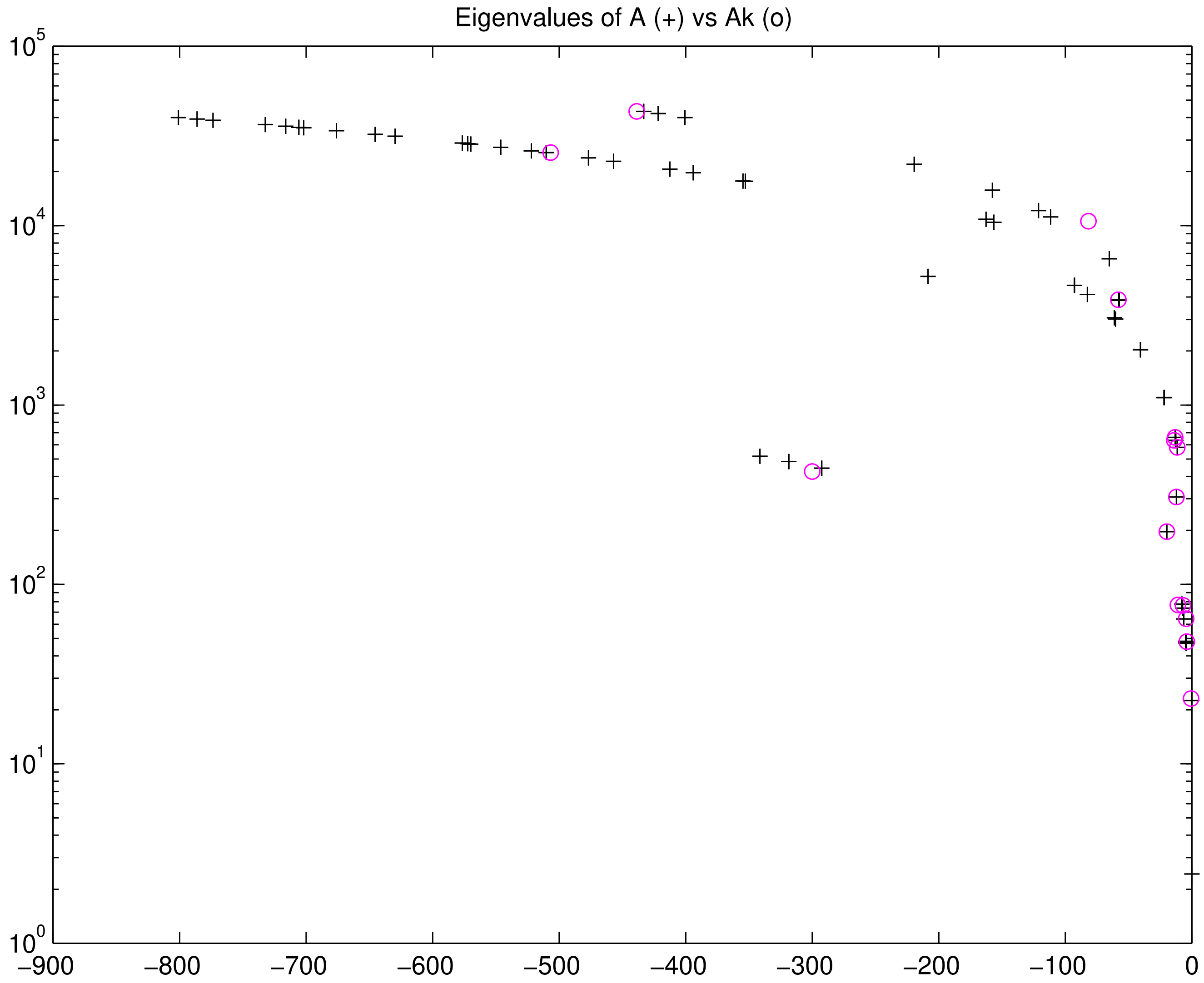


Sigma plot k = 17 , tol = 5e-4



Sigma plot k = 31 , tol = 5e-5


Hankel Singular Values
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
0
20
40
60
80
100
120



Eigenvalues of A (+) vs Ak (o)
10^5
10^4
10^3
10^2
10^1
10^0
−900
−800
−700
−600
−500
−400
−300
−200
−100
0

Balanced Reduction

- Global and Automatic Error Control
- Preserves System Stablility
- Natural for MIMO
- Small Dense Case – well known
- Large Scale – current research

POD in CFD – Incorporating Observations (Balancing)

Lall, Marsden and Glavaski

K. Willcox and J. Peraire

B. Farrell and P. Ioannou (Kalman Filter)

Must Solve Lyapunov Equations

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0$$

- Dense Case: Generalize Bartels - Stewart (72),
- Hammarling(82) - Solve in Factored Form $\mathcal{P} = \mathbf{U}\mathbf{U}^T \qquad \mathcal{Q} = \mathbf{L}\mathbf{L}^T$
- Sparse Case: Iteratively Solve in Low Rank Factored Form,

$$\mathcal{P} \approx \mathbf{U}_k\mathbf{U}_k^T, \quad \mathcal{Q} \approx \mathbf{L}_k\mathbf{L}_k^T$$

When are Low Rank Solutions Expected

Eigenvalue Distribution $\lambda(\mathbf{A})$ implies decay rate:

Cholesky Ordering: $\mathcal{C} = \mathbf{LDL}^*$ $\qquad$ $\mathcal{C}_{ij} = \frac{-1}{\lambda_i + \lambda_j^*}$.

