Proof of the Local Discretization Error Estimate for the Optimal Control Problem in the Presence of Interior Layers[2]

D. Leykekhman and M. Heinkenschloss

This "appendix" contains the proof of Theorem 5.1 in the paper *D. Leykekhman and M. Heinkenschloss: Local Error Analysis of Discontinuous Galerkin Methods for Advection-Dominated Elliptic Linear-Quadratic Optimal Control Problems* [1].

To facilitate readability, the beginning of Section 5 and Subsection 5.1 of [1] is repeated here. Theorem 5.1 of [1] is restated here as Corollary A.2. All references to equations and other results that do not start with "A", refer to the corresponding equations and results in [1].

We will analyze the error between the solution of the infinite dimensional optimal control problem (2.5) and the solution of the discretized problem (3.6) in the presence of interior layers.

The results in, e.g., [28, p. 473] or [29, L. 23.1] describe what parts of the forcing term f influence the exact solution of a single advection dominated PDE at any fixed point $x_0 \in \Omega$: The force term in the entire upstream direction of x_0 influences the exact solution at x_0, but only the force term from within an $\varepsilon|\log\varepsilon|$-neighborhood in the streamline (downwind) direction and within a $\sqrt{\varepsilon}|\log(\varepsilon)|$-neighborhood in the crosswind direction influence exact solution at x_0. The same behavior can be observed from the properties of the corresponding Green's function. In the presence of interior layers only, the exact solution may vary strongly in the crosswind direction, but not in the streamline direction. Since the adjoint equation has similar properties, the same behavior of the solution can be expected from the coupled system. Our main goal of this section is to show that similarly to the case of a single equation (cf. [16]), the interior layers do not pollute the numerical solution to the coupled optimality system (3.7). We will accomplish this by weighted error estimates, where the purpose of the weighting function is essentially to isolate the domains of smoothness from the layers. The analysis is rather technical and in order to avoid unnecessary technicalities we will make several simplifications:

- $\varepsilon \le h$, i.e. we consider only the advection-dominating case.
- The reaction term $r \equiv 1$. This simplification is not essential, the same analysis can be applied to $r(x) \ge 0$ (cf. Lemma 4.2).
- $Y_h = U_h = V_h$.

Consider the optimality systems (2.7) and (3.7). From equation (2.7b) we can conclude that $\lambda = \alpha u$ which leads to the reduced optimality system

$$\alpha a(\varphi, u) + (y, \varphi) = \langle \hat{y}, \varphi\rangle \qquad \forall \varphi \in V, \tag{A.1a}$$

$$a(y, \phi) - (u, \phi) = \langle f, v\rangle \qquad \forall \phi \in V. \tag{A.1b}$$

Similarly, from (3.7b) we obtain $\lambda_h = \alpha u_h$, which leads to reduced discrete optimality system,

$$\alpha a_h(\varphi_h, u_h) = -\langle y_h - \hat{y}, \varphi_h\rangle \qquad \forall \varphi_h \in V_h, \tag{A.2a}$$

$$a_h(y_h, \phi_h) - (u_h, \phi_h) = l_h(\phi_h) \qquad \forall \phi_h \in V_h. \tag{A.2b}$$

[2]Available at
http://www.caam.rice.edu/∼heinken/papers/DLeykekhman_MHeinkenschloss_2010a.html.

The system (A.2) motivates the definition of the reduced bilinear form $\mathcal{A}^{DG}(\cdot,\cdot)$ on $(V_h \times V_h) \times (V_h \times V_h)$ given by

$$\mathcal{A}^{DG}((y_h,u_h),(\phi_h,\varphi_h)) = a_h(y_h,\phi_h) - (u_h,\phi_h) + \alpha a_h(\varphi_h,u_h) + (y_h,\varphi_h). \tag{A.3}$$

The reduced discrete optimality system (A.2) can be written as

$$\mathcal{A}^{DG}((y_h,u_h),(\phi_h,\varphi_h)) = l_h(\phi_h) + \langle \hat{y},\varphi_h\rangle \quad \forall (\phi_h,\varphi_h)\in V_h\times V_h. \tag{A.4}$$

