

Analysis of weak solutions for the fully coupled Stokes-Darcy-Transport problem

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Abstract

This paper analyzes the surface/subsurface flow coupled with transport. The flow is modeled by the coupling of Stokes and Darcy equations. The transport is modeled by a convection-dominated parabolic equation. The two-way coupling between flow and transport is nonlinear and it is done via the velocity field and the viscosity. This problem arises from a variety of natural phenomena such as the contamination of the groundwater through rivers. The main result is existence and stability bounds of a weak solution.

Keywords: miscible displacement, convergence, compactness, Galerkin approach, Beavers-Joseph-Saffman

1 Introduction

The coupled surface/subsurface flow and transport problems appear in a variety of physical phenomena that affect the human health and the environment at a large scale. For instance, pollution of groundwater by transport of contaminants through rivers and lakes is one important environmental problem. This paper deals with the mathematical analysis of the coupled multiphysics problem. In the subsurface the miscible displacement problem is considered whereas the surface is characterized by the Stokes and transport equations.

The coupling of Stokes and Darcy equations has recently been a popular research topic. Standard transmissibility conditions at the interface include the Beavers-Joseph-Saffman law (Beavers and Joseph [1], Saffman [2]), the continuity of normal component of velocity and the balance of forces across the interface. Using these interface conditions, a weak solution has been defined and analyzed in (Layton et al. [3], Discacciati et al [4]). Various numerical methods have been proposed, see for instance a non-exhaustive list [3, 4, 5, 6, 7, 8, 9, 10, 19]. The mathematical analysis of the miscible displacement problem in subsurface was done in a seminal paper by Alt et al. [11], and by others such as Marpeau and Saad [12], Fabrie and Gallouët [13].

The contribution of our work is to analyze the more general coupling of miscible displacement in porous media with surface flow and transport. To our knowledge, it is the first analysis of the fully coupled surface/subsurface problem. Our mathematical analysis also applies to the particular case where the flow problem is loosely coupled to the transport problem. In this loose one-way coupling, the velocity field obtained from the Stokes/Darcy problem becomes an input data for the transport equation. Numerical methods for this particular case have been analyzed by Vassilev and Yotov [14] and Cesmelioglu et al. [15].

The organization of the paper is as follows. Section 2 defines the mathematical model and introduces the necessary assumptions on the data. In Section 3, we formulate the weak problem. The rest of the paper is devoted to the main results of existence, regularity and boundedness of a weak solution and their proofs.

2 Model problem

This section defines the model problem for the coupling of a transport equation with a Stokes/Darcy flow. The coupling is done through the velocity field and the concentration. Let $\Omega \subset \mathbb{R}^2$ denote the region of

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concern subdivided into two subregions as $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 \subset \mathbb{R}^2$ corresponds to the surface region and $\Omega_2 \subset \mathbb{R}^2$ corresponds to the subsurface region. We assume that Ω is an open bounded Lipschitz domain with Lipschitz boundary. Let \mathbf{u} , p and φ denote the fluid velocity in Ω , the Stokes pressure in Ω_1 and the Darcy pressure in Ω_2 , respectively. Denote the boundaries of Ω , Ω_1 and Ω_2 by $\partial\Omega$, $\partial\Omega_1$ and $\partial\Omega_2$. Denote by \mathbf{n} the unit outward normal to $\partial\Omega$. Let $\boldsymbol{\tau}_{12}$ and \mathbf{n}_{12} be a unit tangential vector and a unit normal vector respectively at the interface $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$ and define $\Gamma_i = \partial\Omega_i \setminus \Gamma_{12}$. We assume that $|\Gamma_1| > 0$. Let $Q_T = \Omega \times (0, T)$ and $\Sigma_T = \partial\Omega \times (0, T)$. The coupled surface/subsurface flow is characterized by the Stokes equations in the surface Ω_1

$$-\nabla \cdot (2\mu(c)\mathbf{D}(\mathbf{u}) - p\mathbf{I}) = \Psi, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega_1 \times (0, T), \quad (2)$$

and the Darcy equations in the subsurface Ω_2

$$\mathbf{u} = -\frac{\mathbf{K}}{\mu(c)}(\nabla\varphi - \rho\mathbf{g}), \quad \nabla \cdot \mathbf{u} = \Pi, \quad \text{in } \Omega_2 \times (0, T). \quad (3)$$

The interface conditions are given by the continuity of the flux, the Beavers-Joseph-Saffman law [1, 2] and the balance of forces on $\Gamma_{12} \times (0, T)$.

$$\mathbf{u}|_{\Omega_1} \cdot \mathbf{n}_{12} = \mathbf{u}|_{\Omega_2} \cdot \mathbf{n}_{12}, \quad (4)$$

$$G\mathbf{K}^{-1/2}\mathbf{u}|_{\Omega_1} \cdot \boldsymbol{\tau}_{12} = -2\mu(c)\mathbf{D}(\mathbf{u}|_{\Omega_1})\mathbf{n}_{12} \cdot \boldsymbol{\tau}_{12}, \quad (5)$$

$$((-2\mu(c)\mathbf{D}(\mathbf{u}|_{\Omega_1}) + p)\mathbf{n}_{12}) \cdot \mathbf{n}_{12} = \varphi. \quad (6)$$

The Stokes/Darcy flow is fully coupled to the following transport equation which defines the concentration (fraction of volume) c of a contaminant transported in the domain Ω over the time interval $(0, T)$.

$$\frac{\partial}{\partial t}(\phi c) - \nabla \cdot (\mathbf{F}(\mathbf{u})\nabla c - c\mathbf{u}) = \Lambda, \quad \text{in } Q_T. \quad (7)$$

This system of equations is subject to the following boundary and initial conditions.

$$\mathbf{u} = 0 \text{ on } \Gamma_1 \times (0, T), \quad (8)$$

$$\mathbf{u} \cdot \mathbf{n} = \mathcal{U} \text{ on } \Gamma_2 \times (0, T), \quad (9)$$

$$\mathbf{F}(\mathbf{u})\nabla c \cdot \mathbf{n} = \begin{cases} (c - \mathcal{C})(\mathbf{u} \cdot \mathbf{n}), & \text{on } \{x \in \partial\Omega : \mathcal{U}(x) < 0\} \times (0, T) \\ 0 & \text{on } \{x \in \partial\Omega : \mathcal{U}(x) \geq 0\} \times (0, T) \end{cases}, \quad (10)$$

$$c = c_0, \text{ in } \Omega \times \{0\}. \quad (11)$$

For the uniqueness of the Darcy pressure, we assume

$$\int_{\Omega_2} \varphi = 0. \quad (12)$$

In the following, we define the coefficients of the equations above and set suitable assumptions, which are necessary for the conclusions of the paper, on these coefficients.

- $\mu = \mu(c)$ is the fluid viscosity such that $\mu \in \mathcal{C}^0(\mathbb{R}^+; \mathbb{R}^+)$ and there exists $\mu_L, \mu_U > 0$ satisfying

$$\mu_L < \mu(x) \leq \mu_U \text{ for any } x \in \mathbb{R}^+. \quad (13)$$

- $\mathbf{D}(\mathbf{u})$ is the symmetric rate of strain matrix defined as $\mathbf{D}(\mathbf{u}) = 0.5(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$. The matrix \mathbf{I} is the identity matrix.

Since $\mathbf{D}(\mathbf{u})$ is symmetric, for any $\mathbf{u}, \mathbf{v} \in H^1(\Omega_1)^2$,

$$(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\Omega_1} = (\mathbf{D}(\mathbf{u}), \nabla\mathbf{v})_{\Omega_1}. \quad (14)$$

Furthermore, for any $\mathbf{u} \in H^1(\Omega_1)^2$, there exists $M_1 > 0$ such that

$$\|\nabla\mathbf{u}\|_{L^2(\Omega_1)} \leq M_1 \|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega_1)}. \quad (15)$$

- Ψ , Π and Λ are the source functions such that $\Pi \geq 0$, $\Lambda \geq 0$ and

$$\Psi \in L^2(0, T; L^2(\Omega_1)^2), \Pi \in L^2(0, T; L^2(\Omega_2)), \Lambda \in L^1(0, T; L^\infty(\Omega)) \cap L^2(0, T; (H^1(\Omega))'). \quad (16)$$

- $\mathbf{K} \in L^\infty(\Omega_2)^{2 \times 2}$ is the symmetric positive definite permeability matrix bounded from above and below by $k_U > 0$ and $k_L > 0$:

$$\forall \xi \in \mathbb{R}^2, \quad k_L \xi \cdot \xi \leq \xi \cdot \mathbf{K} \xi \leq k_U \xi \cdot \xi. \quad (17)$$

