

Steady State Couette Flow  
with Viscous Heating

Anthony José Kearsley

October, 1991

TR91-33



# Steady State Couette Flow with Viscous Heating

Anthony José Kearsley \*

November 2, 1992

## Abstract

An exact solution is found for a non-linear problem with thermomechanical coupling, the steady flow of a fluid with viscosity exponentially dependent on temperature, which is sheared between an adiabatic, fixed, inner cylinder and a thermostatted, rotating, outer cylinder. There is a maximum torque above which no steady flow is possible and below which two flows are possible, a high shear and a low shear steady flow for each value of torque.

---

\*Rice University, Department of Computational & Applied Mathematics, P.O. Box 1892, Houston, Texas 77251-1892

# 1 Introduction

Some years ago a method for solving certain simple steady flows of incompressible Newtonian fluids with viscous heating was published [5]. The method depended upon representing the viscosity as exponential in the temperature, ordinarily a very good approximation. At the time, the method was applied to Poiseuille flow in an infinite thermostatted tube, but it was suggested that the method could also be used for Couette flow. We will implement that suggestion here.

Consider a steady shearing flow between a fixed infinite inner cylinder of radius  $R_i$  and an infinite outer cylinder of radius  $R_0$  rotating with an angular velocity of amount  $\Omega$ . The outer cylinder is thermostatted at a fixed temperature which can be called zero without loss of generality. The inner cylinder is thermally insulated from the environment and allowed to come to equilibrium with the fluid so that there is no further heat exchange at the inner cylinder. The fluid is an incompressible Newtonian liquid characterized by a viscosity exponentially dependent on temperature but whose other material properties are all constant. All properties are assumed independent of pressure.

## 2 The Model

Consider a system of cylindrical coordinates oriented in the obvious way. We put the  $r$  and  $z$  components of velocity equal to zero and assume the angular velocity, viscosity, temperature and pressure, etc. to be spatial functions of only the radius. The governing equations are as follows:

Two components of the vector force equation.

$$\frac{dp}{dr} = \rho \frac{u^2}{r} \quad (2.1)$$

$$\frac{1}{r^2} \frac{d}{dr} (\eta r^3 \frac{d}{dr} (\frac{u}{r})) = 0 \quad (2.2)$$

The energy balance equation.

$$\frac{\lambda}{r} \frac{d}{dr} (r T_r) + \eta r^2 (\frac{d}{dr} (\frac{u}{r}))^2 = 0 \quad (2.3)$$

The temperature-viscosity equation.

$$\eta = \eta_0 e^{-\alpha T} \quad (2.4)$$

In these equations,  $p$  is the pressure,  $\rho$  is the density,  $r$  is the radius,  $u$  is the velocity in the  $\theta$  direction,  $\eta$  is the viscosity,  $\lambda$  is the thermal conductivity and  $T$  is the temperature. The viscosity at zero temperature is  $\eta_0$  and  $\alpha$  is a thermal coefficient of viscosity, a material property with the dimension reciprocal temperature. It is clear that the zero of temperature is arbitrary save for an adjustment of the value of  $\eta_0$ . Throughout this work, derivatives of a physical quantity by a variable will be indicated by subscripting the quantity with that variable.

The first of these equations determines the pressure which, once the velocity is known, can be chosen to counterbalance the inertial forces on the fluid, and will not affect the shearing. The pressure will play no role in the calculation of the temperature or of the shear. It can be found by integration given the solution for fluid velocity.

We clarify the mathematical structure of the problem by defining the following non-dimension quantities:

$$\psi = \alpha T$$

$$x = \frac{r}{R_0}, 1 \geq x \geq X = \frac{R_i}{R_0} \quad (2.5)$$

$$v = \frac{u}{\Omega R_0}, 0 \leq v \leq X$$

Then, equations (2.1)-(2.3) are expressed as follows:

$$\frac{d}{dx} \left( x^3 e^{-\psi} \frac{d}{dx} \left( \frac{v}{x} \right) \right) = 0 \quad (2.6)$$

$$\psi_{xx} + \frac{1}{x} \psi_x + k e^{-\psi} x^2 \left( \frac{d}{dx} \left( \frac{v}{x} \right) \right)^2 = 0 \quad (2.7)$$

where

$$k = \frac{\alpha \eta_0 R_0^2 \Omega^2}{\lambda} \quad (2.8)$$

The first integral of equation (2.6) is

$$x^3 \frac{d}{dx} \left( \frac{v}{x} \right) = -J e^{\psi} \quad (2.9)$$

where

$$J = \frac{M}{2\pi \Omega \eta_0 R_0^2} \quad (2.10)$$

Here,  $M$  is the torque acting on a cylinder of fluid per unit of axial length. Of course,  $M$  is independent of radius.

Equations (2.9)-(2.10) can be used to eliminate the variable  $v$  from equation (2.7) to give

$$\psi_{xx} + \frac{1}{x} \psi_x + K \frac{e^{\psi}}{x^4} = 0 \quad (2.11)$$

where

$$K = k J^2 = \frac{\alpha M^2}{4\pi^2 \lambda \eta_0 R_0^2} \quad (2.12)$$

An explicit integral of equation (2.11) has been published [6].

$$\psi = \ln \left[ \frac{2m^2 x^2}{K \cosh^2(m \ln(x) + a)} \right] \quad (2.13)$$

where  $m$  and  $a$  are constants of integration to be determined by the thermal boundary conditions.