Low Rank Approximation - Decay Rates

If $\delta_{k+1} < \epsilon \cdot \delta_1$.

$$\text{If} \quad \mathcal{P} \;\; \text{solves} \;\; \mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^{\mathrm{T}} + \mathbf{B}\mathbf{B}^{\mathrm{T}} = 0,$$

Then there is a matrix $\mathcal{P}_k$ of rank at most mk such that

$$\|\mathcal{P} - \mathcal{P}_k\|_\infty \leq \epsilon \cdot \delta_1 \cdot (\kappa_\infty(\mathbf{X})\|\mathbf{B}\|_1)^2,$$

where $\mathbf{AX} = \mathbf{X}\Lambda$ $\qquad$ (Y.K. Zhou) .

Eigenvalue Decay Rates

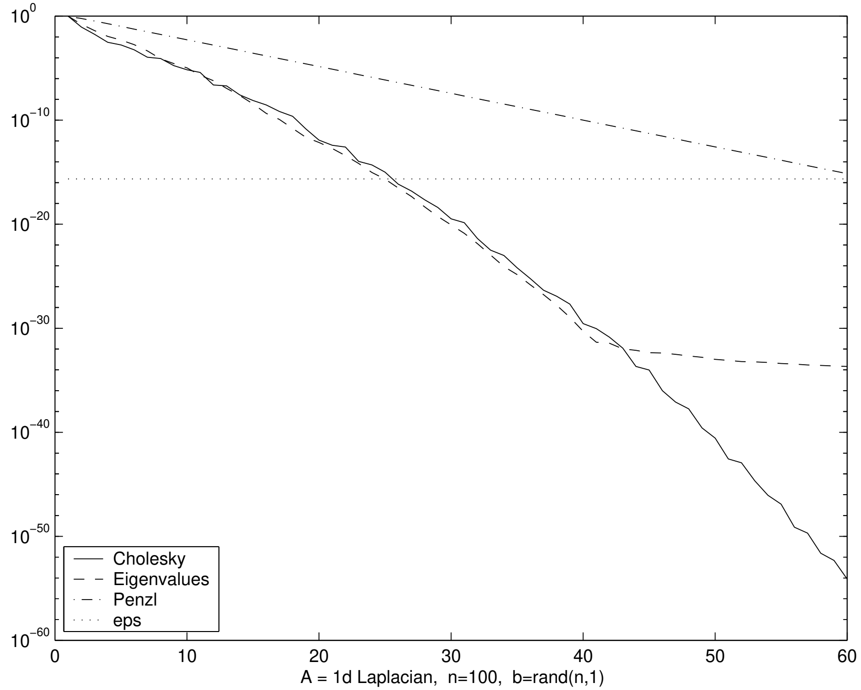
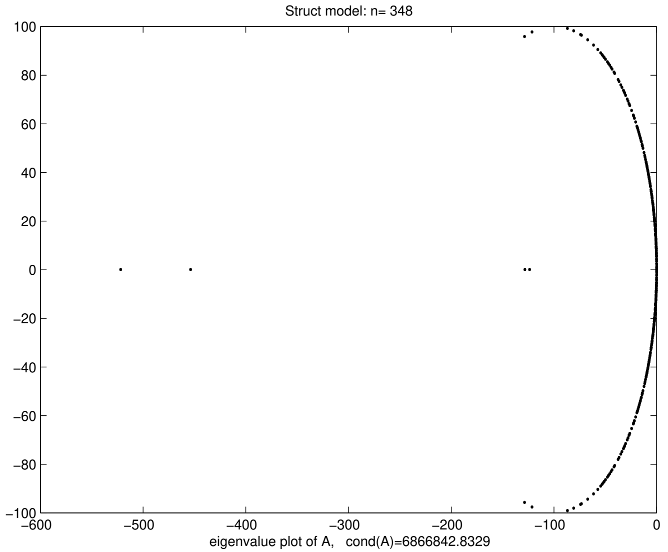