Notice that the discontinuous Galerkin method is consistent, i.e., provided that the exact solution is regular enough (e.g., y and u,λ in H^2), then

$$\mathcal{A}^{DG}((y,u),(\phi_h,\varphi_h)) = l_h(\phi_h) + \langle \hat{y},\varphi_h\rangle \quad \forall (\phi_h,\varphi_h)\in V_h\times V_h. \tag{A.5}$$

In particular, (A.4) and (A.5) imply the Galerkin orthogonality condition

$$\mathcal{A}^{DG}\big((y-y_h,u-u_h),(\phi_h,\varphi_h)\big) = 0 \quad \forall (\phi_h,\varphi_h)\in V_h\times V_h. \tag{A.6}$$

To establish our local error estimates, we define a weight function as in [17]. As before, we use the convention that for two dimensional vectors a and b the cross product is defined by $a\times b := a_1b_2 - a_2b_1$, which is just a z-component of the cross-product if we think of vectors a and b as three dimensional vectors with z component to be zero. We define

$$\Omega_0 = \{x\in\Omega : A_1 \le (x\times\beta) \le A_2\}, \tag{A.7}$$

which is a strip along β of width $|A_2 - A_1|$. The weight function ω is $O(1)$ on Ω_0 and decays exponentially outside of a slightly larger subdomain. More precisely, the weight $\omega\in C^\infty(\Omega)$ is a positive function with the following properties:

$$C_1 \le \omega(x) \le C_2, \qquad \text{for } x\in\Omega_0, \tag{A.8a}$$

$$|\omega(x)| \le Ce^{-((x\times\beta)-A_2)/K\sqrt{h}}, \qquad \text{for } (x\times\beta)\ge A_2, \tag{A.8b}$$

$$|\omega(x)| \le Ce^{-(A_1-(x\times\beta))/K\sqrt{h}}, \qquad \text{for } (x\times\beta)\le A_1. \tag{A.8c}$$

Here C_1 and C_2 are two fixed positive constants, K is a sufficiently large number.

Our weighted error estimate is the following.

THEOREM A.1. *Let (y,u) and (y_h,u_h) satisfy (A.6). If $h\le C_2\alpha$ for some constant C_2 and $\varepsilon\le h$, then there exists a constant C independent of h and ε such that*

$$Q_\omega^2(y-y_h) + \alpha Q_\omega^2(u-u_h) \le C\Big(\min_{\chi_1\in V_h} L_\omega^2(y-\chi_1) + \min_{\chi_2\in V_h}\alpha L_\omega^2(u-\chi_2)\Big), \tag{A.9}$$

where

$$Q_\omega^2(v) := \sum_{\tau\in T_h}\varepsilon\|\omega\nabla v\|_\tau^2 + \|\omega v\|_\tau^2 + \varepsilon\sum_{e\in\mathcal{E}_h} {h_e}^{-1}\|\omega[\![v]\!]\|_e^2 + \frac{1}{2}\sum_{e\in\mathcal{E}_h^0}\|\omega(v^+-v^-)|\beta\cdot\boldsymbol{n}|^{1/2}\|_e^2 + \frac{1}{2}\sum_{e\in\mathcal{E}_h^\partial}\|\omega v|\beta\cdot\boldsymbol{n}|^{1/2}\|_e^2$$

and

$$L_\omega^2(v) := \sum_{\tau\in T_h} h^{-1}\|\omega v\|_\tau^2 + h\|\omega\nabla v\|_\tau^2 + h^3\|\omega\nabla^2 v\|_\tau^2.$$

Theorem A.1 will be proven in the following subsection. To give an application of the above result, let Ω_0 be as in (A.7) and define

$$\Omega_s^+ = \{\ A_1 - sK\sqrt{h}|\log h| \le (x \times \beta) \le A_2 + sK\sqrt{h}|\log h|\ \} \cap \Omega.$$

