# Problem Set 1,Chapter 1

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**Problem 1** *a "constant rate stretch"*

*Solution: To compute the spatial velocity function* $v(x,t)$ *by definition, the first step is to compute the material velocity* $\frac{\partial x(X,t)}{\partial t}$ *of the material particle at position* $X$ *in the undeformed state,i.e.,*

$$\frac{\partial x(X,t)}{\partial t} = \frac{\partial((1+t)X)}{\partial t} = X \tag{1}$$

*the second step is to substitute for the position* $X$ *the position* $X(x,t)$ *in the undeformed state of the particle which is now (time* $= t$*) at* $x$*, i.e., by the definition of the motion here,*

$$X(x,t) = \frac{x}{(1+t)}, \tag{2}$$

*it follows that the spatial velocity*

$$v(x,t) = \frac{x}{(1+t)}. \tag{3}$$

**Problem 2** *1D version of the Reynolds Transport Theorem*

*Solution: 1st step : use the chain rule. Set*

$$F(y,z,t) \equiv \int_y^z dx\Phi(x,t) \tag{4}$$

*then*

$$\int_{x(X_1,t)}^{x(X_2,t)} dx\Phi(x,t) = F(x(X_1,t),x(X_2,t),t) \tag{5}$$

*So*

$$\frac{d}{dt}\int_{x(X_1,t)}^{x(X_2,t)} dx\Phi(x,t) = \frac{\partial F(x(X_1,t),x(X_2,t),t)}{\partial y}\frac{\partial x(X_1,t)}{\partial t} + \frac{\partial F(x(X_1,t),x(X_2,t),t)}{\partial z}\frac{\partial x(X_2,t)}{\partial t} + \frac{\partial F(x(X_1,t),x(X_2,t),t)}{\partial t} \tag{6}$$

*According to the funadmental theorem of calculus,*

$$\frac{\partial F(y,z,t)}{\partial y} = \frac{\partial}{\partial y}\int_y^z dx\Phi(x,t) = -\Phi(y,t) \tag{7}$$

*Similarly,*

$$\frac{\partial F(y,z,t)}{\partial z} = \Phi(z,t) \tag{8}$$

*The rule on the differentiation under the integral sign is applicable( $\Phi$ is continuously differentiable),so*

$$\frac{\partial F(y,z,t)}{\partial t} = \int_y^z dx\frac{\partial \Phi(x,t)}{\partial t} \tag{9}$$

*Putting all of above together,get*

$$\begin{aligned}\frac{d}{dt}\int_{x(X_1,t)}^{x(X_2,t)} dx\Phi(x,t) &= \Phi(x(X_2,t),t)\frac{\partial x(X_2,t)}{\partial t} - \Phi(x(X_1,t),t)\frac{\partial x(X_1,t)}{\partial t} + \\ &\int_{x(X_1,t)}^{x(X_2,t)} dx\frac{\partial \Phi(x,t)}{\partial t}, \end{aligned}\tag{10}$$

*2nd step: Recall from class that*

$$v(x,t) = \frac{\partial x(X,t)}{\partial t}|_{X=X(x,t)} \tag{11}$$

*or*

$$v(x(X,t),t) = \frac{\partial x(X,t)}{\partial t} \tag{12}$$

*In particular,*

$$v(x(X_1,t),t) = \frac{\partial x(X_1,t)}{\partial t} \tag{13}$$

$$v(x(X_2,t),t) = \frac{\partial x(X_2,t)}{\partial t} \tag{14}$$

*Substituting into ( 10),obtain*

$$\begin{aligned}\frac{d}{dt}\int_{x(X_1,t)}^{x(X_2,t)} dx\Phi(x,t) &= \Phi(x(X_2,t),t)v(x(X_2,t),t) - \Phi(x(X_1,t),t)v(x(X_1,t),t) + \\ &\int_{x(X_1,t)}^{x(X_2,t)} dx\frac{\partial \Phi(x,t)}{\partial t}, \end{aligned}\tag{15}$$

*Recognize that the difference of the values of $\Phi v$ at $(x(X_2,t),t)$ and $(x(X_1,t),t)$ is exactly*

$$\int_{x(X_1,t)}^{x(X_2,t)} dx \frac{\partial(\Phi v)}{\partial x},$$

*the Reynolds transport theorem follows from ( 15):*

$$\frac{d}{dt}Q(x(X_1,t),x(X_2,t);t) = \int_{x(X_1,t)}^{x(X_2,t)} dx(\frac{\partial \Phi}{\partial t} + \frac{\partial(\Phi v)}{\partial x})(x,t), \qquad (16)$$

**Problem 3** *Material description of conservation of mass from the spatial description.*

*Solution: By the spatial description of conservation of mass,*

$$\frac{\partial \rho}{\partial t}(x,t) + \frac{\partial(\rho v)}{\partial x})(x,t) = 0 \qquad (17)$$

*which is the integrand on the RHS of ( 16) if you take $\Phi = \rho$. Thus it follows from ( 16) that*

$$\frac{d}{dt}\int_{x(X_1,t)}^{x(X_2,t)} dx \rho(x,t) = 0 \qquad (18)$$

*namely,*

$$\int_{x(X_1,t)}^{x(X_2,t)} dx \rho(x,t) = const \qquad (19)$$

*but this integral is exactly the amount of mass between locations $x(X_1,t)$ and $x(X_2,t)$ at time $t$, that is, the mass of any part is independent of time, which is the Lagrangian form of the mass balance law.*