- ρ is the fluid density, which is a positive constant.
- $\mathbf{g} \in L^\infty(\Omega)^2$ is the vector of gravitational acceleration.
- G that appears in the interface condition (5) is a positive constant determined experimentally [1, 2].
- ϕ is the porosity, which is the ratio of the void volume to the total volume. There exists $\phi_L > 0$ such that

$$\phi(x) = 1 \text{ in } \Omega_1, \quad \phi_L \leq \phi(x) \leq 1 \text{ in } \Omega_2. \quad (18)$$

- \mathbf{F} is the diffusion/dispersion tensor, which is a continuous and bounded function from \mathbb{R}^2 to $\mathbb{R}^{2 \times 2}$, i.e., there exists $F_C > 0$ and $F_B > 0$ such that

$$\mathbf{F}(\mathbf{w}) \text{ is measurable for all } \mathbf{w} \in \mathbb{R}^2, \quad \|\mathbf{F}(\mathbf{w})\| \leq F_C \|\mathbf{w}\|, \quad \|\mathbf{F}(\mathbf{w})\| \leq F_B. \quad (19)$$

In addition, we assume that $\mathbf{F}(\mathbf{w})$ is uniformly positive definite for all $\mathbf{w} \in \mathbb{R}^2$, i.e., there is $\alpha > 0$ such that

$$\mathbf{F}(\mathbf{w}) \xi \cdot \xi \geq \alpha \xi \cdot \xi, \quad \forall \xi \in \mathbb{R}^2. \quad (20)$$

- \mathcal{U} is the boundary flux which belongs to $L^2(0, T; L^2(\Gamma_2))$. Because of the Neumann boundary condition on the subsurface region, the data Π and \mathcal{U} are assumed to satisfy the compatibility condition

$$\int_{\Gamma_2} \mathcal{U} = \int_{\Omega_2} \Pi. \quad (21)$$

We assume that there is a subset of Γ_2 of positive measure, corresponding to an outflow boundary, on which \mathcal{U} is positive. We extend \mathcal{U} outside of Γ_2 by zero and from (8) and (9) we can write:

$$\mathbf{u} \cdot \mathbf{n} = \mathcal{U}, \quad \text{on } \partial\Omega. \quad (22)$$

- \mathcal{C} is the prescribed concentration function on the inflow boundary.

$$\mathcal{C} \in L^\infty(\Sigma_T), \quad \mathcal{C} \geq 0 \text{ a.e. in } \Sigma_T. \quad (23)$$

For any function z , we define the negative part z^- and the positive part z^+ as

$$z^- = \frac{|z| - z}{2}, \quad z^+ = \frac{|z| + z}{2}.$$

Note that $z^+ = \max(0, z)$ and $z^- = \max(0, -z)$. Using these definitions, we rewrite (10) as

$$\mathbf{F}(\mathbf{u}) \nabla c \cdot \mathbf{n} = (\mathcal{C} - c) \mathcal{U}^- \text{ on } (0, T) \times \partial\Omega. \quad (24)$$

- c_0 is the initial concentration such that

$$c_0 \geq 0, \quad c_0 \in L^\infty(\Omega). \quad (25)$$

We conclude this section with additional notation, two trace inequalities, Poincaré's inequality and an important compactness result for Sobolev spaces that we use frequently. For any domain \mathcal{D} , the standard notation for the $L^k(\mathcal{D})$ spaces and Sobolev spaces $H^k(\mathcal{D})$ is used. The L^2 inner-product of two functions is denoted by $(\cdot, \cdot)_{\mathcal{D}}$. The dual space of $H^1(\mathcal{D})$ is denoted by $(H^1(\mathcal{D}))'$ and the duality pairing is denoted by $\langle \cdot, \cdot \rangle_{(H^1(\mathcal{D}))', H^1(\mathcal{D})}$. As usual the Bochner spaces are denoted by $L^k(0, T; H^r(\mathcal{D}))$, for $k \geq 1$. There are constants $M_2, M_4 > 0$ such that for any function $z \in H^1(\mathcal{D})$, we have

$$\|z\|_{L^2(\partial\mathcal{D})} \leq M_2 \|z\|_{H^1(\mathcal{D})}, \quad (26)$$

$$\|z\|_{L^4(\partial\mathcal{D})} \leq M_4 \|z\|_{H^1(\mathcal{D})}. \quad (27)$$

In addition, if $z \in H^1(\mathcal{D})$ such that $z = 0$ on a subset of \mathcal{D} or $\int_{\mathcal{D}} z dx = 0$, then there exists $M_P > 0$ satisfying

$$\|z\|_{L^2(\mathcal{D})} \leq M_P \|\nabla z\|_{L^2(\mathcal{D})}. \quad (28)$$

The following theorem is a special case of the Rellich-Kondrachov theorem [16, Theorem 6.2].

Theorem 2.1. *Let $\mathcal{D} \subset \mathbb{R}^2$ be an open, bounded Lipschitz domain.*

- *If $0 \leq l < 1$, then $H^l(\Omega)$ is compactly embedded in $H^1(\mathcal{D})$.*
- *$H^1(\Omega)$ is compactly embedded in $L^1(\Omega)$.*

The next section introduces the weak solution spaces and the definition of the weak problem.

3 Weak Formulation

Let us define the spaces for the Stokes velocity, the Darcy velocity, the Stokes pressure and the Darcy pressure.

$$\mathbf{X}_1 = \{\mathbf{v} \in H^1(\Omega_1)^2 : \mathbf{v} = 0 \text{ on } \Gamma_1\}, \quad \mathbf{X}_2 = \{\mathbf{v} \in L^2(\Omega_2)^2\},$$

$$R_1 = \{q \in L^2(\Omega_1)\}, \quad R_2 = \{q \in H^1(\Omega_2) : \int_{\Omega_2} q = 0\}.$$

Definition 3.1. *The weak formulation of the coupled flow-transport problem is to find $\mathbf{u}|_{\Omega_1} \in L^2(0, T; \mathbf{X}_1)$, $p \in L^2(0, T; R_1)$, $\varphi \in L^2(0, T; R_2)$ and $c \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q_T)$ such that*

$$t \rightarrow c(\cdot, t) \in C^0([0, T]; (H^1(\Omega))') \quad (29)$$

$$t \rightarrow \frac{\partial c}{\partial t}(\cdot, t) \in L^2(0, T; (H^1(\Omega))') \quad (30)$$

$$c(\cdot, 0) = c_0(\cdot) \text{ a.e. in } \Omega \quad (31)$$

and satisfying for all $\mathbf{v} \in L^2(0, T; \mathbf{X}_1)$, for all $q \in L^2(0, T; R_2)$ and for all $r \in L^2(0, T; R_1)$,

$$\begin{aligned} & \int_0^T \left(2(\mu(c)\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\Omega_1} + \left(\frac{\mathbf{K}}{\mu(c)}(\nabla\varphi - \rho\mathbf{g}), \nabla q\right)_{\Omega_2} - (\nabla \cdot \mathbf{v}, p)_{\Omega_1} + (\nabla \cdot \mathbf{u}, r)_{\Omega_1} + (\varphi, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} \right. \\ & \left. + G(\mathbf{K}^{-1/2}\mathbf{u} \cdot \boldsymbol{\tau}_{12}, \mathbf{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{u} \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} \right) dt = \int_0^T ((\Psi, \mathbf{v})_{\Omega_1} + (\Pi, q)_{\Omega_2} - (\mathcal{U}, q)_{\Gamma_2}) dt, \end{aligned} \quad (32)$$

and for all $\psi \in L^2(0, T; H^1(\Omega))$,

$$\int_0^T \left\langle \phi \frac{\partial c}{\partial t}, \psi \right\rangle_{(H^1(\Omega))', H^1(\Omega)} dt + \int_{Q_T} (\mathbf{F}(\mathbf{u})\nabla c - c\mathbf{u}) \cdot \nabla \psi dx dt + \int_{\Sigma_T} (c\mathcal{U}^+ - c\mathcal{U}^-)\psi d\sigma dt = \int_0^T \langle \Lambda, \psi \rangle_{(H^1(\Omega))', H^1(\Omega)} dt. \quad (33)$$

Remark 3.2. The velocity $\mathbf{u}|_{\Omega_2} \in L^2(0, T; \mathbf{X}_2)$ in the Darcy region Ω_2 is obtained from the Darcy pressure φ by the equation

$$\mathbf{u} = -\frac{\mathbf{K}}{\mu(c)}(\nabla\varphi - \rho\mathbf{g}) \text{ a.e. in } \Omega_2 \times (0, T). \quad (34)$$

Derivation of the weak formulation :