Corollary A.2. *Let Ω be a bounded open convex subset of $\mathbb{R}^n$. If the assumptions of Theorem A.1 are valid, then there exists a constant C independent of y, u and h such that for any $s > 0$ and and mesh sizes $\varepsilon \le h$,*

$$\begin{aligned}&\|y - y_h\|_{\Omega_0} + \alpha\|u - u_h\|_{\Omega_0}\\ &\le C\left(h^{3/2}\|y\|_{2,\Omega_s^+} + h^{s+3/2}\|y\|_{2,\Omega}\right)\\ &\quad + C\alpha\left(h^{3/2}\|u\|_{2,\Omega_s^+} + h^{s+3/2}\|u\|_{2,\Omega}\right).\end{aligned}$$

Note that by Theorem 2.2, $(y, u) \in H^2(\Omega) \times H^2(\Omega)$. The proof of this corollary is the same as that of Corollary 3.3 in [17]. The interpretation of Corollary A.2 is essentially the same that given in [17, p. 4615] and we adapt it here for completeness. The right hand side in the error estimate of Corollary A.2 depends on local and global norms of the state and the adjoint. The local norms associated with $h^{3/2}$ are independent of ε if Ω_s^+ does not contain interior layers. The global norms depend on $\|y\|_{2,\Omega}$ and $\|u\|_{2,\Omega}$ and because of the regularity result in Theorem 2.2 may depend on negative powers of ε. However, they are associated with the higher order terms $h^{s+3/2}$. Thus negative powers of ε can be compensated by h^s for sufficiently large s, provided that for these values of s the subdomain Ω_s^+ does not contain interior layers.

A.1. Preliminary Results. Before we start to prove of Theorem A.1, we collect some preliminary results.

A.1.1. The weight function. Let

$$D_\beta = \beta \cdot \nabla, \quad D_{\beta^\perp} = \beta^\perp \cdot \nabla,$$

and $RO(S, v) = \max_{x \in S} |v(x)| / \min_{x \in S} |v(x)|$. In addition to the properties (A.8) of the weight function ω, we assume that ω satisfies,

(A.10a) $\quad D_\beta \omega(x) = 0, \quad$ for all $x \in \Omega$, i.e. ω is constant in the direction β,

(A.10b) $\quad |D_{\beta^\perp}^\gamma \omega| \le CK^{-\gamma} h^{-\gamma/2} \omega, \;$ for $\; \gamma \le k+1,$

(A.10c) $\quad RO(S, \omega) \le C_\omega, \;$ for any ball S of radius Kh.

The explicit construction of a such function is given in [21].

A.1.2. L^2-projection. For $v \in H^{k+1}(\Omega)$, we let $\widetilde{v}$ denote the local L^2-projection of v onto V_h defined by

$$(v - \widetilde{v}, \chi)_\tau = 0, \quad \forall \chi \in \mathbb{P}^k(\tau) \quad \tau \in T_h.$$

We will use the standard estimates

(A.11a) $\quad \|v - \widetilde{v}\|_{s,\tau} \le Ch_\tau^{k+1-s} |v|_{k+1,\tau}, \quad s = 0, 1,$

(A.11b) $\quad \|v - \widetilde{v}\|_{\partial\tau} \le Ch_\tau^{k+1/2} |v|_{k+1,\tau}.$

A.1.3. Superapproximation. Next we will need a superapproximation result. The proof of this result is essentially contained in [16], Lemma 2.1 or [21], Lemma 2.2.

LEMMA A.3. *Let $v \in V_h$ and set $E_\omega(v) = \omega^2 v - \widetilde{\omega^2 v}$. There exists a constant C independent of h and v, such that*

$$\begin{aligned} & h^{-2}\|\omega^{-1}E_\omega(v)\|_\tau^2 + \|\omega^{-1}\nabla E_\omega(v)\|_\tau^2 + h^2\|\omega^{-1}\nabla^2 E_\omega(v)\|_\tau^2 \\ & \leq Ch^{-1}K^{-2}\|\omega v\|_\tau^2. \end{aligned} \tag{A.12}$$

A.2. Lemmas. The proof of Theorem A.1 follows from the following three lemmas. The proofs of these lemmas are rather standard although technical. For an easier flow, we first state the lemmas and provide the proofs at the end of this section.