To find the velocity distribution, (2.9) and (2.10) may be used to substitute for the exponential in (2.11) and the result integrated. Upon taking into account that the velocity is zero at the outer cylinder, ( $v = 0$  when  $x = 1$ ), one can express the velocity as follows:

$$\frac{v}{x} = \frac{2J}{K} m (\tanh(a) - \tanh(m \ln(x) + a)) \quad (2.14)$$

Equations (2.13) and (2.14) are exact expressions for the temperature and velocity of the fluid in this flow. To complete the solution there only remains the task of evaluating the constants of integration,  $m$  and  $a$ , from the thermal boundary conditions.

The thermal conditions to be satisfied can be expressed as follows:

$$\psi = 0 \text{ at } x = 1$$

$$\psi_x = 0 \text{ at } x = X < 1$$

These conditions, when applied to equation (2.13) lead to simultaneous equations which determine  $m$  and  $a$ , the thermal constants of integration.

$$\frac{K}{2} = \frac{m^2}{\cosh^2(a)} \quad (2.15)$$

$$\tanh(m \ln(X) + a) = \frac{1}{m} \quad (2.16)$$

It is easy to see that  $m$  and  $a$  must be of the same sign and that if  $(m, a)$  is a solution of these equations, then  $(-m, -a)$  is also a solution. Inspection of equations (2.13) and (2.14) then shows that  $m$  and  $a$  may be taken as positive without loss of generality.

Unfortunately, there does not seem to be a convenient, explicit, functional representation of the solutions of these equations and it seems necessary to resort to numerical calculations. A complete parametric solution is best displayed by using equation (2.16) to express  $a$  in terms of  $m$ .

$$\cosh(a) = \frac{m \cosh(m \ln(X)) - \sinh(m \ln(X))}{\sqrt{m^2 - 1}} \quad (2.17)$$

With a little manipulation, this result can be put in the following form:

$$a = \frac{1}{2} \ln \left[ \frac{m+1}{m-1} X^{-2m} \right] \quad (2.18)$$

Then, we can write the following equation by substitution into equation (2.15):

$$\sqrt{\frac{K}{2}} = \frac{m}{\cosh\left(\frac{1}{2} \ln\left(\frac{m+1}{m-1} X^{-2m}\right)\right)} \quad (2.19)$$

Given  $K$  and  $X$ , this can be used to find  $m$ .

Figure 1 is a plot of the right hand side of equation (2.19) versus  $m$  for a few values of  $X$ . From this plot it is easy to see that for each value of  $X$  there is a limiting value of  $K$  beyond which there are no solutions. In general, for values of  $K$  less than the critical value there will be two values of  $m$  which are solutions and at the critical  $K$  there will be only one. This implies that there is no steady solution to the problem for torques greater than a maximum torque depending on the geometry and physical properties.

Equations (2.13) and (2.14) can be rewritten in terms of a single parameter,  $m$ , and

physical quantities using the definitions of equations (2.9), (2.10) and (2.12).

$$T_D = \frac{\alpha T}{2} = \ln \left[ \frac{xG(m, X)}{G(m, \frac{x}{X})} \right] \quad (2.20)$$

$$\left(\frac{u}{r}\right)_D = \frac{2\pi R_0^2 \eta_0 u}{Mr} = -\frac{\sinh(m \ln(x))G(m, X)}{mG(m, \frac{x}{X})} \quad (2.21)$$

where, for convenience, we have defined the function  $G(\cdot, \cdot)$  as follows:

$$G(m, x) = m \cosh(m \ln(x)) - \sinh(m \ln(x)) \quad (2.22)$$

These equations combined with the physical evaluation of  $M$  through equations (2.12), (2.15) and (2.16) constitute a complete solution.

From a practical point of view, there are three natural physical measurements that characterize this solution, the torque on the rotor, the angular velocity of the rotor and the temperature difference between the inner and outer cylinders. Of course, the temperature of the of the thermostatted outer cylinder must also be measured, but only to set the zero of the temperature scale being used and it is not pertinent to this discussion.

These quantities may be expressed in terms of the physical properties of the fluid and the geometry. By evaluating equations (2.20) and (2.21) at the inner and outer cylinder ( $x = X$  and  $x = 1$ , respectively), one can show the following:

$$M_D = \sqrt{\frac{\alpha}{2\lambda\eta_0}} \frac{M}{2\pi R_0^2} = \frac{m\sqrt{m^2 - 1}}{G(m, X)} \quad (2.23)$$

$$\Omega_D = \sqrt{\frac{\alpha\eta_0}{2\lambda}} R_0 \Omega = -\frac{\sqrt{m^2 - 1} \sinh(m \ln(X))}{m} \quad (2.24)$$

$$\Delta T_D = \frac{\alpha \Delta T}{2} = \ln \left[ \frac{XG(m, X)}{m} \right] \quad (2.25)$$

where  $\Delta T$  is the temperature difference. In these equations, the quantities on the left are dimensionless.