Let $\psi \in L^2(0, T; H^1(\Omega))$. Multiply (7) by ψ , integrate over Q_T and use Green's formula:

$$\int_{Q_T} \frac{\partial}{\partial t}(\phi c)\psi dxdt + \int_{Q_T} (\mathbf{F}(\mathbf{u})\nabla c - c\mathbf{u}) \cdot \nabla\psi dxdt - \int_{\Sigma_T} (\mathbf{F}(\mathbf{u})\nabla c - c\mathbf{u}) \cdot \mathbf{n}\psi d\sigma dt = \int_0^T \langle \Lambda, \psi \rangle_{(H^1(\Omega))', H^1(\Omega)} dt.$$

Assuming $\phi \frac{\partial c}{\partial t} \in L^2(0, T; H^1(\Omega)')$, and observing from (24) that

$$(\mathbf{F}(\mathbf{u})\nabla c - c\mathbf{u}) \cdot \mathbf{n} = \mathbf{F}(\mathbf{u})\nabla c \cdot \mathbf{n} - c(\mathbf{u} \cdot \mathbf{n})^+ + c(\mathbf{u} \cdot \mathbf{n})^- = \mathcal{C}\mathcal{U}^- - c\mathcal{U}^+,$$

we obtain:

$$\int_0^T \langle \frac{\phi \partial c}{\partial t}, \psi \rangle_{H^1(\Omega)', H^1(\Omega)} dt + \int_{Q_T} (\mathbf{F}(\mathbf{u})\nabla c - c\mathbf{u}) \cdot \nabla\psi dxdt + \int_{\Sigma_T} (c\mathcal{U}^+ - c\mathcal{U}^-)\psi d\sigma dt = \int_0^T \langle \Lambda, \psi \rangle_{(H^1(\Omega))', H^1(\Omega)} dt.$$

Hence (33). The weak formulation for the flow part is gathered similarly as in [17].

4 Existence of a weak solution

The following theorem gives the main result of this paper which is the existence of a weak solution.

Theorem 4.1. *There exists a weak solution $(\mathbf{u}, p, \varphi, c)$ to the problem defined in Definition 3.1. In addition, there exists a constant $M_f > 0$ depending on the data Ψ and Π such that (\mathbf{u}, φ) satisfies*

$$2\mu_L \|\mathbf{D}(\mathbf{u})\|_{L^2(0, T; L^2(\Omega_1))}^2 + \|\mathbf{K}^{\frac{1}{2}} \nabla \varphi\|_{L^2(0, T; L^2(\Omega_2))}^2 \leq M_f T, \quad (35)$$

and c satisfies

$$0 \leq c(x, t) \leq \left\| \frac{\Lambda}{\phi} \right\|_{L^1(0, T; L^\infty(\Omega))} + \max(\|c_0\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)}) \text{ a.e. } (x, t) \in Q_T. \quad (36)$$

We will show this existence result by first restricting the problem to the space of divergence-free functions \mathbf{V}_1 .

$$\mathbf{V}_1 = \{\mathbf{v} \in \mathbf{X}_1 : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega_1\}.$$

Using this space we define another variational formulation of (32) where the Stokes pressure term p is eliminated. Find $\mathbf{u}|_{\Omega_1} \in L^2(0, T; \mathbf{V}_1)$, and $\varphi \in L^2(0, T; R_2)$ such that for all $\mathbf{v} \in L^2(0, T; \mathbf{V}_1)$ and for all $q \in L^2(0, T; R_2)$,

$$\begin{aligned} \int_0^T \left(2(\mu(c)\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\Omega_1} + \left(\frac{\mathbf{K}}{\mu(c)}(\nabla\varphi - \rho\mathbf{g}), \nabla q\right)_{\Omega_2} + (\varphi, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} + G(\mathbf{K}^{-1/2}\mathbf{u} \cdot \boldsymbol{\tau}_{12}, \mathbf{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \right. \\ \left. - (\mathbf{u} \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} \right) dt = \int_0^T ((\Psi, \mathbf{v})_{\Omega_1} + (\Pi, q)_{\Omega_2} - (\mathcal{U}, q)_{\Gamma_2}) dt. \end{aligned} \quad (37)$$

We will first prove the following existence theorem for this new problem.

Theorem 4.2. *There exist $\mathbf{u}|_{\Omega_1} \in L^2(0, T; \mathbf{V}_1)$, $\varphi \in L^2(0, T; R_2)$ and $c \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q_T)$ satisfying (29), (30), (31), (37), (33) and (34).*

The proof follows a similar technique as in [11, 12] and is based on a Galerkin approach and consists of several steps. We first set some notation for the discrete approximation we use. Then we prove an intermediate result and derive necessary estimates for the approximations. After proving the Theorem 4.2, we finally deduce the main result Theorem 4.1 by recovering the Stokes pressure p that we lose by restricting to the space \mathbf{V}_1 .

4.1 Approximate solution

For a fixed positive integer N , let $\Delta t = \frac{T}{N}$. Let $t_i = i\Delta t$, $i = 0, \dots, N$. Next for any Banach space B and for any $z \in L^1(0, T; B)$, define averages at each time step by

$$\bar{z}_0^N = 0, \quad \bar{z}_i^N = \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} z(t) dt, \quad i = 1, \dots, N. \quad (38)$$

We apply this averaging technique to the source terms Λ , Ψ , Π , the boundary flux \mathcal{U} and the inflow concentration \mathcal{C} to obtain

$$\bar{\Lambda}^N = (\bar{\Lambda}_0^N, \dots, \bar{\Lambda}_N^N), \quad \bar{\Psi}^N = (\bar{\Psi}_0^N, \dots, \bar{\Psi}_N^N), \quad \bar{\Pi}^N = (\bar{\Pi}_0^N, \dots, \bar{\Pi}_N^N), \quad \bar{\mathcal{U}}^N = (\bar{\mathcal{U}}_0^N, \dots, \bar{\mathcal{U}}_N^N), \quad \bar{\mathcal{C}}^N = (\bar{\mathcal{C}}_0^N, \dots, \bar{\mathcal{C}}_N^N).$$

Observe that for any $z \in L^\infty(0, T; B)$,

$$\|\bar{z}_i^N\|_B = \left\| \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} z(x, t) dt \right\|_B \leq \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} \|z(x, t)\|_B dt \leq \|z\|_{L^\infty(0, T; B)}.$$

Hence

$$\|\bar{z}_i^N\|_B \leq \|z\|_{L^\infty(0, T; B)}, \quad i = 0, \dots, N. \quad (39)$$

Also for any $z \in L^p(0, T; B)$, $1 \leq p < \infty$, Hölder's inequality and Fubini's theorem imply,

$$\|\bar{z}_i^N\|_B \leq \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} \|z(x, t)\|_B dt \leq \frac{1}{(\Delta t)^{\frac{1}{p}}} \left(\int_{(i-1)\Delta t}^{i\Delta t} \|z(x, t)\|_B^p dt \right)^{\frac{1}{p}}.$$

Therefore, $1 \leq p < \infty$,

$$\|\bar{z}_i^N\|_B \leq \frac{1}{(\Delta t)^{\frac{1}{p}}} \|z\|_{L^p(0, T; B)}, \quad i = 0, \dots, N. \quad (40)$$

Now we will introduce an intermediate problem to (37) and (33).

Proposition 4.3. *For $n = 0, \dots, N-1$, given $C_n^N \in L^2(\Omega)$, there exists a unique $(\mathbf{U}_{n+1}^N, \Phi_{n+1}^N) \in \mathbf{V}_1 \times R_2$ satisfying*

$$(P) \left\{ \begin{array}{l} \forall \mathbf{v} \in \mathbf{V}_1, \forall q \in R_2, \quad 2(\mu(C_n^N) \mathbf{D}(\mathbf{U}_{n+1}^N), \mathbf{D}(\mathbf{v}))_{\Omega_1} + (\frac{\mathbf{K}}{\mu(C_n^N)} (\nabla \Phi_{n+1}^N - \rho \mathbf{g}), \nabla q)_{\Omega_2} + (\Phi_{n+1}^N, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ \quad + G(\mathbf{K}^{-1/2} \mathbf{U}_{n+1}^N \cdot \boldsymbol{\tau}_{12}, \mathbf{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{U}_{n+1}^N \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} \\ \quad = (\bar{\Psi}_{n+1}^N, \mathbf{v})_{\Omega_1} + (\bar{\Pi}_{n+1}^N, q)_{\Omega_2} - (\bar{\mathcal{U}}_{n+1}^N, q)_{\Gamma_2}. \end{array} \right.$$

On Ω_2 , define $\mathbf{U}_{n+1}^N \in \mathbf{X}_2$ as

$$\mathbf{U}_{n+1}^N = -\frac{\mathbf{K}}{\mu(C_n^N)} (\nabla \Phi_{n+1}^N - \rho \mathbf{g}) \text{ in } \Omega_2. \quad (41)$$

Then

$$\nabla \cdot \mathbf{U}_{n+1}^N = \bar{\Pi}_{n+1}^N \text{ in } \Omega_2, \quad (42)$$

and

$$\mathbf{U}_{n+1}^N \cdot \mathbf{n} = \bar{\mathcal{U}}_{n+1}^N \text{ in } \Gamma_2. \quad (43)$$