LEMMA A.4. *Let $(y_h, u_h) \in V_h \times V_h$ and $\varepsilon \leq h$. If the penalty parameter σ is sufficiently large, then for K large enough we have*

$$Q_\omega^2(y_h) + \alpha Q_\omega^2(u_h) \leq 2\mathcal{A}^{DG}((y_h, u_h), (\omega^2 y_h, \omega^2 u_h)).$$

LEMMA A.5. *If the assumptions of Lemma A.4 are valid, then for K large enough we have*

$$\mathcal{A}^{DG}((y_h, u_h), (\omega^2 y_h - \widetilde{\omega^2 y_h}, \omega^2 u_h - \widetilde{\omega^2 u_h})) \leq \frac{1}{8}\left(Q_\omega^2(y_h) + \alpha Q_\omega^2(u_h)\right).$$

LEMMA A.6. *If in addition to the assumptions of Lemma A.4 and Lemma A.5, there exists a constant C_2 such that $h \leq C_2\alpha$ and (y, u) and (y_h, u_h) satisfy* (A.6), *then for K sufficiently large there exists a constant C independent of h such that*

$$\mathcal{A}^{DG}((y, u), (\widetilde{\omega^2 y_h}, \widetilde{\omega^2 u_h})) \leq \frac{1}{4}\left(Q_\omega^2(y_h) + \alpha Q_\omega^2(u_h)\right) + C\left(L_\omega^2(y) + \alpha L_\omega^2(u)\right).$$

A.3. Proof of Theorem A.1. The properties of ω and the assumption $\varepsilon \leq h$ imply

$$Q_\omega^2(v - \chi) \leq CL_\omega^2(v - \chi) \text{ for all } v, \chi \in V.$$

Using this inequality and the triangle inequality, we have

$$\begin{aligned} & Q_\omega^2(y - y_h) + \alpha Q_\omega^2(u - u_h) \\ & \leq Q_\omega^2(y_h - \chi_1) + \alpha Q_\omega^2(u_h - \chi_2) + C(L_\omega^2(y - \chi_1) + \alpha L_\omega^2(u - \chi_2)), \end{aligned} \tag{A.13}$$

for all $\chi_1, \chi_2 \in V_h$. Hence, it is enough to show that for any $\chi_1, \chi_2 \in V_h$,

$$Q_\omega^2(y_h - \chi_1) + \alpha Q_\omega^2(u_h - \chi_2) \leq C(L_\omega^2(y - \chi_1) + \alpha L_\omega^2(u - \chi_2)).$$

If (y_h, u_h) and (y, u) satisfy the orthogonality property (A.6), then $(y_h - \chi_1, u_h - \chi_2)$ and $(y - \chi_1, u - \chi_2)$ satisfy the orthogonality property (A.6). Therefore, it is sufficient to establish

$$Q_\omega^2(y_h) + \alpha Q_\omega^2(u_h) \leq C(L_\omega^2(y) + \alpha L_\omega^2(u)),$$

for any (y_h, u_h) and (y, u) that satisfy the orthogonality property (A.6). By Lemma A.4,

$$Q_\omega^2(y_h) + \alpha Q_\omega^2(u_h) \le 2\mathcal{A}^{DG}((y_h, u_h), (\omega^2 y_h, \omega^2 u_h)).$$

We add and subtract $\mathcal{A}^{DG}((y_h, u_h), (\widetilde{\omega^2 y_h}, \widetilde{\omega^2 u_h}))$ and use the orthogonality property (A.6). Thus,

$$\begin{aligned}\mathcal{A}^{DG}((y_h, u_h), (\omega^2 y_h, \omega^2 u_h)) &= \mathcal{A}^{DG}((y_h, u_h), (\omega^2 y_h - \widetilde{\omega^2 y_h}, \omega^2 u_h - \widetilde{\omega^2 u_h})) \\ &\quad + \mathcal{A}^{DG}((y, u), (\widetilde{\omega^2 y_h}, \widetilde{\omega^2 u_h})).\end{aligned} \tag{A.14}$$

Applying Lemma A.5 and Lemma A.6 to the right hand side of (A.14) completes the proof of Theorem A.1. □

A.4. Proofs of Lemmas A.4 to A.6.

