

Problem Set 3, Chapter 1

October 7, 1997

1. Solution: By the definition of the homogeneous deformation $x(X, t)$, it follows that

$$\begin{aligned}\nabla_x(X) &= A, \\ \det(A) &= \lambda,\end{aligned}\tag{1}$$

where

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\tag{2}$$

moreover,

$$\begin{aligned}X - X_0 &= A^{-1}(x - x_0), \\ \nabla_X(x) &= A^{-1}.\end{aligned}\tag{3}$$

So the Lagrangian strain tensor

$$\begin{aligned}L(X) &= \frac{1}{2}(\nabla_x(X)^T \nabla_x(X) - I) \\ &= \begin{bmatrix} (\lambda^2 - 1)/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\end{aligned}\tag{4}$$

and the Eulerian strain tensor

$$\begin{aligned}e(x) &= \frac{1}{2}(I - \nabla_X(x)^T \nabla_X(x)) \\ &= \begin{bmatrix} (1 - 1/\lambda^2)/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\end{aligned}\tag{5}$$

Furthermore, the pairs of principal strain (eigenvalue) and axis (eigenvector) of L are $(0, (0, 0, 1)^T)$, $(0, (0, 1, 0)^T)$, $((\lambda^2 - 1)/2, (1, 0, 0)^T)$, and the pairs of the principal strain and axis of e are $(0, (0, 0, 1)^T)$, $(0, (0, 1, 0)^T)$, $((1 - 1/\lambda^2)/2, (1, 0, 0)^T)$.

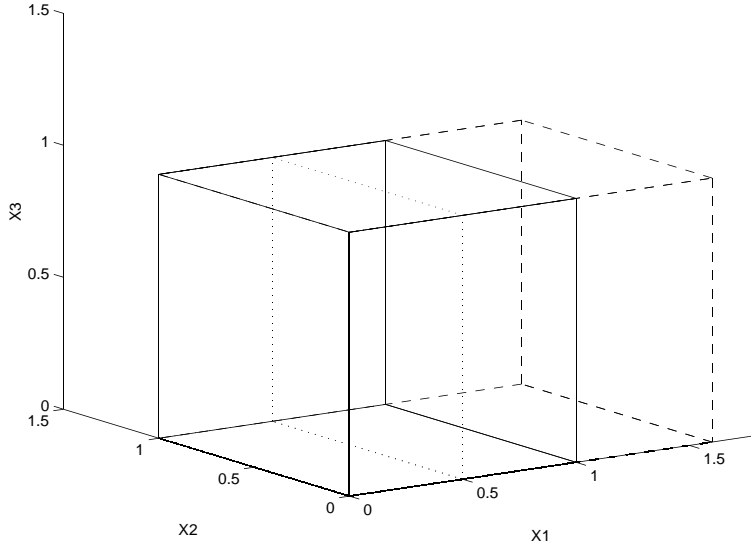


FIG. 1.

$1/\lambda^2)/2, (1, 0, 0)^T$.

The uniform extension is illustrated in the figure 1.

2. Solution: By the definition of the homogeneous deformation $x(X, t)$, it follows that the Lagrangian strain tensor

$$\begin{aligned} L &= \frac{1}{2}(A^T A - I) \\ &= \frac{1}{2} \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (6)$$

and the Eulerian strain tensor

$$\begin{aligned} e &= \frac{1}{2}(I - (A^{-1})^T A^{-1}) \\ &= \frac{1}{2} \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & -\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (7)$$

Furthermore, the pairs of principal strain (eigenvalue) and axis (eigenvector) of L are $(0, (0, 0, 1)^T)$, $((\gamma^2 + \gamma(\gamma^2 + 4)^{1/2})/4, (2/(\gamma + (\gamma^2 + 4)^{1/2}), 1, 0)^T)$, $((\gamma^2 - \gamma(\gamma^2 + 4)^{1/2})/4, (2/(\gamma - (\gamma^2 + 4)^{1/2}), 1, 0)^T)$, and the pairs of the principal strain and axis of e are $(0, (0, 0, 1)^T)$, $((-\gamma^2 +$

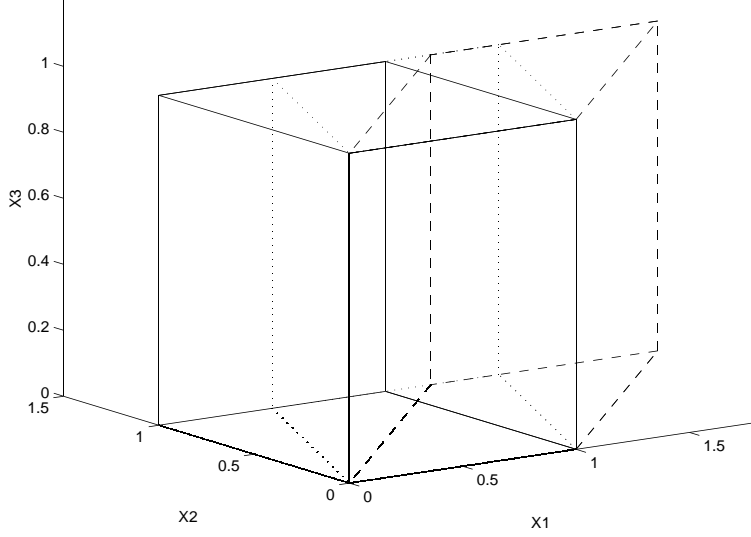


FIG. 2.