Figure 1: Comparison of Estimates: Symmetric A

Eigenvalue Decay Rates



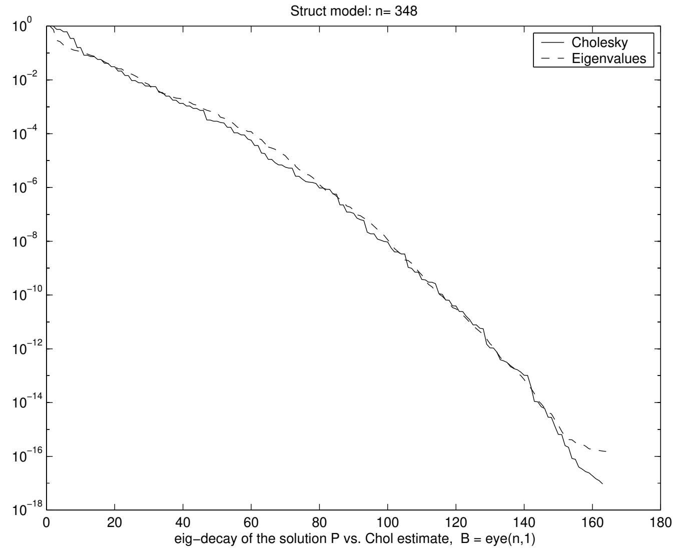


Figure 2: Eigen-decay rate vs Cholesky estimates

Eigenvalue Decay Rates

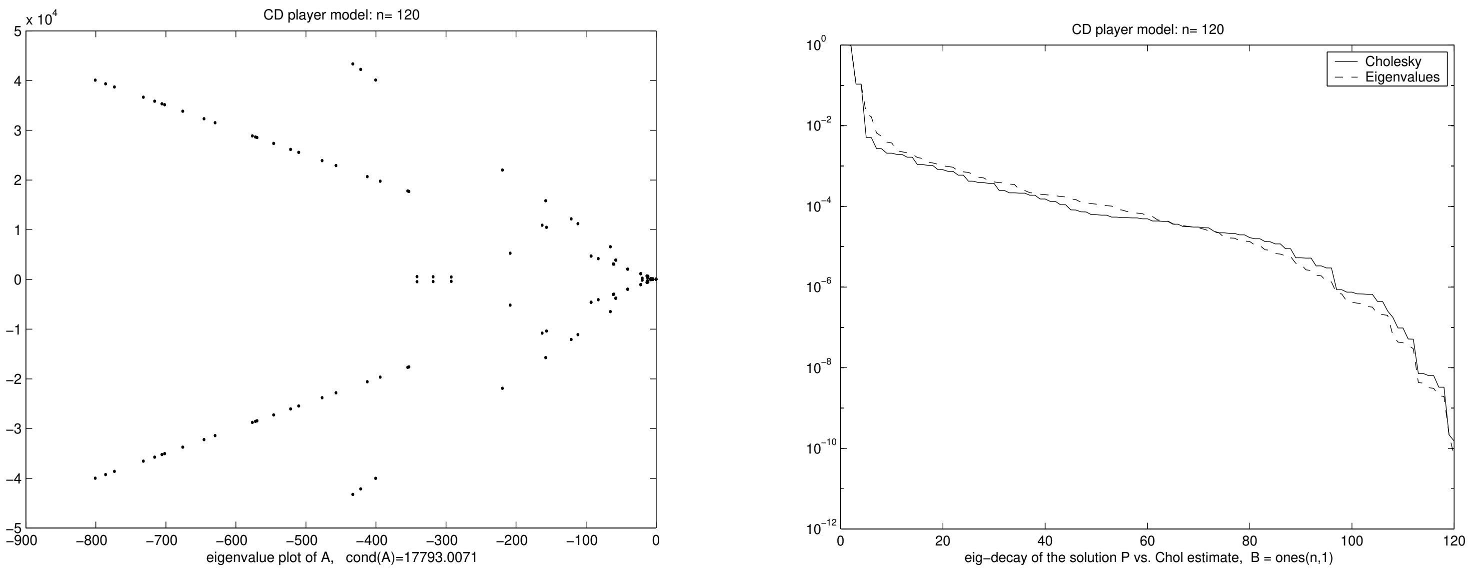


Figure 3: Eigen-decay rate vs Cholesky estimates

Balanced Reduction via Projection

From $[\mathbf{X}, \mathbf{S}, \mathbf{Y}] = \mathrm{svd}(\mathbf{U}^T \mathbf{L})$, $\qquad \mathcal{P} = \mathbf{U}\mathbf{U}^T \quad \mathcal{Q} = \mathbf{L}\mathbf{L}^T$

Construct $\mathbf{V}, \mathbf{W}$: $\qquad \underline{\mathcal{P}\mathbf{W}_k = \mathbf{V}_k\mathbf{S}_k \text{ and } \mathcal{Q}\mathbf{V}_k = \mathbf{W}_k\mathbf{S}_k}$

Reduced model: $\mathbf{A}_k = \mathbf{W}_k^T \mathbf{A}\mathbf{V}_k, \quad \mathbf{B}_k = \mathbf{W}_k^T\mathbf{B}, \quad \mathbf{C}_k = \mathbf{C}\mathbf{V}_k.$

$$\begin{aligned} 0 = \mathbf{W}_k^T(\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)\mathbf{W}_k &= \mathbf{A}_k\mathbf{S}_k + \mathbf{S}_k\mathbf{A}_k^T + \mathbf{B}_k\mathbf{B}_k^T \\ 0 = \mathbf{V}_k^T(\mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C})\mathbf{V}_k &= \mathbf{A}_k^T\mathbf{S}_k + \mathbf{S}_k\mathbf{A}_k + \mathbf{C}_k^T\mathbf{C}_k \end{aligned}$$

<u>Reduced model is balanced and asymptotically stable for every k.</u>

Balancing by Subspace Projection

Develop a Projection Method for Large Lyapunov Equations

Get Approximate Balancing

from

Best Rank-k Lyapunov Solutions

Large Scale Lyapunov Equations

- Iterative - Wachpress
- Krylov - Saad; Hu and Reichel
- Quadrature - Gudmundsson and Laub
- Subspace Iteration - Hodel and Tennison, VanDooren, Y. Zhou and S.