Proof of Lemma A.4. Using the identities

$$\int_\tau (\beta \cdot \nabla v)\omega^2 v = -\frac{1}{2}\int_\tau v^2 \omega(\beta \cdot \nabla \omega) + \frac{1}{2}\int_{\partial\tau} (\beta \cdot n)\omega^2 v^2 \tag{A.15a}$$

$$\sum_{\tau \in T_h} \int_{\partial\tau} (\beta \cdot n)\omega^2 v^2 = \sum_{e \in \mathcal{E}_h} \int_e |\beta \cdot n|\omega^2 (v^-)^2 - \sum_{e \in \mathcal{E}_h} \int_e |\beta \cdot n|\omega^2 (v^+)^2 \tag{A.15b}$$

$$\begin{aligned}&\frac{1}{2}\sum_{\tau \in T_h} \int_{\partial\tau} (\beta \cdot n)\omega^2 v^2 + \sum_{e \in \mathcal{E}^0} \int_e (v^+ - v^-)\omega^2 v^+ |\beta \cdot n| + \sum_{e \in \mathcal{E}_\partial^-} \int_e \omega^2 v^2 |\beta \cdot n| \\ &= \frac{1}{2}\sum_{e \in \mathcal{E}^0} \|\omega(v^+ - v^-)|\beta \cdot n|^{1/2}\|_e^2 + \frac{1}{2}\sum_{e \in \mathcal{E}^\partial} \|\omega v |\beta \cdot n|^{1/2}\|_e^2,\end{aligned} \tag{A.15c}$$

we have

$$\begin{aligned}&Q_\omega^2(y_h) + \alpha Q_\omega^2(u_h) = \mathcal{A}^{DG}((y_h, u_h), (\omega^2 y_h, \omega^2 u_h)) \\ &- 2\varepsilon \sum_{\tau \in T_h} \int_\tau \omega y_h \nabla\omega \cdot \nabla y_h + 2\varepsilon \sum_{e \in \mathcal{E}_h} \int_e \{\omega \nabla y_h\} [\![\omega y_h]\!] + 2\varepsilon \sum_{e \in \mathcal{E}_h} \int_e \{y_h \nabla \omega\} [\![\omega y_h]\!] \\ &- 2\alpha\varepsilon \sum_{\tau \in T_h} \int_\tau \omega u_h \nabla\omega \cdot \nabla u_h + 2\alpha\varepsilon \sum_{e \in \mathcal{E}_h} \int_e \{\omega \nabla u_h\} [\![\omega u_h]\!] + 2\alpha\varepsilon \sum_{e \in \mathcal{E}_h} \int_e \{u_h \nabla \omega\} [\![\omega u_h]\!].\end{aligned}$$

We recall that $\beta = (\beta_1, \beta_2)^T$, $\beta^\perp = (-\beta_2, \beta_1)^T$, and use

$$\nabla\omega = \frac{1}{|\beta|^2}\left(\beta\ \beta^\perp\right)\left(\beta\ \beta^\perp\right)^T \nabla\omega = (\omega_{x_1}, \omega_{x_2})^T,$$

where

$$\omega_{x_1} = \frac{1}{|\beta|^2}\left(\beta_1(\beta \cdot \nabla\omega) - \beta_2(\beta^\perp \cdot \nabla\omega)\right), \quad \omega_{x_2} = \frac{1}{|\beta|^2}\left(\beta_2(\beta \cdot \nabla\omega) + \beta_1(\beta^\perp \cdot \nabla\omega)\right).$$

By the assumptions of this section $\beta \cdot \nabla\omega = 0$ and the expressions above simplify to

$$\omega_{x_1} = -\frac{\beta_2}{|\beta|^2}\left(\beta^\perp \cdot \nabla\omega\right), \quad \omega_{x_2} = \frac{\beta_1}{|\beta|^2}\left(\beta^\perp \cdot \nabla\omega\right).$$