Furthermore, there is a constant $M > 0$ independent of \mathbf{U}_{n+1}^N and Φ_{n+1}^N such that

$$2\mu_L \|\mathbf{D}(\mathbf{U}_{n+1}^N)\|_{L^2(\Omega_1)}^2 + \|\mathbf{K}^{\frac{1}{2}} \nabla \Phi_{n+1}^N\|_{L^2(\Omega_2)}^2 \leq M. \quad (44)$$

Proof. The proof of the existence and the uniqueness of $(\mathbf{U}_{n+1}^N, \Phi_{n+1}^N)$ and the bound (44) can be found in [3, 18] or in [17] by removing the nonlinearity from the Navier-Stokes equations. To obtain (42), let $\mathbf{v} = 0$ and $q \in \mathcal{C}_0^\infty(\Omega_2)$ in (P). Then using (41), we have

$$-(\mathbf{U}_{n+1}^N, \nabla q)_{\Omega_2} = (\bar{\Pi}_{n+1}^N, q)_{\Omega_2}.$$

So in the distributional sense, we obtain (42), that is,

$$\nabla \cdot \mathbf{U}_{n+1}^N = \bar{\Pi}_{n+1}^N \text{ in } \Omega_2.$$

To show (43), let $\mathbf{v} \in \mathcal{C}_0^\infty(\Omega_1)^2$ and $q = 0$ in (P). Then

$$2(\mu(C_n^N) \mathbf{D}(\mathbf{U}_{n+1}^N), \mathbf{D}(\mathbf{v}))_{\Omega_1} = (\bar{\Psi}_{n+1}^N, \mathbf{v})_{\Omega_1},$$

and thus together with (14) the definition of weak derivatives yields

$$-2\nabla \cdot (\mu(C_n^N) \mathbf{D}(\mathbf{U}_{n+1}^N)) = \bar{\Psi}_{n+1}^N \text{ in } \Omega_1, \quad (45)$$

in the distributional sense. Multiplying this by $\mathbf{v} \in \mathbf{X}_1$, integrating over Ω_1 and using Green's formula

$$(2(\mu(C_n^N) \mathbf{D}(\mathbf{U}_{n+1}^N), \mathbf{D}(\mathbf{v}))_{\Omega_1} - (2\mu(C_n^N) \mathbf{D}(\mathbf{U}_{n+1}^N) \mathbf{n}, \mathbf{v})_{\partial\Omega_1}) = (\bar{\Psi}_{n+1}^N, \mathbf{v})_{\Omega_1}. \quad (46)$$

Next multiply (42) by $q \in R_2$ and use Green's formula to get

$$-(\mathbf{U}_{n+1}^N, \nabla q)_{\Omega_2} + (\mathbf{U}_{n+1}^N \cdot \mathbf{n}, q)_{\partial\Omega_2} = (\bar{\Pi}_{n+1}^N, q)_{\Omega_2}.$$

Adding this to (46) and comparing the sum with (P) and using (41) yields,

$$\begin{aligned} & (2\mu(C_n^N) \mathbf{D}(\mathbf{U}_{n+1}^N) \cdot \mathbf{n}, \mathbf{v})_{\partial\Omega_1} + (\Phi_{n+1}^N, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} + G(\mathbf{K}^{-1/2} \mathbf{U}_{n+1}^N \cdot \boldsymbol{\tau}_{12}, \mathbf{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \\ & - (\mathbf{U}_{n+1}^N \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} - (\mathbf{U}_{n+1}^N \cdot \mathbf{n}, q)_{\partial\Omega_2} = -(\bar{\mathcal{U}}_{n+1}^N, q)_{\Gamma_2}. \end{aligned}$$

Letting $\mathbf{v} = 0$ in this equation and choosing q such that $q = 0$ on Γ_{12} , we have

$$(\mathbf{U}_{n+1}^N \cdot \mathbf{n}, q)_{\Gamma_2} = (\bar{\mathcal{U}}_{n+1}^N, q)_{\Gamma_2}.$$

Therefore, (43) holds. □

Proposition 4.4. For $n = 0, 1, \dots, N$, given C_n^N there exists $C_{n+1}^N \in H^1(\Omega)$ satisfying

$$0 \leq C_{n+1}^N(x) \leq \Delta t \left\| \frac{\bar{\Lambda}_{n+1}^N}{\phi} \right\|_{L^\infty(\Omega)} + \max \left(\|C_n^N\|_{L^\infty(\Omega)}, \|\bar{\mathcal{C}}_{n+1}^N\|_{L^\infty(\partial\Omega)} \right) \text{ a.e. } x \in \Omega, \quad (47)$$

and for all $\psi \in H^1(\Omega)$,

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi(C_{n+1}^N - C_n^N) \psi dx & + \int_{\Omega} (\mathbf{F}(\mathbf{U}_{n+1}^N) \nabla C_{n+1}^N - C_{n+1}^N \mathbf{U}_{n+1}^N) \cdot \nabla \psi dx \\ & + \int_{\partial\Omega} (C_{n+1}^N (\bar{\mathcal{U}}_{n+1}^N)^+ - \bar{\mathcal{C}}_{n+1}^N (\bar{\mathcal{U}}_{n+1}^N)^-) \psi d\sigma = \int_{\Omega} \bar{\Lambda}_{n+1}^N \psi dx. \end{aligned} \quad (48)$$

where \mathbf{U}_{n+1}^N is defined in Proposition 4.3.

Proof. For convenience we drop the superscript N for the proof. Let

$$\mathcal{M} = \Delta t \left\| \frac{\bar{\Lambda}_{n+1}}{\phi} \right\|_{L^\infty(\Omega)} + \max(\|C_n\|_{L^\infty(\Omega)}, \|\bar{C}_{n+1}\|_{L^\infty(\partial\Omega)}).$$

We define a bounded function H on \mathbb{R} by

$$H(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } 0 \leq x \leq \mathcal{M}, \\ \mathcal{M}, & \text{if } x \geq \mathcal{M}. \end{cases}$$

We will show the existence of $C_{n+1} \in H^1(\Omega)$ such that for all $\psi \in H^1(\Omega)$,

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi(C_{n+1} - C_n)\psi dx + \int_{\Omega} \mathbf{F}(\mathbf{U}_{n+1})\nabla C_{n+1} \cdot \nabla \psi dx - \int_{\Omega} H(C_{n+1})\mathbf{U}_{n+1} \cdot \nabla \psi dx \\ + \int_{\partial\Omega} (C_{n+1}(\bar{\mathcal{U}}_{n+1})^+ - \bar{C}_{n+1}(\bar{\mathcal{U}}_{n+1})^-)\psi d\sigma = \int_{\Omega} \bar{\Lambda}_{n+1}\psi dx. \end{aligned} \quad (49)$$

Observe that the solution to (49) solves (47) and (48) if $0 \leq C_{n+1} \leq \mathcal{M}$ a.e. in Ω . We will use Schauder's fixed point theorem [20, p.154] to show that such a solution exists. Define an operator $\theta : L^2(\Omega) \rightarrow L^2(\Omega)$ by $\theta(w) = v$ where v is the unique function of $H^1(\Omega)$ such that for any $\psi \in H^1(\Omega)$,

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi v \psi dx + \int_{\Omega} \mathbf{F}(\mathbf{U}_{n+1})\nabla v \cdot \nabla \psi dx + \int_{\partial\Omega} v(\bar{\mathcal{U}}_{n+1})^+ \psi d\sigma \\ = \frac{1}{\Delta t} \int_{\Omega} \phi C_n \psi dx + \int_{\Omega} H(w)\mathbf{U}_{n+1} \cdot \nabla \psi dx + \int_{\partial\Omega} \bar{C}_{n+1}(\bar{\mathcal{U}}_{n+1})^- \psi d\sigma + \int_{\Omega} \bar{\Lambda}_{n+1}\psi dx \end{aligned} \quad (50)$$

Clearly, any fixed point of (50) is a solution to (49). Well-definition of θ comes from Lax-Milgram theorem. Indeed, let us define a bilinear form \mathcal{B} by

$$\mathcal{B}(v, \psi) = \frac{1}{\Delta t} \int_{\Omega} \phi v \psi dx + \int_{\Omega} \mathbf{F}(\mathbf{U}_{n+1})\nabla v \cdot \nabla \psi dx + \int_{\partial\Omega} v(\bar{\mathcal{U}}_{n+1})^+ \psi d\sigma,$$

and a linear form \mathcal{L} by

$$\mathcal{L}(\psi) = \frac{1}{\Delta t} \int_{\Omega} \phi C_n \psi dx + \int_{\Omega} H(w)\mathbf{U}_{n+1} \cdot \nabla \psi dx + \int_{\partial\Omega} \bar{C}_{n+1}(\bar{\mathcal{U}}_{n+1})^- \psi d\sigma + \int_{\Omega} \bar{\Lambda}_{n+1}\psi dx.$$

