Problem Set 4, Chapter 1

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1. <u>Solution</u>: (a)By the definition of the cross product,it follows that

$$u \times v = Uv, \tag{1}$$

where

$$U = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}, \tag{2}$$

so,

$$\begin{aligned} u \times (av) &= U(av) \\ (au) \times v &= (aU)v = U(av) \\ a(u \times v) &= a(Uv) = U(av), \end{aligned} \tag{3}$$

(a) follows.

For (b), by the definition we have that

$$\begin{aligned} (u \times v)_1 &= u_2 v_3 - u_3 v_2 \\ &= -(v \times u)_1 \\ (u \times v)_2 &= u_3 v_1 - u_1 v_3 \\ &= -(v \times u)_2 \\ (u \times v)_3 &= u_1 v_2 - u_2 v_1 \\ &= -(v \times u)_3. \end{aligned} \tag{4}$$

For (c), by (b)

$$\begin{aligned} v \times v &= -v \times v \\ v \times v &= 0 \end{aligned} \tag{5}$$

For (d), it is a straightforward verification.

2. Solution: By the definition of the cross product,we have that

$$\begin{aligned}(u\times v)_1 &= u_2v_3-u_3v_2\\ (u\times v)_2 &= u_3v_1-u_1v_3\\ (u\times v)_3 &= u_1v_2-u_2v_1\end{aligned} \tag{6}$$

so

$$\nabla(u\times v)_{\cdot,j} = \begin{pmatrix} \dfrac{\partial(u_2v_3-u_3v_2)}{\partial x_j} \\ \dfrac{\partial(u_3v_1-u_1v_3)}{\partial x_j} \\ \dfrac{\partial(u_1v_2-u_2v_1)}{\partial x_j} \end{pmatrix} \tag{7}$$

By expanding the partial derivatives and combining the two derivatives w.r.t. u and v,the results follow.

3. Solution: A straightforward verification by problem 2.

4. Solution: By the definition of the material derivative,it follows that

$$\begin{aligned}\tfrac{D}{Dt}(x\times v) &= \frac{\partial(x\times v)}{\partial t}+(\nabla(x\times v))v\\ &= \frac{dx}{dt}\times v+x\times\frac{dv}{dt}+(-v\times I+x\times\nabla v)v\\ &= v\times v+x\times\frac{dv}{dt}+(-v\times v)+x\times\nabla vv\\ &= x\times(\frac{dv}{dt}+\nabla vv)\\ &= x\times\frac{Dv}{Dt}\end{aligned} \tag{8}$$

where we have used the results of Problems 1-3.

5. Solution: Straightforward verification by definition.

6. Solution: Straightforward verification by definition.