Other physical quantities of some interest can be expressed as combinations of those above, for example, the ratio of viscosities at the inner and outer radius; the power,  $\Omega M$ ; the ratio of torque and angular velocity,  $\frac{M}{\Omega}$ ; and the ratio of power to temperature difference,  $\frac{\Omega M}{\Delta T}$ .

Tables I and II are calculated from these equations for two examples, a thick shearing layer ( $X = .8$ ) and a thin shearing layer ( $X = .95$ ). In each example, the quantities associated with the maximum torque are indicated by asterisks.

Several features of these tables are worth a comment. The last two columns are dimensionless quantities which do not vanish at zero torque (i.e.  $m = 1$ ). They were evaluated in the limit by solution of the simpler problem with viscosities independent of temperature.

$$\frac{M_D}{\Omega_D} = \frac{2X^2}{1 - X^2} \quad (2.26)$$

$$\frac{M_D \Omega_D}{T_D} = \frac{2(1 - X^2)}{X^2 - 1 - 2 \ln(X)} \quad (2.27)$$

At infinite power, all quantities go to zero or infinity except that of the last column. That limit is easily evaluated by taking limits as  $m$  goes to infinity.

It seemed quite surprising to find that for the cases tabulated here, the viscosity ratio at the maximum torque varies barely 1 percent. In fact, this ratio ranges barely 14 percent for a range of  $X$  from (.001) to (.999).

Figures 2 and 3 illustrate the dimensionless temperature and velocity profiles across the gap for typical low shear and high shear flows associated with the same torque. As one would expect, for the high shear flow, shearing tends to concentrate in a boundary layer at the hot inner radius. The temperature profile of the high shear flow shows a slight point of inflection. The low shear flow looks very much like a slightly distorted version of the case

with viscosity independent of temperature.

Figure 4 is a plot of  $\Omega_D$  versus  $M_D$  for the same geometry as for Figures 2 and 3. It illustrates the limiting maximum torque occurring at a finite  $\Omega$ .

### 3 Conclusions

It is interesting to compare this solution to the solution of a similar problem with a viscosity-temperature law designed to linearize Equation (2.11), the energy equation [3]. In that case, as here, a limiting torque results but, in contrast, the solution for a given torque is unique, a consequence of the linear nature of the problem.

It can be shown that our solution, in the limit as  $R_i$  grows infinite and  $X$  approaches 1, approaches the solution for shearing between thermostatted planes given by Nahme [7]. In that limit, the surface of  $R_i$  corresponds to the midpoint of the gap between the thermostatted planes. To that extent, our solution includes that of Nahme.

Of considerable interest is a calculation appearing in the literature for plane shearing of fluid with an Arrhenius type of viscosity-temperature law [2]. That calculation leads to a sigmoidal curve for a plot corresponding to our Figure 4. As a result, it suggests that in principal as many as three possible values of  $\Omega$  can occur for a given  $M$ .

Our exact solution is quite remarkable in that it describes a very complex physical situation. It seems that the method could be generalized quite easily to solve the problem with other simple boundary conditions, perhaps allowing some heat conduction at the inner radius, etc. It is not difficult to think of possible practical applications. For instance, this solution might well shed some light on the action of journal bearings or it may be a

useful model applicable to shear bands in deforming ductile metals [1]. It also suggests the possibility of constructing various instruments to measure physical properties of liquids.

A few comments are in order on the possibility of achieving the high shear condition. Is it possible to operate an apparatus at points on the curve of Figure 4 at values of  $\Omega_D$  above that at the torque maximum? This is a question of stability. Ordinarily, for Couette flows of fluids with constant viscosity, the non-linearities of the inertial effects lead to an instability first appearing as vortices. It has been shown that these so-called Taylor vortices may be suppressed by adding a rigid body rotation to the fluid affecting only the pressure [8] and we do not consider them here. The thermal coupling to the mechanical equations through the viscosity adds another non-linearity which may lead to instability in our case. For thermostatted plane flow as in the Nahme case, the work of Joseph [4] suggests that for fixed controlled shearing forces there is a thermal instability active at all shear rates above that at the maximum shear force. This suggests that in our case it would not be possible to achieve the high shear states by fixing the torque. As a rule, however, situations with dead loading are much more susceptible to instability than when deformation is controlled. It is not clear without further study that an apparatus which controls  $\Omega_D$  or even controls power might not be perfectly stable at high values of  $M_D$ . A proper study of these questions is not trivial.

## Acknowledgments

I wish to thank Professor Søren Jensen for his supervision and generous guidance. I must also thank Dr. E. A. Kearsley, who suggested the problem to me and contributed considerable help.