By the properties of ω (A.10b), the Cauchy-Schwarz inequality and the assumption $\varepsilon \le h$, we have

$$\begin{aligned}\varepsilon \int_\tau \omega v \nabla\omega \cdot \nabla v &= \frac{\varepsilon}{|\beta|^2}\int_\tau \omega v(-\beta_2(\beta^\perp \cdot \nabla\omega)v_{x_1} + \beta_1(\beta^\perp \cdot \nabla\omega)v_{x_2}) \\ &\le \frac{\varepsilon}{|\beta|^2} K^{-1}h^{-1/2}\|\omega v\|_\tau \left((|\beta_2| + |\beta_1|)\|\omega\nabla v\|_\tau\right) \\ &\le CK^{-1}\left(\|\omega v\|_\tau^2 + \varepsilon\|\omega\nabla v\|_\tau^2\right).\end{aligned}$$

Similarly, we can show

$$\varepsilon \sum_{\tau\in T_h}\int_\tau \omega y_h \nabla\omega \cdot \nabla y_h + 2\varepsilon \sum_{e\in\mathcal{E}_h}\int_e \{\omega\nabla y_h\}[\![\omega y_h]\!] + 2\varepsilon \sum_{e\in\mathcal{E}_h}\int_e \{y_h\nabla\omega\}[\![\omega y_h]\!] \le \frac{C}{K}Q_\omega^2(y_h)$$

and

$$\begin{aligned}&\varepsilon \sum_{\tau\in T_h}\int_\tau \omega u_h \nabla\omega \cdot \nabla u_h + 2\varepsilon \sum_{e\in\mathcal{E}_h}\int_e \{\omega\nabla u_h\}[\![\omega u_h]\!] + 2\varepsilon \sum_{e\in\mathcal{E}_h}\int_e \{u_h\nabla\omega\}[\![\omega u_h]\!] \\ &\le \frac{C}{K}Q_\omega^2(u_h).\end{aligned}$$

Thus for K large enough we have

$$Q_\omega^2(y_h) + \alpha Q_\omega^2(u_h) \le 2\widetilde{\mathcal{A}}^{DG}((y_h, u_h), (\omega^2 y_h, \omega^2 u_h)).$$

□

Proof of Lemma A.5. Recall the notation $E_\omega(v) = \omega^2 v - \widetilde{\omega^2 v}$. Since $U_h = V_h$, the definition of the projection implies

$$(u_h, E_\omega(y_h)) = (y_h, E_\omega(u_h)) = 0.$$

To prove Lemma A.5, we need to estimate

$$\begin{aligned}&\widetilde{\mathcal{A}}^{DG}\big((y_h, u_h), (E_\omega(y_h), E_\omega(u_h))\big) \\ &= a_h(y_h, E_\omega(y_h)) - (u_h, E_\omega(y_h)) + \alpha a_h(E_\omega(u_h), u_h) + (y_h, E_\omega(u_h)) \\ &= a_h(y_h, E_\omega(y_h)) + \alpha a_h(E_\omega(u_h), u_h).\end{aligned} \tag{A.16}$$

Following the proof of Theorem 3.1 in [16] we can show

$$a_h(y_h, E_\omega(y_h)) + \alpha a_h(E_\omega(u_h), u_h) \le CK^{-1}(Q_\omega^2(y_h) + \alpha Q_\omega^2(u_h)),$$

which together with (A.16) for K large enough gives the desired estimate. □

Proof of Lemma A.6. We have

$$\widetilde{\mathcal{A}}^{DG}((y,u),(\widetilde{\omega^2 y_h},\widetilde{\omega^2 u_h})) = a_h(y,\widetilde{\omega^2 y_h}) - (u,\widetilde{\omega^2 y_h}) + \alpha a_h(\widetilde{\omega^2 u_h},u) + (y,\widetilde{\omega^2 u_h}).$$