Balanced Reduction – Large Scale:

- Lanczos - Golub and Boley
- Restarting – Kasenally et. al.
- Low Rank Smith – Penzl, Li, White , Benner

Solve by Projection

$$\mathbf{A}^T\mathbf{V}_k = \mathbf{V}_k\mathbf{H}^T + \mathbf{F}$$

$$\mathbf{A}(\mathcal{P}\mathbf{V}_k) + (\mathcal{P}\mathbf{V}_k)\mathbf{H}^T + \mathbf{B}(\mathbf{B}^T\mathbf{V}_k) = -\mathcal{P}\mathbf{F} = \mathcal{O}(\gamma_{k+1})$$

$$\mathcal{P}_k = \mathbf{U}_k\mathbf{U}_k^T \quad \mathbf{U}_k = \mathbf{V}_k\mathbf{\Gamma}_k^{1/2}$$

<u>One sided Projected Equation - Residual Correction</u>

$$\mathbf{AZ} + \mathbf{ZH}^T + \mathbf{B}(\mathbf{B}^T\mathbf{V}_k) = 0$$

$$\mathbf{Z} \approx \mathcal{P}\mathbf{V}_k$$

Implicit Restarting – Iterated Subspace Projection

Construct $\mathbf{AV} = \mathbf{VH} + \mathbf{F}$ with $\mathbf{V}^T\mathbf{F} = \mathbf{0}$

Iterate until convergence:

1. Solve Projected Lyapunov equation of order $2k$
   $\mathbf{H}\hat{\mathbf{P}} + \hat{\mathbf{P}}\mathbf{H}^T + \hat{\mathbf{B}}\hat{\mathbf{B}}^T = \mathbf{0}$
2. <u>Compress</u> order k by deleting the k smallest eigenvalues
   – get $\mathbf{H}_k$ and $\mathbf{U}_k = \mathbf{V}_k\mathbf{\Gamma}_k$ with $\mathbf{P}_k = \mathbf{U}_k\mathbf{U}_k^T$.
3. Solve special Sylvester equation for residual correction
   $\mathbf{AZ} + \mathbf{ZH}_k^T + \mathbf{M} = \mathbf{0}$
4. <u>Expand</u> subspace to a model of order $2k$
   $[\mathbf{V}, \mathbf{R}] = qr([\mathbf{V}_k, \mathbf{Z}])$ $\qquad$ $\mathbf{H} = \mathbf{V}^T\mathbf{AV}$

Key – Special Sylvester Equation – ARPACK

$$\mathbf{AZ} + \mathbf{ZH}^T + \mathbf{M} = 0$$

$\mathbf{H}$ – stable $k \times k$ matrix, $k << n$

$$\begin{bmatrix} \mathbf{A} & \mathbf{M} \\ 0 & -\mathbf{H}^T \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \mathbf{R}$$

$$\mathbf{Z} = \mathbf{V}_1 \mathbf{V}_2^{-1}$$

$\mathbf{V}_2$ nonsingular iff $\sigma(\mathbf{R}) = \sigma(-\mathbf{H}^T)$

Performance of Implicit Restarting Method

- Simple Test Problem

  A 2d Discrete Convection Diffusion

  n = 10,000 , k = 30

  4 iterations

- Crossover Direct vs. Iterative around n = 400

- No Proof of Convergence – 3-4 Major Iters typical

Low Rank Smith

R.A. Smith (68) Variant of ADI

- Wachpress(88)
- Calvetti, Levenberg, Reichel (97)
- Penzl (99) (00)
- Li, White, et. al. (00)

Low Rank Smith

Convert to Stein Equation:

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \Longleftrightarrow \quad \mathcal{P} = \mathbf{A}_\mu \mathcal{P} \mathbf{A}_\mu^T + \mathbf{B}_\mu \mathbf{B}_\mu^T,$$

where $\quad \mathbf{A}_\mu \quad$ a Cayley Transform of $\mathbf{A}$

Solution: $\quad \mathcal{P} = \mathcal{P}_\infty, \qquad \mathcal{P}_m = \sum_{j=0}^m \mathbf{A}_\mu^j \mathbf{B}_\mu \mathbf{B}_\mu^T (\mathbf{A}_\mu^j)^T$

Convergence: $\quad \mathcal{P} - \mathcal{P}_m = \mathcal{O}(\rho(\mathbf{A}_\mu)^{2m}) \to 0,$

since $\rho(\mathbf{A}_\mu) < 1.$

Modified Low Rank Smith

LR - Smith: Update Factored Form $\mathcal{P}_m = U_m U_m^T$: $\quad (Penzl)$

$$\begin{aligned} \mathbf{U}_{m+1} &= [\mathbf{A}_\mu \mathbf{U}_m, \mathbf{B}_\mu] \\ &= [\mathbf{A}_\mu^{m+1} \mathbf{B}_\mu, \mathbf{U}_m] \end{aligned}$$

Modified LR - Smith: Update and Truncate SVD (S. Gugercin)

$$\begin{aligned} \mathbf{B} &\leftarrow \mathbf{A}_\mu \mathbf{B}; \\ [\mathbf{V}, \mathbf{S}, \mathbf{Q}] &= \mathrm{svd}([\mathbf{A}_\mu \mathbf{B}, \mathbf{U}_\mathrm{m}]); \\ \mathbf{U}_{m+1} &\leftarrow \mathbf{V}_k \mathbf{S}_k; \quad (\sigma_{k+1} < tol \cdot \sigma_1) \end{aligned}$$

Low Rank Smith is Quadrature approx to Grammian

Performance of Modified Low Rank Smith

$$\frac{\|\mathcal{P}_S - \mathcal{P}_{MS}\|}{\|\mathcal{P}_S\|} < tol, \quad \frac{\|\mathcal{Q}_S - \mathcal{Q}_{MS}\|}{\|\mathcal{Q}_S\|} < tol.$$