## References

- [1] T. J. BURNS Approximate Linear Stability of a Model of Adiabatic Shear Band Formation *Quart. Appl. Math.* XLIII:65–84, 1985.
- [2] S. H. DAVIS, G. A. KRIEGSMAN, R. L. LAURENCE and S. ROSENBLAT Multiple solutions and Hysteresis in Steady Parallel Viscous Flows *Phys. Fluids*, 26:1177–1182, 1983.
- [3] T. IA. GRAZDOVSKII and S. A. REGIRER, Motion of Newtonian Liquid Between Rotating Coaxial Cylinders in the Presence of Thermal Processes Affecting the Viscous Properties *Soviet Physics (Technical Physics)*, 1:1493–1571, 1957.
- [4] D. D. JOSEPH, Variable Viscosity Effects on the Flow and Stability of Flow in Channels *Phys. Fluids*, 7:1761–1771, 1964.
- [5] E. A. KEARSLEY, The Viscous Heating Correction for Viscometer Flows *Trans. Soc. Rheology*, VI:253–261, 1962.
- [6] E. A. KEARSLEY, Solutions of the Equation  $\psi_{xx} + \frac{1}{x}\psi_x + Kx^n e^\psi = 0$  *Journ. Research Nat. Bur. Stand.* , 67B:1146–1161, 1963.
- [7] R. NAHME, Beitrage zur hydrodynamischen Theorie der Lagerreibung *Ing. Archiv* 11:191–209, 1940.
- [8] G. I. TAYLOR Motion of solids in Fluids when the Flow is not Irrotational *Proc. Roy. Soc. (London)* A222:289–343, 1916.

Table I

Dimensionless Variables for Case  $X = .8$

| $m$      | $M_D$  | $\Omega_D$ | $\Delta T_D$ | $\eta_i/\eta_0$ | $\Omega_D M_D$ | $\frac{M_D}{\Omega_D}$ | $\Omega_D \frac{M_D}{\Delta T_D}$ |
|----------|--------|------------|--------------|-----------------|----------------|------------------------|-----------------------------------|
| 1.000    | 0.0000 | 0.0000     | 0.000        | 1.000           | 0.0000         | 3.5556                 | 8.3442                            |
| 1.001    | 0.0358 | 0.0101     | 4.32E-5      | 1.000           | 3.60E-4        | 3.5555                 | 8.3441                            |
| 1.010    | 0.1134 | 0.0319     | 4.34E-4      | 0.999           | 3.62E-3        | 3.5534                 | 8.3431                            |
| 1.100    | 0.3650 | 0.1033     | 4.52E-3      | 0.991           | 3.77E-2        | 3.5334                 | 8.3328                            |
| 1.200    | 0.5257 | 0.1498     | 9.46E-3      | 0.981           | 7.87E-2        | 3.5093                 | 8.3204                            |
| 1.300    | 0.6548 | 0.1880     | 1.48E-2      | 0.971           | 1.23E-1        | 3.4834                 | 8.3070                            |
| 1.500    | 0.8708 | 0.2542     | 2.67E-2      | 0.948           | 2.21E-1        | 3.4262                 | 8.2774                            |
| 2.000    | 1.3005 | 0.3995     | 6.34E-2      | 0.881           | 5.20E-1        | 3.2556                 | 8.1879                            |
| 2.500    | 1.6430 | 0.5382     | 1.09E-1      | 0.803           | 8.84E-1        | 3.0526                 | 8.0795                            |
| 3.000    | 1.9205 | 0.6794     | 1.64E-1      | 0.720           | 1.305          | 2.827                  | 7.956                             |
| 3.500    | 2.1411 | 0.8269     | 2.26E-1      | 0.636           | 1.770          | 2.588                  | 7.822                             |
| 4.000    | 2.3059 | 0.9836     | 2.95E-1      | 0.553           | 2.268          | 2.344                  | 7.680                             |
| 4.500    | 2.4334 | 1.152      | 3.71E-1      | 0.477           | 2.792          | 2.103                  | 7.535                             |
| 5.000    | 2.4970 | 1.335      | 4.51E-1      | 0.406           | 3.332          | 1.871                  | 7.390                             |
| *5.785   | 2.5369 | 1.6552     | 5.86E-1      | 0.310           | 4.199          | 1.5327                 | 7.1655                            |
| 6.000    | 2.5343 | 1.751      | 6.25E-1      | 0.287           | 4.439          | 1.447                  | 7.106                             |
| 6.500    | 2.5089 | 1.991      | 7.17E-1      | 0.238           | 4.996          | 1.260                  | 6.970                             |
| 7.000    | 2.4109 | 2.256      | 8.12E-1      | 0.197           | 5.553          | 1.091                  | 6.841                             |
| 7.500    | 2.3958 | 2.549      | 9.09E-1      | 0.162           | 6.107          | 0.940                  | 6.718                             |
| 8.000    | 2.3167 | 2.874      | 1.008        | 0.113           | 6.657          | 0.806                  | 6.602                             |
| 8.500    | 2.2274 | 3.234      | 1.109        | 0.109           | 7.203          | 0.689                  | 6.493                             |
| 9.000    | 2.1301 | 3.636      | 1.212        | 0.089           | 7.749          | 0.586                  | 6.391                             |
| 10.00    | 1.924  | 4.580      | 1.420        | 0.0584          | 8.813          | .0420                  | 6.207                             |
| 20.00    | 0.439  | 443.31     | 3.595E0      | 0.001           | 2.00E+1        | 0.0101                 | 5.283                             |
| $\infty$ | 0.0000 | $\infty$   | $\infty$     | 0.0E0           | $\infty$       | 0.0000                 | 4.4810                            |

\* marks the row of values associated with the limiting maximum torque.