First we prove the following estimates for the terms $a_h(y,\widetilde{\omega^2 y_h})$ and $a_h(\widetilde{\omega^2 u_h},u)$ are essentially estimated in the proof of Theorem 3.1 in [16] by

$$a_h(y,\widetilde{\omega^2 y_h}) \le \frac{C}{\delta}L_\omega^2(y) + CK^{-1}Q_\omega^2(y_h) + C\delta Q_\omega^2(y_h), \tag{A.17a}$$

$$a_h(\widetilde{\omega^2 u_h},u_h) \le \frac{C}{\delta}L_\omega^2(u) + CK^{-1}Q_\omega^2(u_h) + C\delta Q_\omega^2(u_h). \tag{A.17b}$$

The estimates (A.17) are essentially contained in the proof of Theorem 3.1 in [16]. Therefore, we only sketch the proof of (A.17a). Recall from (3.3) that

$$\begin{aligned}
a_h(y,\widetilde{\omega^2 y_h}) =& \varepsilon\sum_{\tau\in T_h}(\nabla y,\nabla\widetilde{\omega^2 y_h})_\tau \\
&+\varepsilon\sum_{e\in\mathcal{E}_h}\left(\frac{\sigma}{h_e}([\![y]\!],[\![\widetilde{\omega^2 y_h}]\!])_e - (\{\nabla_h y\},[\![\widetilde{\omega^2 y_h}]\!])_e - ([\![y]\!],\{\nabla_h\widetilde{\omega^2 y_h}\})_e\right)\\
&+\sum_{\tau\in T_h}(\beta\cdot\nabla y + ry,\widetilde{\omega^2 y_h})_\tau\\
&+\sum_{e\in\mathcal{E}_h^0}\left(y^+-y^-,|\mathbf{n}\cdot\beta|\widetilde{\omega^2 y_h}^+\right)_e + \sum_{e\in\mathcal{E}_h^-}\left(y^+,\widetilde{\omega^2 y_h}^+|\mathbf{n}\cdot\beta|\right)_e.
\end{aligned} \tag{A.18}$$

By the Cauchy-Schwarz and the triangle inequalities we have

$$\begin{aligned}
&\varepsilon\int_\tau \nabla y\cdot\nabla\widetilde{\omega^2 y_h} \le \varepsilon\|\omega\nabla y\|_\tau\|\omega^{-1}\nabla\widetilde{\omega^2 y_h}\|_\tau\\
&\le \frac{2\varepsilon}{\delta}\|\omega\nabla y\|_\tau^2 + \delta\varepsilon\left(\|\omega^{-1}\nabla E_\omega(y_h)\|_\tau^2 + \|\omega^{-1}\nabla(\omega^2 y_h)\|_\tau^2\right).
\end{aligned} \tag{A.19}$$

By the superapproximation result (A.12) and the assumption $\varepsilon \le h$

$$\varepsilon\|\omega^{-1}\nabla E_\omega(y_h)\|_\tau^2 \le CK^{-2}\|\omega y_h\|_\tau^2. \tag{A.20}$$

By the properties (A.10a,b) of ω

$$\begin{aligned}
\|\omega^{-1}\nabla(\omega^2 y_h)\|_\tau^2 &= \|\omega\nabla y_h\|_\tau^2 + 4\|\nabla\omega y_h\|_\tau^2\\
&= \|\omega\nabla y_h\|_\tau^2 + 4\|(D_{\beta^\perp}\omega)y_h + (D_\beta\omega)y_h\|_\tau^2\\
&\le \|\omega\nabla y_h\|_\tau^2 + 4\|(D_{\beta^\perp}\omega)y_h\|_\tau^2 \le \|\omega\nabla y_h\|_\tau^2 + Ch^{-1}K^{-2}\|\omega y_h\|_\tau^2.
\end{aligned} \tag{A.21}$$

Combining (A.19)–(A.21) and using $\varepsilon \le h$ gives

$$\begin{aligned}
\varepsilon\sum_{\tau\in T_h}(\nabla y,\nabla\widetilde{\omega^2 y_h})_\tau &\le \frac{2}{\delta}\sum_{\tau\in T_h}h\|\omega\nabla y\|_\tau^2 + \sum_{\tau\in T_h}\delta\varepsilon\|\omega\nabla y_h\|_\tau^2 + \left(\frac{C\delta}{K^2}+\frac{C\varepsilon\delta}{hK^2}\right)\|\omega y_h\|_\tau^2,\\
&\le \frac{2}{\delta}L_\omega^2(y) + \left(\delta + \frac{C\delta}{K^2}\right)Q_\omega^2(y_h).
\end{aligned}$$