Then from the Cauchy-Schwarz's inequality, (18), (19), (27) and (40)

$$\begin{aligned} |\mathcal{B}(v, \psi)| &\leq \frac{1}{\Delta t} \|v\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} + \|\mathbf{F}(\mathbf{U}_{n+1})\nabla v\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} + \|v\|_{L^4(\partial\Omega)} \|(\bar{\mathcal{U}}_{n+1})^+\|_{L^2(\partial\Omega)} \|\psi\|_{L^4(\partial\Omega)} \\ &\leq \frac{1}{\Delta t} \|v\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)} + F_B \|v\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)} + M_4^2 \|v\|_{H^1(\Omega)} \|\bar{\mathcal{U}}_{n+1}\|_{L^2(\partial\Omega)} \|\psi\|_{H^1(\Omega)} \\ &= \left(\frac{1}{\Delta t} + F_B + \frac{M_4^2}{\Delta t^{\frac{1}{2}}} \|\mathcal{U}\|_{L^2(0,T;L^2(\partial\Omega))} \right) \|v\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}. \end{aligned}$$

Thus, \mathcal{B} is continuous. Coercivity of \mathcal{B} also follows from (18) and (20).

$$\begin{aligned} \mathcal{B}(v, v) &= \frac{1}{\Delta t} \int_{\Omega} \phi v^2 dx + \int_{\Omega} \mathbf{F}(\mathbf{U}_{n+1})\nabla v \cdot \nabla v dx + \int_{\partial\Omega} \bar{\mathcal{U}}^+ v^2 d\sigma \\ &\geq \frac{\phi_L}{\Delta t} \|v\|_{L^2(\Omega)}^2 + \alpha \|\nabla v\|_{L^2(\Omega)}^2 \geq \min\left(\frac{\phi_L}{\Delta t}, \alpha\right) \|v\|_{H^1(\Omega)}^2. \end{aligned}$$

Finally using the bound on the function H , the Cauchy-Schwarz's inequality, (18), (26), (39) and (40), we show that \mathcal{L} is continuous

$$\begin{aligned} |\mathcal{L}(\psi)| &\leq \frac{1}{\Delta t} \|C_n\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} + \mathcal{M} \|\mathbf{U}_{n+1}\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} + \|\bar{C}_{n+1}\|_{L^\infty(\partial\Omega)} \|\bar{\mathbf{U}}_{n+1}^-\|_{L^2(\partial\Omega)} \|\psi\|_{L^2(\partial\Omega)} \\ &+ \|\bar{\Lambda}_{n+1}\|_{H^1(\Omega)'} \|\psi\|_{H^1(\Omega)} \leq \left(\frac{1}{\Delta t} \|C_n\|_{L^2(\Omega)} + \mathcal{M} \|\mathbf{U}_{n+1}\|_{L^2(\Omega)} \right. \\ &\quad \left. + \frac{1}{\Delta t^{\frac{1}{2}}} (M_2 \|\mathcal{C}\|_{L^\infty(\Sigma_T)} \|\mathcal{U}\|_{L^2(0,T;L^2(\partial\Omega))} + \|\Lambda\|_{L^2(0,T;H^1(\Omega)')}) \right) \|\psi\|_{H^1(\Omega)}. \end{aligned}$$

Hence from Lax-Milgram's theorem there exists a unique $v \in H^1(\Omega)$ such that $\mathcal{B}(v, \psi) = \mathcal{L}(\psi)$ for any $\psi \in H^1(\Omega)$.

Schauder's theorem requires us to show that θ is continuous and $\theta(L^2(\Omega))$ is relatively compact in $L^2(\Omega)$. The relative compactness property will follow from Rellich-Kondrachov theorem [16, see remark 6.3] once we show that $\theta(L^2(\Omega))$ is bounded in $H^1(\Omega)$.

In (50), take $\psi = v$,

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi v^2 dx &+ \int_{\Omega} \mathbf{F}(\mathbf{U}_{n+1}) \nabla v \cdot \nabla v dx + \int_{\partial\Omega} (\bar{\mathbf{U}}_{n+1})^+ v^2 d\sigma \\ &= \frac{1}{\Delta t} \int_{\Omega} C_n v dx + \int_{\Omega} H(w) \mathbf{U}_{n+1} \cdot \nabla v dx + \int_{\partial\Omega} \bar{C}_{n+1} (\bar{\mathbf{U}}_{n+1})^- v d\sigma + \int_{\Omega} \bar{\Lambda}_{n+1} v dx. \end{aligned}$$

Therefore, by positiveness of the third term, boundedness of H , (18), (39), (20), (40), (26) and Young's inequality, we obtain for any $\delta_i, i = 1, 2, 3$, that

$$\begin{aligned} \frac{\phi_L}{\Delta t} \|v\|_{L^2(\Omega)}^2 + \alpha \|\nabla v\|_{L^2(\Omega)}^2 &\leq \frac{1}{\Delta t} \|C_n\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \mathcal{M} \|\mathbf{U}_{n+1}\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\quad + \|\bar{C}_{n+1}\|_{L^\infty(\partial\Omega)} \|\bar{\mathbf{U}}_{n+1}^-\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} + \|\bar{\Lambda}_{n+1}\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq \frac{\phi_L}{4\Delta t} \|v\|_{L^2(\Omega)}^2 + \frac{\alpha}{4} \|\nabla v\|_{L^2(\Omega)}^2 + \frac{1}{M_2^2} \min\left(\frac{\alpha}{4}, \frac{\phi_L}{4\Delta t}\right) \|v\|_{H^1(\Omega)}^2 + \mathcal{A}. \end{aligned}$$

where

$$\mathcal{A} = \frac{1}{\phi_L \Delta t} \|C_n\|_{L^2(\Omega)}^2 + \frac{1}{\alpha} \mathcal{M}^2 \|\mathbf{U}_{n+1}\|_{L^2(\Omega)}^2 + \frac{M_2^2}{\min\left(\frac{\alpha}{4}, \frac{\phi_L}{4\Delta t}\right)} \left(\|\mathcal{C}\|_{L^\infty(\Sigma_T)}^2 \|\mathcal{U}\|_{L^2(0,T;L^2(\partial\Omega))}^2 + \|\Lambda\|_{L^2(0,T;H^1(\Omega)')}^2 \right).$$

Hence, we have

$$\frac{\phi_L}{2\Delta t} \|v\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\nabla v\|_{L^2(\Omega)}^2 \leq M, \quad (51)$$

for a constant M that does not depend on w . Therefore $\|v\|_{H^1(\Omega)} \leq \left(\frac{M}{\min(\frac{\phi_L}{2\Delta t}, \frac{\alpha}{2})} \right)^{\frac{1}{2}}$ meaning $\theta(L^2(\Omega))$ is bounded in $H^1(\Omega)$.

Now let us show continuity of θ . Let $\{w_k\}_k$ be a sequence in $L^2(\Omega)$ such that $w_k \rightarrow w$ in $L^2(\Omega)$. Let $v_k = \theta(w_k)$. We will show that $v_k \rightarrow \theta(w)$ in $L^2(\Omega)$ by using the estimate (51). First from [21, p.68] convergence of $\{w_k\}_k$ to w in $L^2(\Omega)$ implies that there exists a subsequence $w_{k_j}, w_{k_j} \rightarrow w$ a.e. in Ω as $j \rightarrow \infty$. As $H(w)$ is bounded and continuous in w , $H(w_{k_j}) \rightarrow H(w)$ a.e. in Ω as $j \rightarrow \infty$. Then by the Lebesgue dominated convergence theorem,

$$H(w_{k_j}) \rightarrow H(w) \text{ strongly in } L^2(\Omega). \quad (52)$$

By (51), $\{v_{k_j}\}_j$ is bounded in $H^1(\Omega)$ so there exists a subsequence still denoted by $\{v_{k_j}\}_j$ such that

$$v_{k_j} \rightarrow v \text{ weakly in } H^1(\Omega) \quad (53)$$

for some $v \in H^1(\Omega)$. As $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, again, up to a subsequence,

$$v_{k_j} \rightarrow v \text{ strongly in } L^2(\Omega). \quad (54)$$

Since the trace function is continuous from $L^2(\Omega)$ to $L^2(\partial\Omega)$,

$$v_{k_j} \rightarrow v \text{ strongly in } L^2(\partial\Omega). \quad (55)$$

We consider (50) with v_{k_j} and w_{k_j} in place of v and w . With the above convergence results (52), (53), (54), (55), we pass to the limit in (50) to get $v = \theta(w)$. Hence $v_{k_j} \rightarrow v = \theta(w)$ strongly in $L^2(\Omega)$. Similarly, every subsequence of $\{v_k\}_k$ converging in $L^2(\Omega)$ has limit $\theta(w)$. Therefore $\{v_k\}_k$ has a unique accumulation point. As $\theta(L^2(\Omega))$ is relatively compact in $L^2(\Omega)$, $\theta(w_k) = v_k \rightarrow \theta(w)$ in $L^2(\Omega)$. Hence θ is continuous and we can conclude that there exists a fixed point $C_{n+1} \in H^1(\Omega)$ satisfying (49).