Table II

Dimensionless Variables for Case  $X = .95$

| $m$      | $M_D$  | $\Omega_D$ | $\Delta T_D$ | $\eta_i/\eta_0$ | $\Omega_D M_D$ | $\frac{M_D}{\Omega_D}$ | $\Omega_D \frac{M_D}{\Delta T_D}$ |
|----------|--------|------------|--------------|-----------------|----------------|------------------------|-----------------------------------|
| 1.00     | 0.0000 | 0.0000     | 0.0000       | 1.000           | 0.000          | 18.513                 | 38.336                            |
| 1.01     | 0.1347 | 7.27E-3    | 2.55E-5      | 1.000           | 9.79E-4        | 18.512                 | 38.336                            |
| 1.10     | 0.4352 | 2.35E-2    | 2.67E-4      | 0.999           | 0.0102         | 18.506                 | 38.334                            |
| 1.50     | 1.060  | 0.057      | 1.59E-3      | 0.997           | 0.0609         | 18.473                 | 38.316                            |
| 2.00     | 1.639  | 0.089      | 3.81E-3      | 0.992           | 0.1459         | 18.418                 | 38.288                            |
| 3.00     | 2.660  | 0.146      | 1.01E-2      | 0.980           | 0.3874         | 18.262                 | 38.210                            |
| 5.00     | 4.515  | 0.254      | 3.02E-2      | 0.941           | 1.147          | 17.774                 | 37.962                            |
| 6.00     | 5.379  | 0.308      | 4.39E-2      | 0.916           | 1.658          | 17.449                 | 37.797                            |
| 7.00     | 6.199  | 0.363      | 5.99E-2      | 0.887           | 2.251          | 17.076                 | 37.604                            |
| 8.00     | 6.974  | 0.419      | 7.81E-2      | 0.855           | 2.920          | 16.658                 | 37.389                            |
| 9.00     | 7.700  | 0.475      | 9.85E-2      | 0.821           | 3.659          | 16.202                 | 37.151                            |
| 10.0     | 8.375  | 0.533      | 0.121        | 0.785           | 4.464          | 15.712                 | 36.893                            |
| 11.0     | 8.997  | 0.592      | 0.146        | 0.748           | 5.328          | 15.193                 | 36.617                            |
| 12.0     | 9.566  | 0.653      | 0.172        | 0.709           | 6.245          | 14.653                 | 36.327                            |
| 15.0     | 10.945 | 0.846      | 0.262        | 0.593           | 9.257          | 12.942                 | 35.383                            |
| 20.0     | 12.21  | 1.214      | 0.440        | 0.415           | 14.834         | 10.064                 | 33.696                            |
| *23.734  | 12.467 | 1.5398     | 0.5917       | 0.362           | 19.1962        | 8.096                  | 32.444                            |
| 25.0     | 12.442 | 1.662      | 0.646        | 0.275           | 20.684         | 7.484                  | 32.032                            |
| 30.0     | 11.941 | 2.221      | 0.869        | 0.176           | 26.521         | 5.377                  | 30.507                            |
| 35.0     | 11.011 | 2.926      | 1.105        | 0.110           | 32.223         | 3.763                  | 29.168                            |
| 40.0     | 9.872  | 3.825      | 1.348        | 0.068           | 37.763         | 2.581                  | 28.023                            |
| 50.0     | 7.499  | 6.458      | 1.846        | 0.025           | 48.434         | 1.161                  | 26.242                            |
| 60.0     | 5.426  | 10.829     | 2.352        | 0.009           | 58.754         | 0.5011                 | 24.983                            |
| 75.0     | 3.158  | 23.413     | 3.116        | 0.002           | 73.933         | 0.1349                 | 23.725                            |
| 90.0     | 1.760  | 50.556     | 3.883        | 0.0004          | 88.983         | 0.0348                 | 22.915                            |
| $\infty$ | 0.0000 | $\infty$   | $\infty$     | 0.00E00         | $\infty$       | 0.0000                 | 19.496                            |

\* marks the row of values associated with the limiting maximum torque.