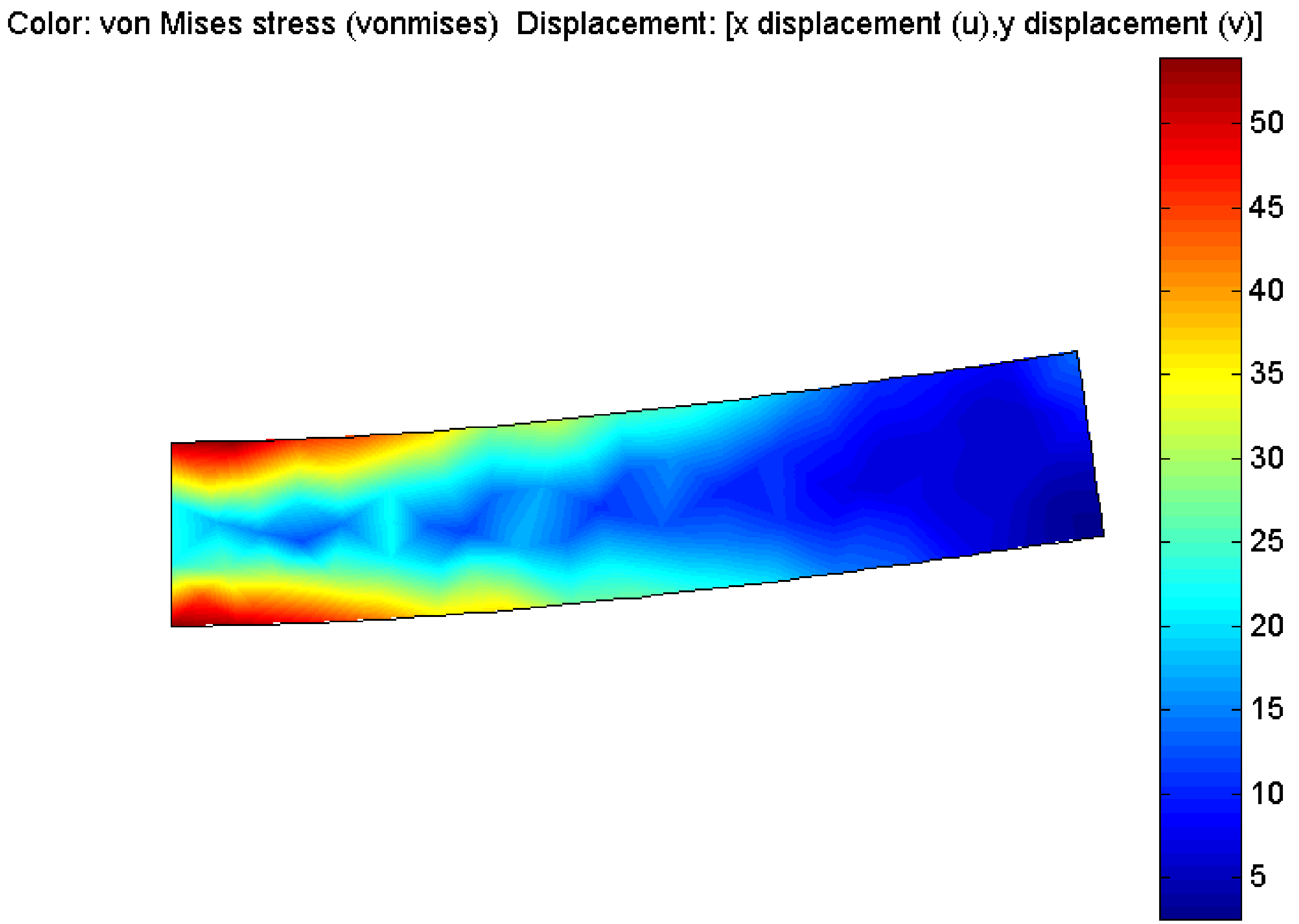
tol = SVD drop tol, S = LR Smith, MS = Modified LR Smith