Next, we will show that $0 \leq C_{n+1} \leq \mathcal{M}$ a.e. in Ω which proves (47) and also implies that $H(C_{n+1}) = C_{n+1}$. This will give (48).

Let us first show $C_{n+1} \geq 0$ a.e. in Ω . From Stampacchia [22, p.50], $C_{n+1}^- \in H^1(\Omega)$. In (49), let $\psi = -C_{n+1}^-$

$$\begin{aligned} -\frac{1}{\Delta t} \int_{\Omega} \phi(C_{n+1} - C_n) C_{n+1}^- dx &+ \int_{\Omega} H(C_{n+1}) \mathbf{u} \cdot \nabla C_{n+1}^- dx - \int_{\Omega} \mathbf{F}(U_{n+1}) \nabla C_{n+1} \cdot \nabla C_{n+1}^- dx \\ &- \int_{\partial\Omega} (C_{n+1} \bar{U}_{n+1}^+ - \bar{C}_{n+1} \bar{U}_{n+1}^-) C_{n+1}^- d\sigma + \int_{\Omega} \bar{\Lambda}_{n+1} C_{n+1}^- dx = 0. \end{aligned}$$

Observe that for any function z ,

$$zz^- = \begin{cases} -(z^-)^2, & \text{if } z < 0, \\ 0, & \text{otherwise} \end{cases} = -(z^-)^2.$$

Similarly, $\mathbf{F}(U_{n+1}) \nabla C_{n+1} \cdot \nabla C_{n+1}^- = -\mathbf{F}(U_{n+1}) \nabla C_{n+1}^- \cdot \nabla C_{n+1}^-$. The second term in the equation vanishes since for $C_{n+1} \leq 0$, $H(C_{n+1}) = 0$ and for $C_{n+1} \geq 0$, $C_{n+1}^- = 0$. Therefore,

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi(C_{n+1}^-)^2 dx + \frac{1}{\Delta t} \int_{\Omega} \phi C_n C_{n+1}^- dx + \int_{\Omega} F(U_{n+1}) \nabla C_{n+1}^- \cdot \nabla C_{n+1}^- dx + \int_{\partial\Omega} (C_{n+1}^-)^2 \bar{U}_{n+1}^+ d\sigma \\ + \int_{\partial\Omega} \bar{C}_{n+1} \bar{U}_{n+1}^- C_{n+1}^- d\sigma + \int_{\Omega} \bar{\Lambda}_{n+1} C_{n+1}^- dx = 0. \end{aligned}$$

Observe that $C_0 \geq 0$ and $\bar{U}_{n+1}^-, \bar{U}_{n+1}^+, \bar{C}_{n+1}, \bar{\Lambda}_{n+1} \geq 0$, for all $n \geq 0$. This together with (20) shows that

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi(C_1^-)^2 dx + \frac{1}{\Delta t} \int_{\Omega} \phi C_0 C_1^- dx + \alpha \int_{\Omega} |\nabla C_1^-| dx + \int_{\partial\Omega} (C_1^-)^2 \bar{U}_1^+ d\sigma \\ + \int_{\partial\Omega} \bar{C}_1 \bar{U}_1^- C_1^- d\sigma + \int_{\Omega} \bar{\Lambda}_1 C_1^- dx = 0, \end{aligned}$$

in which all the terms except the first one are nonnegative. Hence $\frac{1}{\Delta t} \int_{\Omega} \phi(C_1^-)^2 dx \leq 0$. This implies $C_1^- = 0$ a.e. in Ω as $\phi > 0$. In other words, $C_1 \geq 0$ a.e. in Ω . Then an induction argument shows that $C_n \geq 0$ a.e. in Ω for all $n \geq 0$.

Now we will show $C_{n+1} \leq \mathcal{M}$ a.e. in Ω by proving that $(C_{n+1} - \mathcal{M})^+ = 0$ a.e. in Ω . As before, $(C_{n+1} - \mathcal{M})^+ \in H^1(\Omega)$. So let $\psi = (C_{n+1} - \mathcal{M})^+$ in (49).

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi(C_{n+1} - C_n) (C_{n+1} - \mathcal{M})^+ dx - \int_{\Omega} H(C_{n+1}) U_{n+1} \cdot \nabla (C_{n+1} - \mathcal{M})^+ dx \\ + \int_{\Omega} \mathbf{F}(U_{n+1}) \nabla C_{n+1} \cdot \nabla (C_{n+1} - \mathcal{M})^+ dx + \int_{\partial\Omega} (C_{n+1} \bar{U}_{n+1}^+ - \bar{C}_{n+1} \bar{U}_{n+1}^-) (C_{n+1} - \mathcal{M})^+ d\sigma \\ - \int_{\Omega} \bar{\Lambda}_{n+1} (C_{n+1} - \mathcal{M})^+ dx = 0. \quad (56) \end{aligned}$$

Note that

$$\begin{aligned}\mathbf{F}(\mathbf{U}_{n+1})\nabla C_{n+1} \cdot \nabla(C_{n+1} - \mathcal{M})^+ &= \mathbf{F}(\mathbf{U}_{n+1})\nabla(C_{n+1} - \mathcal{M}) \cdot \nabla(C_{n+1} - \mathcal{M})^+ \\ &= \mathbf{F}(\mathbf{U}_{n+1})\nabla(C_{n+1} - \mathcal{M})^+ \cdot \nabla(C_{n+1} - \mathcal{M})^+.\end{aligned}$$

So the third term in (56) is positive by (20). Now let

$$I = - \int_{\Omega} H(C_{n+1})\mathbf{U}_{n+1} \cdot \nabla(C_{n+1} - \mathcal{M})^+ dx + \int_{\partial\Omega} (C_{n+1}\bar{u}_{n+1}^+ - \bar{c}_{n+1}\bar{u}_{n+1}^-)(C_{n+1} - \mathcal{M})^+ d\sigma.$$

From the definition of H , we have

$$H(C_{n+1})\mathbf{U}_{n+1} \cdot \nabla(C_{n+1} - \mathcal{M})^+ = \mathcal{M}\mathbf{U}_{n+1} \cdot \nabla(C_{n+1} - \mathcal{M})^+ \text{ a.e. in } \Omega.$$

This, the Stokes' formula, (42) and (43) gives

$$\begin{aligned}I &= \int_{\Omega} \mathcal{M}\nabla \cdot \mathbf{U}_{n+1}(C_{n+1} - \mathcal{M})^+ dx - \int_{\partial\Omega} \mathcal{M}\bar{u}_{n+1}(C_{n+1} - \mathcal{M})^+ d\sigma \\ &\quad + \int_{\partial\Omega} (C_{n+1}\bar{u}_{n+1}^+ - \bar{c}_{n+1}\bar{u}_{n+1}^-)(C_{n+1} - \mathcal{M})^+ d\sigma \\ &= \int_{\Omega_2} \mathcal{M}\bar{\Pi}_{n+1}(C_{n+1} - \mathcal{M})^+ dx - \int_{\partial\Omega} \mathcal{M}\bar{u}_{n+1}^+(C_{n+1} - \mathcal{M})^+ d\sigma + \int_{\partial\Omega} \mathcal{M}\bar{u}_{n+1}^-(C_{n+1} - \mathcal{M})^+ d\sigma \\ &\quad + \int_{\partial\Omega} C_{n+1}\bar{u}_{n+1}^+(C_{n+1} - \mathcal{M})^+ d\sigma - \int_{\partial\Omega} \bar{c}_{n+1}\bar{u}_{n+1}^-(C_{n+1} - \mathcal{M})^+ d\sigma \\ &= \int_{\Omega_2} \mathcal{M}\bar{\Pi}_{n+1}(C_{n+1} - \mathcal{M})^+ dx + \int_{\partial\Omega} (C_{n+1} - \mathcal{M})\bar{u}_{n+1}^+(C_{n+1} - \mathcal{M})^+ d\sigma + \int_{\partial\Omega} (\mathcal{M} - \bar{c}_{n+1})\bar{u}_{n+1}^-(C_{n+1} - \mathcal{M})^+ d\sigma.\end{aligned}$$