Figure 1

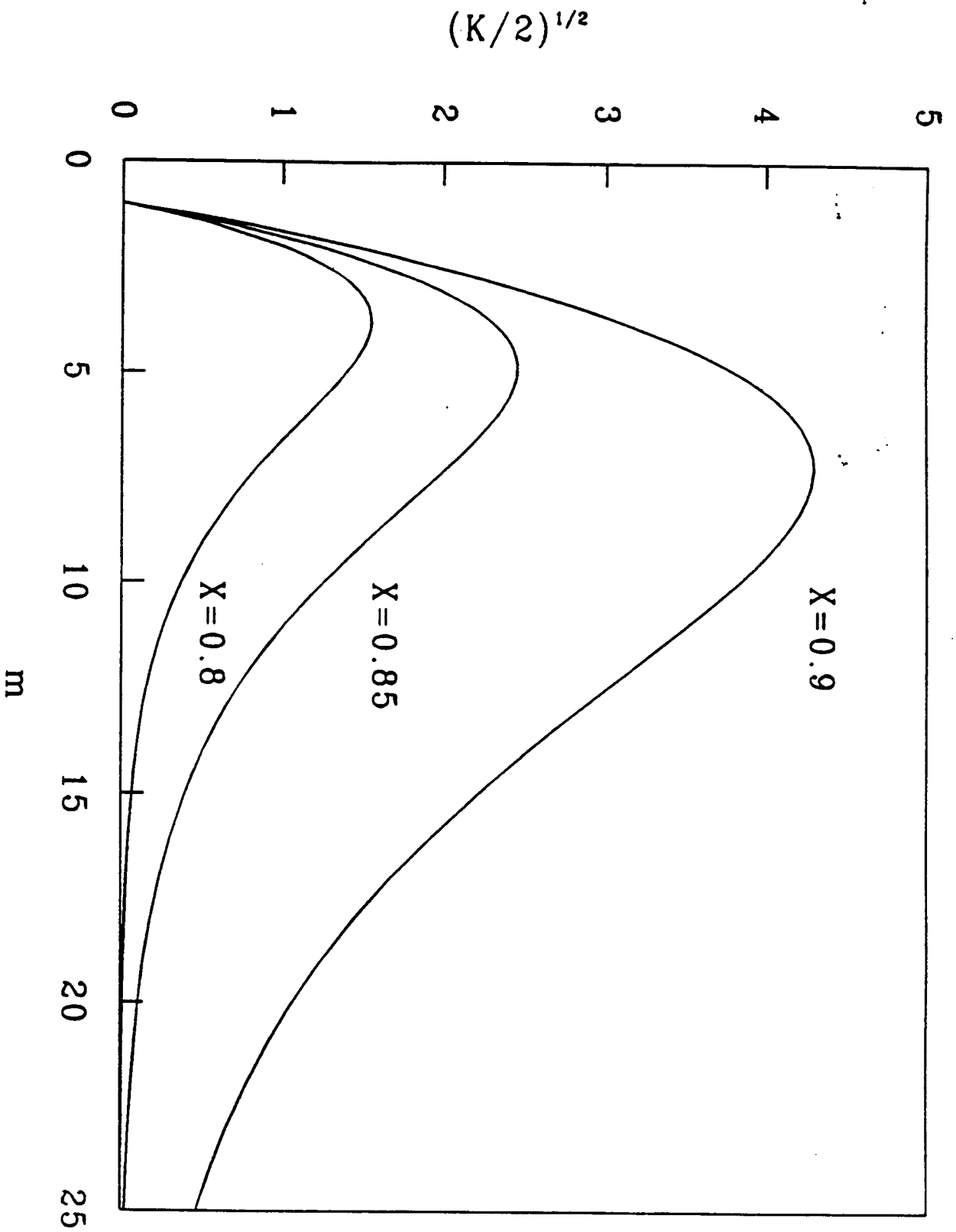


Figure 2

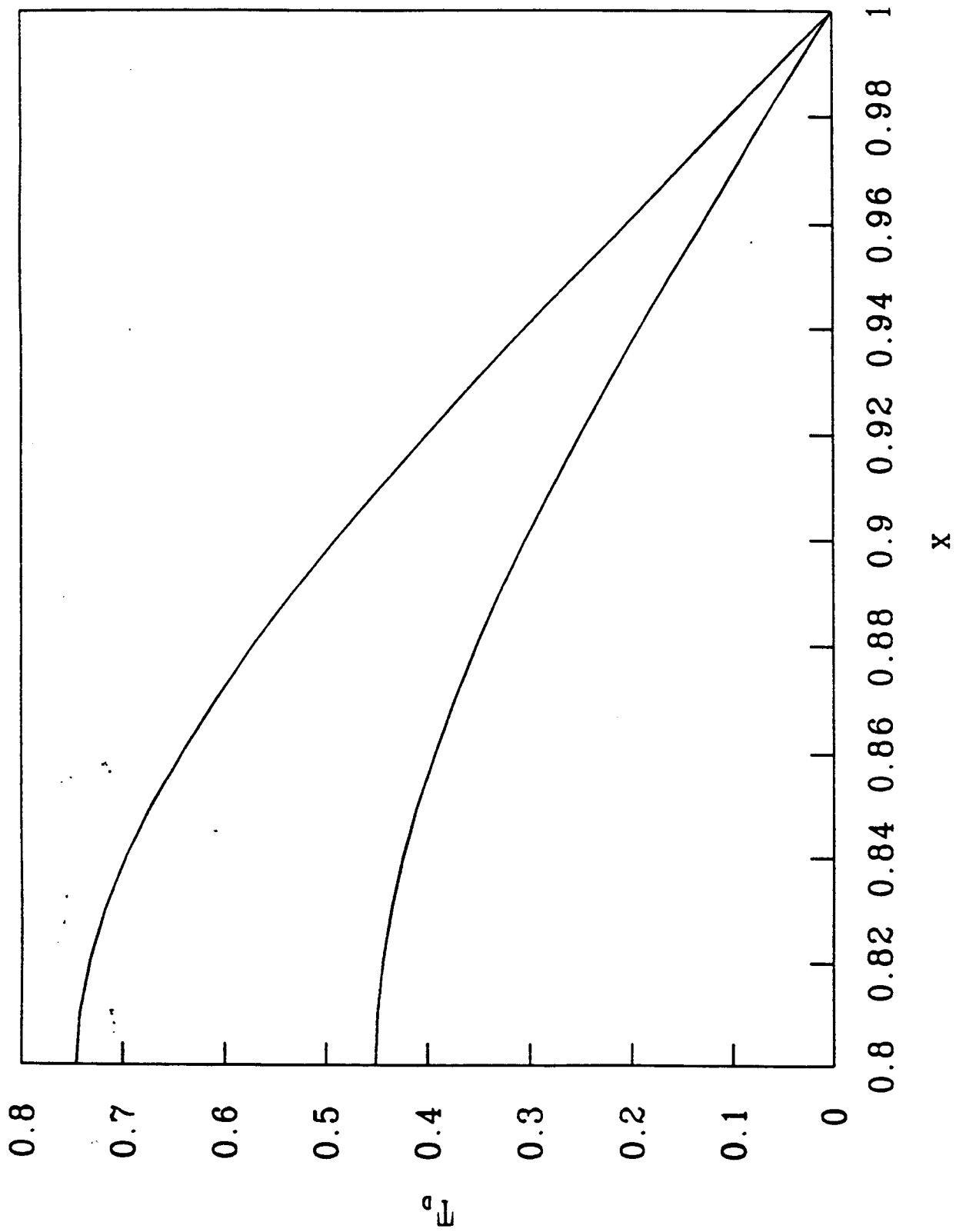


Figure 3

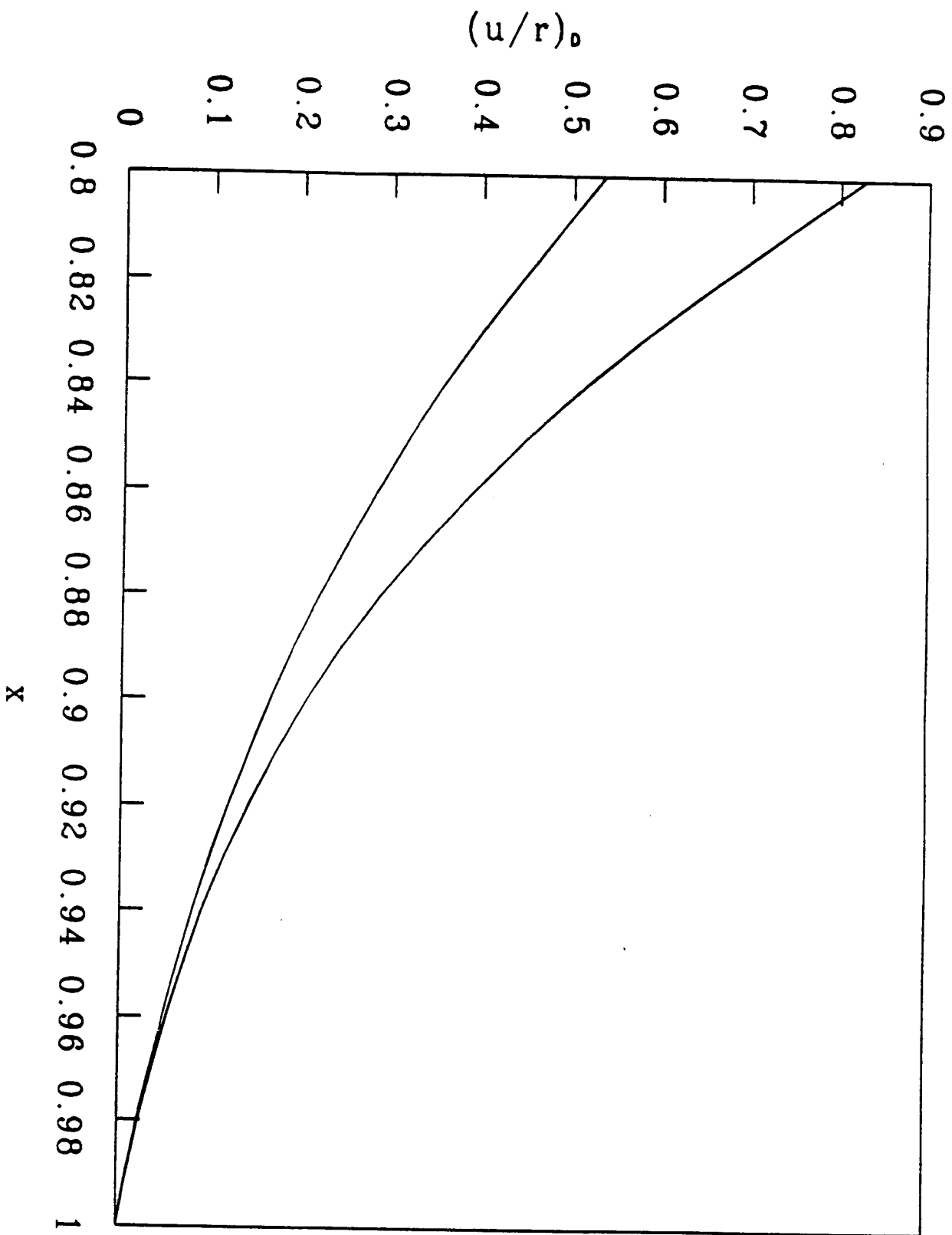
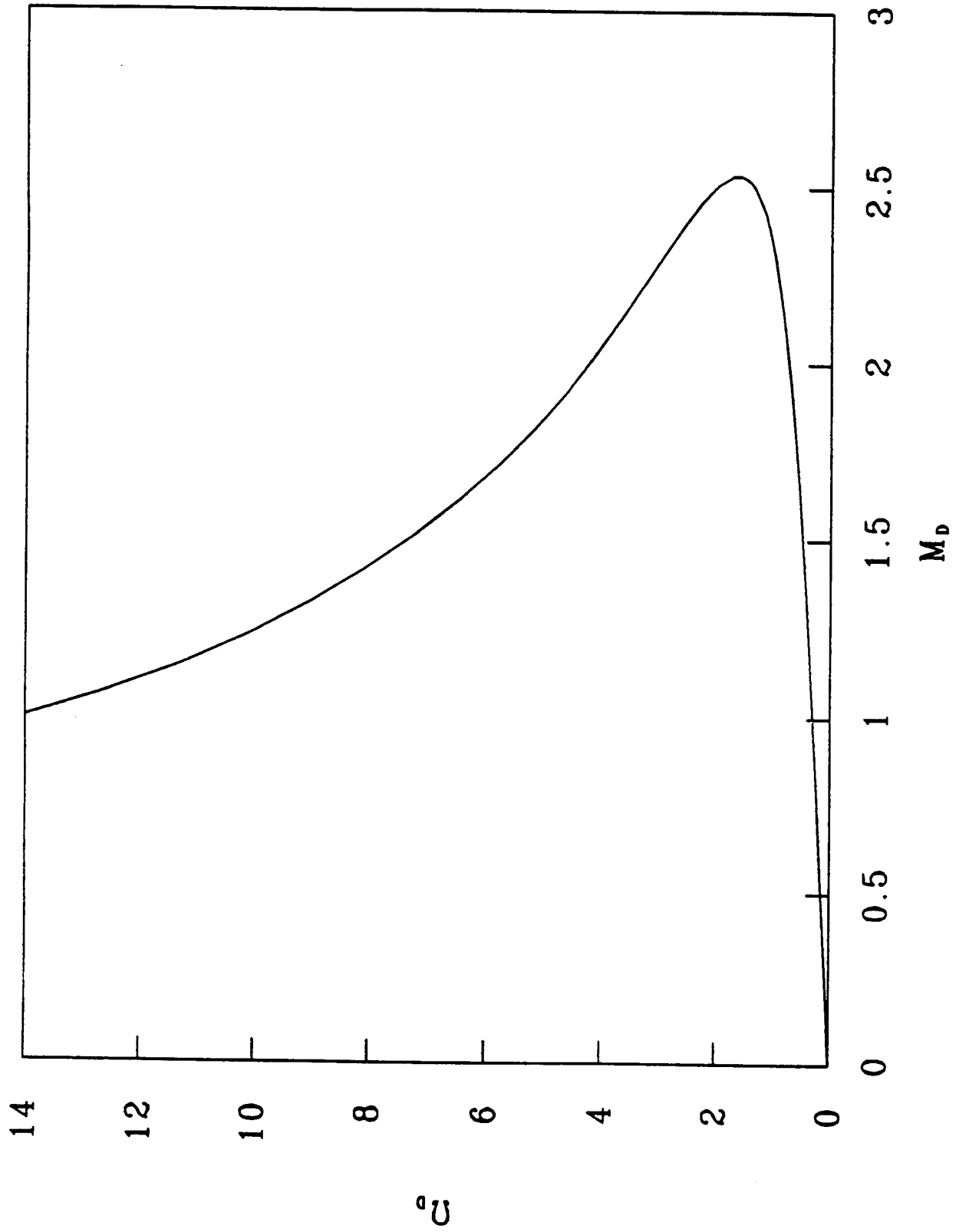


Figure 4



## Captions to Figures

FIG. 1. A plot of  $(K/2)^{1/2}$ , which is proportional to torque, against the parameter  $m$  for various values of the ratio of the radii. The maxima of these curves correspond to the limiting values of torque beyond which there are no stable solutions. Smaller values of torque correspond to two distinct values of  $m$ .

FIG. 2. The dimensionless temperature across the gap between cylinders. The two curves are for the high shear and low shear modes at the same torque. In this case  $K = (12.47)$  and  $X = (.8)$ .

FIG. 3. The dimensionless angular velocity across the gap between cylinders. The two curves are for the high shear and low shear under same conditions as for FIG. 2.

FIG. 4. A plot of dimensionless angular velocity vs. dimensionless torque for a ratio of radii of  $(0.8)$ . The existence of a limiting value of torque and the double values of velocity for a given torque are clear.