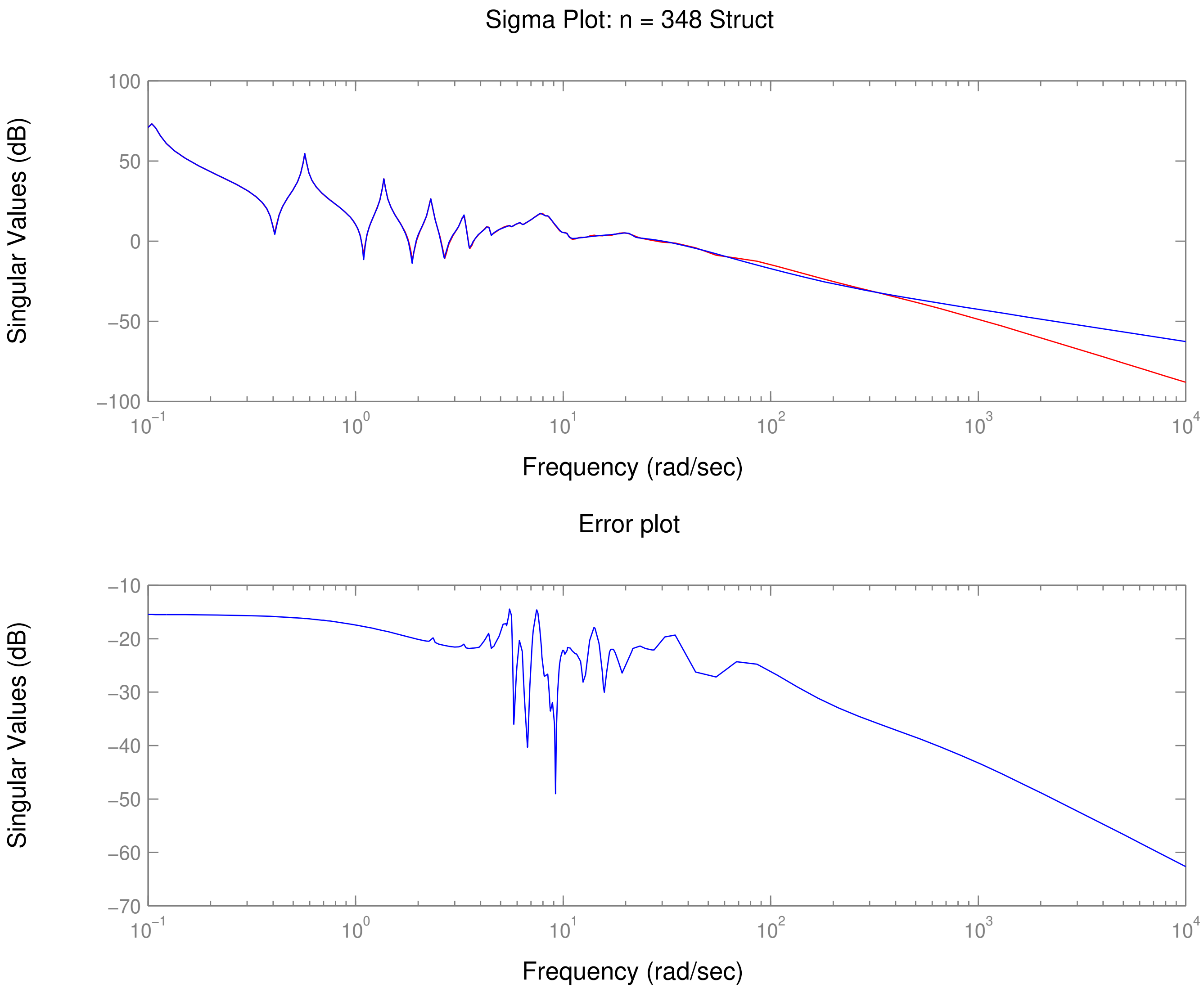
Storage - Cols. $\mathbf{U}_k$, $\mathcal{P} \approx \mathbf{U}_k \mathbf{U}_k^T$