Note that $\mathcal{M}, \bar{\Pi}_{n+1}, (C_{n+1} - \mathcal{M})^+$ and \bar{u}_{n+1}^+ are nonnegative and $\bar{c}_{n+1} \leq \mathcal{M}$. These together with the fact that $(C_{n+1} - \mathcal{M})(C_{n+1} - \mathcal{M})^+ = ((C_{n+1} - \mathcal{M})^+)^2$ yields $I \geq 0$. Then from (56) we conclude that

$$\int_{\Omega} (\phi(C_{n+1} - C_n) - \Delta t \bar{\Lambda}_{n+1})(C_{n+1} - \mathcal{M})^+ dx \leq 0.$$

As $C_n + \Delta t \frac{\bar{\Lambda}_{n+1}}{\phi} \leq \mathcal{M}$ a.e. in Ω , $C_{n+1} - C_n - \Delta t \frac{\bar{\Lambda}_{n+1}}{\phi} \geq C_{n+1} - \mathcal{M}$ a.e. in Ω . Hence $\int_{\Omega} \phi((C_{n+1} - \mathcal{M})^+)^2 dx \leq 0$ yielding

$$(C_{n+1} - \mathcal{M})^+ = 0 \text{ a.e. in } \Omega.$$

□

Let $C_0^N = c_0$, $\Phi_0^N = 0$, $\mathbf{U}_0^N = \mathbf{0}$ and define from Prop. 4.3 and Prop. 4.4:

$$C^N = (C_0^N, \dots, C_N^N), \Phi^N = (\Phi_0^N, \dots, \Phi_N^N), \mathbf{U}^N = (\mathbf{U}_0^N, \dots, \mathbf{U}_N^N).$$

Now we will define constant and linear interpolation operators for the approximations of $\bar{\Lambda}^N, \bar{\Pi}^N, \bar{\Psi}^N, \bar{\mathcal{C}}^N$ and $C^N, \mathbf{U}^N, \Phi^N$.

Definition 4.5. Let B be a Banach space. For $\xi = (\xi_0, \dots, \xi_N) \in B^{N+1}$, define $I_0\xi, I_1\xi : [0, T] \rightarrow B$ by

$$I_0\xi(t) = \begin{cases} \xi_0, & t = 0 \\ \xi_{n+1}, & \text{if } n\Delta t < t \leq (n+1)\Delta t, \quad n = 0, \dots, N-1 \end{cases}$$

and

$$I_1\xi(t) = \left(1 + n - \frac{t}{\Delta t}\right) \xi_n + \left(\frac{t}{\Delta t} - n\right) \xi_{n+1}, \text{ if } n\Delta t \leq t \leq (n+1)\Delta t, \quad n = 0, \dots, N-1.$$

Also define \widetilde{I}_0 an extension of the constant interpolation operator such that

$$\widetilde{I}_0 \boldsymbol{\xi}(t) = \begin{cases} \xi_0 & t \in [-\Delta t, 0] \\ \xi_{n+1} & t \in (n\Delta t, (n+1)\Delta t], n = 0, \dots, N-1. \end{cases}$$

Observe that $I_1 \boldsymbol{\xi}$ is continuous and,

$$\frac{\partial}{\partial t} I_1 \boldsymbol{\xi}(t) = \frac{1}{\Delta t} (\xi_{n+1} - \xi_n), \text{ if } n\Delta t < t < (n+1)\Delta t, \quad n = 0, \dots, N-1. \quad (57)$$

Also for $1 \leq p < \infty$,

$$\|I_0 \boldsymbol{\xi}\|_{L^p(0,T;B)} = \left(\int_0^T \|I_0 \boldsymbol{\xi}(t)\|_B^p dt \right)^{\frac{1}{p}} = \left(\sum_{n=1}^N \int_{(n-1)\Delta t}^{n\Delta t} \|\xi_n\|_B^p dt \right)^{\frac{1}{p}} = (\Delta t \sum_{n=1}^N \|\xi_n\|_B^p)^{\frac{1}{p}}, \quad (58)$$

$$\|I_0 \boldsymbol{\xi}\|_{L^\infty(0,T;B)} = \text{ess sup}_{t \in (0,T)} \|I_0 \boldsymbol{\xi}(t)\|_B = \max_{n=1, \dots, N} \|\xi_n\|_B = \|I_1 \boldsymbol{\xi}\|_{L^\infty(0,T;B)}. \quad (59)$$

The following result can be found in [12]. For completeness, we sketch the proof.

Lemma 4.6. For $z \in L^p(0, T; B)$, let $\bar{z}^N = (\bar{z}_0^N, \dots, \bar{z}_N^N)$ where \bar{z}_i^N is defined as in (38). Then for $1 \leq p \leq \infty$,

$$\|I_0 \bar{z}^N\|_{L^p(0,T;B)} \leq \|z\|_{L^p(0,T;B)}. \quad (60)$$

Furthermore, For $1 \leq p < \infty$,

$$I_0 \bar{z}^N \rightarrow z \text{ strongly in } L^p(0, T; B) \text{ as } N \rightarrow \infty. \quad (61)$$

Proof. For $p < \infty$, Hölder's inequality and (58) implies

$$\|I_0 \bar{z}^N\|_{L^p(0,T;B)} = \left(\Delta t \sum_{n=1}^N \left\| \frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} z(t) dt \right\|_B^p \right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^N \int_{(n-1)\Delta t}^{n\Delta t} \|z(t)\|_B^p dt \right)^{\frac{1}{p}} = \left(\int_0^T \|z(t)\|_B^p dt \right)^{\frac{1}{p}}.$$

Also by (59) and (39),

$$\|I_0 \bar{z}^N\|_{L^\infty(0,T;B)} = \max_{n=1, \dots, N} \|\bar{z}_n^N\|_B \leq \frac{1}{\Delta t} \max_{n=1, \dots, N} \int_{(n-1)\Delta t}^{n\Delta t} \|z(t)\|_B dt = \|z\|_{L^\infty(0,T;B)}.$$

Therefore the result holds for $1 \leq p \leq \infty$. We will first show the last result for $z \in \mathcal{C}([0, T]; B)$. Then we will conclude by a density argument as $\mathcal{C}(0, T; B)$ is dense in $L^p(0, T; B)$.

Let $\epsilon_0 > 0$. Let χ_n denote the characteristic function on the interval $(n\Delta t, (n+1)\Delta t)$. Then

$$\begin{aligned} \|I_0 \bar{z}^N - z\|_{L^p(0,T;B)}^p &= \int_0^T \|I_0 \bar{z}^N(t) - z(t)\|_B^p dt = \int_0^T \chi_n(t) \|z_{n+1} - z(t)\|_B^p dt \\ &= \int_0^T \chi_n(t) \left\| \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} z(s) ds - z(t) \right\|_B^p dt = \int_0^T \chi_n(t) \left\| \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} (z(s) - z(t)) ds \right\|_B^p dt \\ &\leq \int_0^T \chi_n(t) \left(\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} \|z(s) - z(t)\|_B ds \right)^p dt \end{aligned}$$

As any continuous function on a compact set is uniformly continuous, z is uniformly continuous on $[0, 1]$. So for any $\epsilon_1 > 0$, we can find $\delta > 0$ such that for any $t, s \in [0, 1]$, $|t - s| < \delta$ implies $\|z(t) - z(s)\|_B < \epsilon_1$. Let N_0 be such that $\frac{T}{N_0} < \delta$. Then for all $N \geq N_0$, $\Delta t < \delta$. Thus for $0 < \epsilon_1 < T^{-\frac{1}{p}} \epsilon_0$,

$$\|I_0 \bar{z}^N - z\|_{L^p(0,T;B)}^p \leq \int_0^T \chi_n(t) \epsilon_1^p dt = \epsilon_1^p \Delta t \leq T \epsilon_1^p < \epsilon_0^p \quad (62)$$

yielding $I_0 \bar{z}^N \rightarrow z$ in $L^p(0, T; B)$ for any $z \in \mathcal{C}([0, T]; B)$. The result then follows by the density. \square

Integrating (48) and (P) from $n\Delta t$ to $(n+1)\Delta t$, summing from $n=0$ to $n=N-1$ and using (41), we have the following definition of the approximate solution to the Stokes-Darcy-transport problem.