To estimate the third term in (A.18) we recall that $E_\omega(y_h) = \omega^2 y_h - \widetilde{\omega^2 y_h}$ and define S_e to be the union of triangles that share the edge e. Similarly by the triangle and the Cauchy-Schwarz inequalities, the properties of ω, the trace inequality (3.8a), and the superapproximation we have

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h} \varepsilon \int_e \{\nabla_h y\} [\![\widetilde{\omega^2 y_h}]\!]_e \le \sum_{e\in\mathcal{E}_h} \varepsilon \int_e \omega \{\nabla_h y\} \left([\![\omega y_h]\!] \|_e + \omega^{-1} [\![E_\omega(y_h)]\!]_e \right) \\
&\le \sum_{e\in\mathcal{E}_h} \frac{\varepsilon}{h} \left(h^{1/2} \|\omega \nabla y\|_{S_e} + h^{3/2} \|\omega \nabla^2 y\|_{S_e} \right) \left(\|[\![\omega y_h]\!]\|_e + \|\omega^{-1} [\![E_\omega(y_h)]\!]\|_e \right) \\
&\le \frac{C}{\delta} \left(h \|\omega \nabla y\|^2 + h^3 \|\nabla^2 y\|^2 \right) + C K^{-1} Q_\omega^2(y_h) + C\delta \sum_{e\in\mathcal{E}_h} \frac{\varepsilon}{h} \|[\![\omega y_h]\!]\|_e^2.
\end{aligned}
$$

The other terms in (A.18) can be estimated similarly to obtain (A.17a). The estimate (A.17b) is proven analogously.

Thus, to estimate $\widetilde{\mathcal{A}}^{DG}((y,u),(\widetilde{\omega^2 y_h},\widetilde{\omega^2 u_h}))$ we only need to estimate $-(u,\widetilde{\omega^2 y_h}) + (y,\widetilde{\omega^2 u_h})$. Using the Cauchy-Schwarz inequality and the superapproximation result (A.12) we have

$$
\begin{aligned}
(u,\widetilde{\omega^2 y_h}) &= \sum_{\tau\in T_h} (u,\widetilde{\omega^2 y_h})_\tau \le \sum_{\tau\in T_h} \|\omega u\|_\tau \left(\|\omega y_h\|_\tau + \|\omega^{-1} E_\omega(y_h)\|_\tau \right) \\
&\le \frac{2C_2\alpha}{h\delta} \sum_{\tau\in T_h} \|\omega u\|_\tau^2 + \delta \sum_{\tau\in T_h} \|\omega y_h\|_\tau^2 + \delta \sum_{\tau\in T_h} \|\omega^{-1} E_\omega(y_h)\|_\tau^2 \\
&\le \frac{2C_2\alpha}{\delta} L_\omega^2(u) + \sum_{\tau\in T_h} \left(\delta + \frac{C\delta}{K^2} \right) \|\omega y_h\|_\tau^2 \\
&\le \frac{2C_2\alpha}{\delta} L_\omega^2(u) + \left(\delta + \frac{C\delta}{K^2} \right) Q_\omega^2(y_h).
\end{aligned}
$$

Similarly we can obtain

$$
(y,\widetilde{\omega^2 u_h}) \le \frac{C}{\delta} L_\omega^2(y) + \alpha \left(C_2\delta + \frac{C\delta}{K^2} \right) Q_\omega^2(u_h).
$$

Now we complete the proof by choosing K sufficiently large and taking δ sufficiently small. □

REFERENCES

[1] D. Leykekhman and M. Heinkenschloss, *Local Error Analysis of Discontinuous Galerkin Methods for Advection-Dominated Elliptic Linear-Quadratic Optimal Control Problems.*