	n	LR Smith	Modified
CD120	120	70	25
ISS	270	210	106
Penzl	1006	300	19

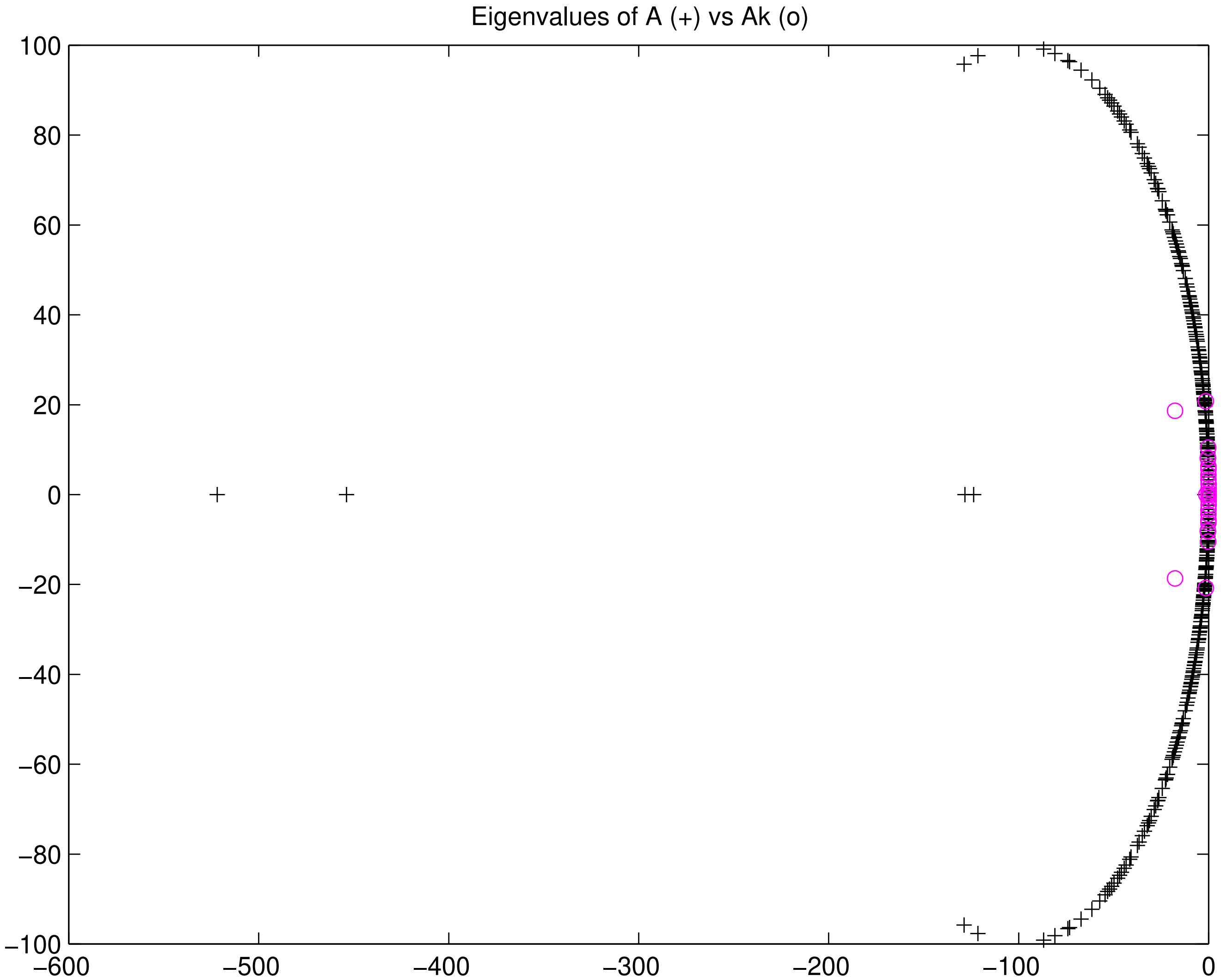
Clamped Beam Example

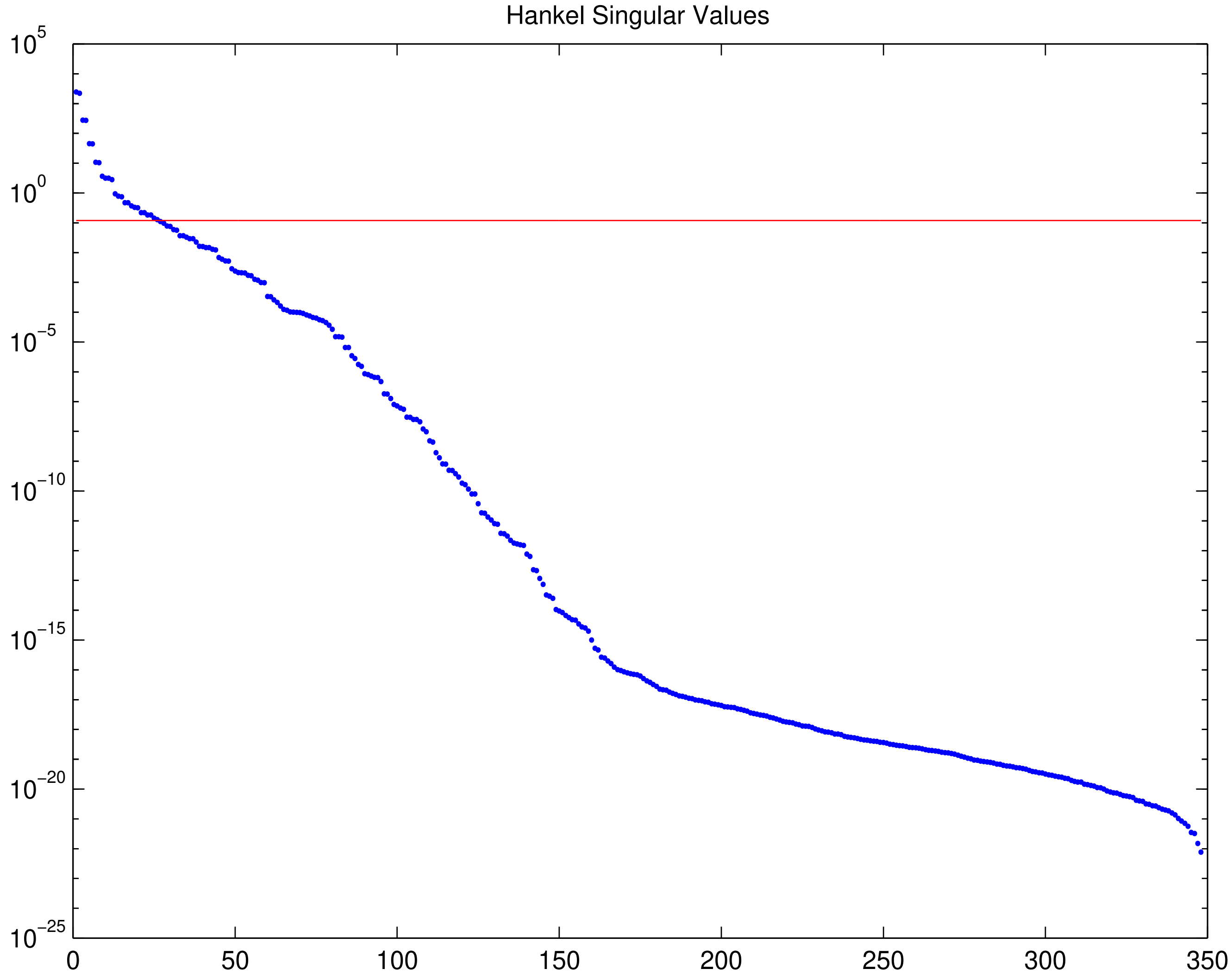
Thanks Comsol


Color: von Mises stress (vonmises) Displacement: [x displacement (u),y displacement (v)]
50
45
40
35
30
25
20
15
10
5



Struct. Sigma plot: n = 348 k = 27 tol = 5e-5


Eigenvalues of A (+) vs Ak (o)
100
80
60
40
20
0
−20
−40
−60
−80
−100
−600
−500
−400
−300
−200
−100
0


Hankel Singular Values
10^5
10^0
10^−5
10^−10
10^−15
10^−20
10^−25
0
50
100
150
200
250
300
350