$\gamma(\gamma^2+4)^{1/2}/4, (2/(-\gamma+(\gamma^2+4)^{1/2}), 1, 0)^T), ((-\gamma^2-\gamma(\gamma^2+4)^{1/2})/4, (2/(-\gamma-(\gamma^2+4)^{1/2}), 1, 0)^T).$

The pure shear is illustrated in the figure 2.

Extra credit: Suppose v is an eigenvector of L with eigenvalue λ :

$$\begin{aligned} Lv &= \frac{1}{2}(A^T A - I)v \\ &= \lambda v. \end{aligned} \quad (8)$$

Let $w = Av$, so $v = A^{-1}w$ and it follows that:

$$\begin{aligned} \frac{1}{2}(A^T A - I)A^{-1}w &= \lambda A^{-1}w \\ \frac{1}{2}(I - (A^{-1})^T A^{-1})w &= \lambda(A^{-1})^T A^{-1}w. \end{aligned} \quad (9)$$

$$(I - (1 + 2\lambda)(A^{-1})^T A^{-1})w = 0,$$

where we have used $(A^{-1})^T = (A^T)^{-1}$ to multiply on left in the first equality to get the second equality. Thus, add $2\lambda w$ to both sides and divide by $(1 + 2\lambda)$ to get

$$\begin{aligned} ew &= \frac{1}{2}(I - (A^{-1})^T A^{-1})w \\ &= \lambda/(1 + 2\lambda)w, \end{aligned} \quad (10)$$

which means that (λ, v) is a pair of Lagrangian principal strain and axis if and only if $(\lambda/(1+2\lambda), Av)$ is a pair of Eulerian principal strain

and axis.

3. Solution: By the definition of the deformation $x(X, t)$, it follows that

$$\frac{\partial x}{\partial t}(X, t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & f'(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} X, \quad (11)$$

and

$$X(x, t) = \begin{bmatrix} 1/(1+t) & 0 & 0 \\ 0 & 1/f(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} x, \quad (12)$$

so the spatial velocity

$$v(x, t) = \frac{\partial x}{\partial t}(X(x, t), t) \quad (13)$$

$$= \begin{bmatrix} 1/(1+t) & 0 & 0 \\ 0 & f'(t)/f(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} x \quad (14)$$

To make the motion volume-preserving,

$$\begin{aligned} \nabla \cdot v(x, t) &= \frac{\partial(x_1/(1+t))}{\partial x_1} + \frac{\partial(f'(t)x_2/f(t))}{\partial x_2} \\ &= \frac{1}{1+t} + \frac{f'(t)}{f(t)} \\ &= 0 \end{aligned} \quad (15)$$

and it implies that

$$f(t) = \frac{c}{1+t}.$$

The above argument is reversible, hence motion of this form is volume-preserving if and only if $f(t)$ has the form given above.

4. Solution: By the definition of $x(X, t)$, it follows that

$$\begin{aligned} \nabla_x(X) &= I + tA, \\ d(t) &= \det(I + tA) \end{aligned} \quad (16)$$

where $d(0) = 1$, thus by continuity argument there is a neighborhood of zero such that $d(t) > 0$ for t in this neighborhood, so $\det(I + tA)$ is positive and $(I + tA)$ is invertible. Thus the formula defines a motion for small t by definition. Moreover, the material velocity

$$\begin{aligned}\frac{\partial x}{\partial t}(X, t) &= AX, \\ X(x, t) &= (I + tA)^{-1}x,\end{aligned}\tag{17}$$

so the spatial velocity

$$v(x, t) = A(I + tA)^{-1}x.\tag{18}$$

Extra credit By volume preserving,

$$\begin{aligned}\nabla \cdot v(x, t) &= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \\ &= (A(I + tA)^{-1})_{11} + (A(I + tA)^{-1})_{22} + (A(I + tA)^{-1})_{33} \\ &= \text{trace}(A(I + tA)^{-1}) \\ &= 0,\end{aligned}\tag{19}$$

that is, the motion is volume-preserving if the trace of matrix $A(I + tA)^{-1}$ is zero.