Passivity Preserving Model Reduction

Interpolation via Invariant Subspace Projection

$$\begin{bmatrix} \mathbf{A} & & \mathbf{B} \\ & -\mathbf{A}^T & -\mathbf{C}^T \\ \mathbf{C} & \mathbf{B}^T & \mathbf{D}+\mathbf{D}^T \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{0} \end{bmatrix} \mathbf{R}.$$

Construct $\mathbf{X} = \mathbf{V}\hat{\mathbf{X}}, \quad \mathbf{Y} = \mathbf{W}\hat{\mathbf{Y}}, \quad \hat{\mathbf{A}} := \mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \mathbf{W}^T\mathbf{V} = \mathbf{I}$

$$\begin{bmatrix} \hat{\mathbf{A}} & & \hat{\mathbf{B}} \\ & -\hat{\mathbf{A}}^T & -\hat{\mathbf{C}}^T \\ \hat{\mathbf{C}} & \hat{\mathbf{B}}^T & \mathbf{D}+\mathbf{D}^T \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}} \\ \hat{\mathbf{Y}} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{X}} \\ \hat{\mathbf{Y}} \\ \mathbf{0} \end{bmatrix} \mathbf{R}.$$

Positive Real Lemma – Sufficiency

Positive Real Lemma

Sufficiency: $\hat{\mathbf{A}}$ *is stable and there exists* $\hat{P} \succeq \mathbf{0}$, $\mathbf{C}_o$, $\mathbf{W}_o$ such that

$$\begin{aligned} \hat{\mathbf{A}}\hat{P} + \hat{P}\hat{\mathbf{A}}^T &= -\mathbf{C}_o^T\mathbf{C}_o, \\ \hat{\mathbf{C}}\hat{P} - \hat{\mathbf{B}}^T &= -\mathbf{W}_o^T\mathbf{C}_o, \\ \mathbf{D} + \mathbf{D}^T &= \mathbf{W}_o^T\mathbf{W}_o, \end{aligned}$$

then $\mathbf{G}$ *is positive real.*

Riccati Eqns., Minimal Solutions, and Balancing

Full Solution: Interpolate n spectral zeros

$$\begin{aligned}\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T &= -(\mathbf{B} - \mathcal{P}\mathbf{C}^T)\mathcal{D}^{-1}(\mathbf{B}^T - \mathbf{C}\mathcal{P}) \\ \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} &= -(\mathbf{C}^T - \mathcal{Q}\mathbf{B})\mathcal{D}^{-1}(\mathbf{C} - \mathbf{B}^T\mathcal{Q})\end{aligned}$$

with

$$\mathcal{P} = -\mathbf{X}\mathbf{Y}^{-1} \succ \mathbf{0}, \quad \mathcal{Q} = -\mathbf{Y}\mathbf{X}^{-1} \succ \mathbf{0}.$$

Approximate Solution: Interpolate $k << n$ spectral zeros
Low rank approximate solutions:

$$\mathcal{P}_k = -\mathbf{X}(\mathbf{X}^T\mathbf{Y})^{-1}\mathbf{X}^T \quad \text{and} \quad \mathcal{Q}_k = -\mathbf{Y}(\mathbf{Y}^T\mathbf{X})^{-1}\mathbf{Y}^T$$

Positive Real Balancing

Interpolate n spectral zeros in <u>right half plane</u>

$$\mathcal{P} \text{ is } \underline{\text{minimal}} \text{ and } \mathcal{Q} = \mathcal{P}^{-1} \text{ is maximal}$$

Interpolate n spectral zeros in <u>left half plane</u>

$$\mathcal{Q} \text{ is } \underline{\text{minimal}} \text{ and } \mathcal{P} = \mathcal{Q}^{-1} \text{ is maximal}$$

<u>**Minimal $\mathcal{P}, \mathcal{Q}$: Pos. Real Balancing**</u>

Definition:

$\mathcal{Q}$ is maximal if $\mathcal{Q} \succeq \hat{\mathcal{Q}}$ for any other solution $\hat{\mathcal{Q}}$

$\mathcal{Q}$ is minimal if $\mathcal{Q} \preceq \hat{\mathcal{Q}}$ for any other solution $\hat{\mathcal{Q}}$

Second Order Systems

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} &= \mathbf{B}\mathbf{u} \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \quad \rightarrow \quad \begin{aligned} \hat{\mathbf{M}}\ddot{\hat{\mathbf{x}}} + \hat{\mathbf{G}}\dot{\hat{\mathbf{x}}} + \hat{\mathbf{K}}\hat{\mathbf{x}} &= \hat{\mathbf{B}}\mathbf{u} \\ \hat{\mathbf{y}}(t) &= \hat{\mathbf{C}}\hat{\mathbf{x}}(t) \end{aligned}$$

where $\hat{\mathbf{M}} = \mathbf{W}^T\mathbf{M}\mathbf{V}$, etc.

Key Point: Preserve Second Order Form

– DO NOT convert to First Order Sys.

Keeps Physical Meaning - can be built

Use state Grammian $\mathcal{P} = \int_o^\infty \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau$ (via Quadrature)

Can Bound $\|y - \hat{y}\|$ in $\mathcal{H}_2$-norm

Generalize Passivity Preserving Idea (C. Teng)

Summary

Projection – Powerful technique for solving large problems

Papers -
`http://www.caam.rice.edu/~sorensen/Tech_Reports.html`

- Model Reduction via Approximate Balancing
  Candidate for robust software
- Provides rigorous bound on system approximation error
- Produces stable reduced model
- Can be made fully automatic