Definition 4.7. (Definition of the approximate solution) For all $\mathbf{v} \in L^2(0, T; \mathbf{V}_1)$ and for all $q \in L^2(0, T; R_2)$,

$$\begin{aligned} & \int_0^T \left(2(\mu(\widetilde{I_0 C_{\Delta t}^N}) \mathbf{D}(I_0 \mathbf{U}^N), \mathbf{D}(\mathbf{v}))_{\Omega_1} + \left(\frac{\mathbf{K}}{\mu(\widetilde{I_0 C_{\Delta t}^N})} (\nabla I_0 \Phi^N - \rho \mathbf{g}), \nabla q \right)_{\Omega_2} + (I_0 \Phi^N, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} \right. \\ & \left. + G(\mathbf{K}^{-1/2} I_0 \mathbf{U}^N \cdot \boldsymbol{\tau}_{12}, \mathbf{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (I_0 \mathbf{U}^N \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} \right) dt = \int_0^T \left((I_0 \bar{\Psi}^N, \mathbf{v})_{\Omega_1} + (I_0 \bar{\Pi}^N, q)_{\Omega_2} - (I_0 \bar{\mathcal{U}}^N, q)_{\Gamma_2} \right) dt \end{aligned} \quad (63)$$

where

$$I_0 \mathbf{U}^N = - \frac{\mathbf{K}}{\mu(\widetilde{I_0 C_{\Delta t}^N})} (\nabla I_0 \Phi^N - \rho \mathbf{g}) \text{ in } \Omega_2 \times (0, T). \quad (64)$$

and the concentration equation is defined as

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial}{\partial t} I_1 C^N, \psi \right\rangle_{(H^1(\Omega)'), H^1(\Omega)} dt - \int_{Q_T} I_0 C^N I_0 \mathbf{U}^N \cdot \nabla \psi dx dt + \int_{Q_T} \mathbf{F}(I_0 \mathbf{U}^N) \nabla I_0 C^N \cdot \nabla \psi dx dt \\ & + \int_{\Sigma_T} (I_0 C^N (I_0 \bar{\mathcal{U}}^N)^+ - I_0 \bar{C}^N (I_0 \bar{\mathcal{U}}^N)^-) \psi d\sigma dt - \int_0^T \langle I_0 \bar{\Lambda}^N, \psi \rangle_{(H^1(\Omega)'), H^1(\Omega)} dt = 0, \end{aligned} \quad (65)$$

for all $\psi \in L^2(0, T; H^1(\Omega))$. The function $\widetilde{I_0 C_{\Delta t}^N}$ denotes the translated function: $\widetilde{I_0 C_{\Delta t}^N}(x, t) = \widetilde{I_0 C^N}(x, t - \Delta t)$. Furthermore, multiplying by Δt and summing from $n=0$ to $N-1$ both sides of the bound (44), we obtain

$$2\mu_L \|\mathbf{D}(I_0 \mathbf{U}^N)\|_{L^2(0, T; L^2(\Omega_1))}^2 + \|\mathbf{K}^{\frac{1}{2}} \nabla(I_0 \Phi^N)\|_{L^2(0, T; L^2(\Omega_2))}^2 \leq M_f T. \quad (66)$$

We will pass to the limit in this definition. First we need some bounds for the approximate solution, which are derived in the next section.

5 Stability bounds

The first proposition of this section gives a uniform L^∞ -bound for $I_0 C^N$ which will be used when passing to the limit. A slightly more general version of this result can be found in [12].

Proposition 5.1. For $n=0, \dots, N$

$$0 \leq C_n^N(x) \leq \mathcal{N}, \text{ a.e. } x \in \Omega, \quad (67)$$

where \mathcal{N} is the right-hand side of (36), i.e.,

$$\mathcal{N} = \left\| \frac{\bar{\Lambda}}{\phi} \right\|_{L^1(0, T; L^\infty(\Omega))} + \max(\|c_0\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)}).$$

Proof. For readability again, we drop the superscript N . Using (47) and (39) recursively, for all $n=1, \dots, N$, we obtain

$$\begin{aligned} 0 \leq C_n(x) & \leq \Delta t \left\| \frac{\bar{\Lambda}_n}{\phi} \right\|_{L^\infty(\Omega)} + \max(\|C_{n-1}\|_{L^\infty(\Omega)}, \|\bar{\mathcal{C}}_n\|_{L^\infty(\partial\Omega)}) \\ & \leq \Delta t \left\| \frac{\bar{\Lambda}_n}{\phi} \right\|_{L^\infty(\Omega)} + \max \left(\left(\Delta t \left\| \frac{\bar{\Lambda}_{n-1}}{\phi} \right\|_{L^\infty(\Omega)} + \max(\|C_{n-2}\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)}) \right), \|\mathcal{C}\|_{L^\infty(\Sigma_T)} \right) \\ & \leq \Delta t \left\| \frac{\bar{\Lambda}_{n-1}}{\phi} \right\|_{L^\infty(\Omega)} + \Delta t \left\| \frac{\bar{\Lambda}_n}{\phi} \right\|_{L^\infty(\Omega)} + \max(\|C_{n-2}\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)}) \leq \dots \\ & \leq \Delta t \sum_{i=1}^n \left\| \frac{\bar{\Lambda}_i}{\phi} \right\|_{L^\infty(\Omega)} + \max(\|C_0\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)}) \end{aligned}$$

for a.e. $x \in \Omega$. Observe from the proof of (39) that we have

$$\Delta t \sum_{i=1}^n \left\| \frac{\bar{\Lambda}_i}{\phi} \right\|_{L^\infty(\Omega)} \leq \sum_{i=1}^n \int_{(i-1)\Delta t}^{i\Delta t} \left\| \frac{\Lambda(t)}{\phi} \right\|_{L^\infty(\Omega)} dt \leq \int_0^T \left\| \frac{\Lambda(t)}{\phi} \right\|_{L^\infty(\Omega)} dt = \left\| \frac{\Lambda}{\phi} \right\|_{L^1(0,T;L^\infty(\Omega))}.$$

Then the result follows from this and the assumption that $C_0 = c_0$. \square

Remark 5.2. *It is trivial to deduce a uniform bound for $I_0 C^N$ and $I_1 C^N$. Indeed, fix $t \in (0, T)$ and let $\bar{n} = \lceil \frac{t}{\Delta t} \rceil$ where for a real number s , $\lceil s \rceil = \min\{n \in \mathcal{N} : s \leq n\}$. Then $I_0 C^N(x, t) = C_{\bar{n}}^N(x)$, and $I_1 C^N(x, t) = (\bar{n} - 1 - \frac{t}{\Delta t}) C_{\bar{n}}^N(x) + (1 + \frac{t}{\Delta t} - \bar{n}) C_{\bar{n}}^N(x)$. Thus, we have*

$$0 \leq I_0 C^N(x, t) \leq \mathcal{N}, \quad 0 \leq I_1 C^N(x, t) \leq \mathcal{N}, \quad \text{a.e. } x \in \Omega, \forall t \in (0, T). \quad (68)$$

The next proposition gives uniform bounds for the terms related to the Stokes-Darcy flow.

Proposition 5.3. *There exists a constant \mathcal{M} independent of N such that*

$$\|I_0 \mathbf{U}^N\|_{L^2(0,T;L^2(\Omega))} \leq \mathcal{M}. \quad (69)$$

Furthermore,

$$\|I_0 \bar{\Lambda}^N\|_{L^1(Q_T)} \leq \|\Lambda\|_{L^1(Q_T)}, \quad (70)$$

$$\|I_0 \bar{\Lambda}^N\|_{L^2(0,T;(H^1(\Omega))')} \leq \|\Lambda\|_{L^2(0,T;H^1(\Omega))'}, \quad (71)$$

$$\|I_0 \bar{\mathcal{U}}^N\|_{L^1(\Sigma_T)} \leq \|\mathcal{U}\|_{L^1(\Sigma_T)}, \quad (72)$$

$$\|I_0 \bar{\mathcal{U}}^N\|_{L^2(\Sigma_T)} \leq \|\mathcal{U}\|_{L^2(\Sigma_T)}. \quad (73)$$

Proof. The estimates (70), (71), (72) and (73) are easy consequences of (60). Let us prove (69). Take $\mathbf{v} = \mathbf{U}_{n+1}^N$ and $q = \Phi_{n+1}^N$ in (P). Then

$$\begin{aligned} 2(\mu(C_n^N) \mathbf{D}(\mathbf{U}_{n+1}^N), \mathbf{D}(\mathbf{U}_{n+1}^N))_{\Omega_1} &+ \left(\frac{\mathbf{K}}{\mu(C_n^N)} (\nabla \Phi_{n+1}^N - \rho \mathbf{g}), \nabla \Phi_{n+1}^N \right)_{\Omega_2} + G(\mathbf{K}^{-1/2} \mathbf{U}_{n+1}^N \cdot \boldsymbol{\tau}_{12}, \mathbf{U}_{n+1}^N \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \\ &= (\bar{\Psi}_{n+1}^N, \mathbf{U}_{n+1}^N)_{\Omega_1} + (\bar{\Pi}_{n+1}^N, \Phi_{n+1}^N)_{\Omega_2} - (\bar{\mathcal{U}}_{n+1}^N, \Phi_{n+1}^N)_{\Gamma_2}. \end{aligned}$$

Positivity of the third term and the properties (13) and (17) imply

$$\begin{aligned} 2\mu_L \|\mathbf{D}(\mathbf{U}_{n+1}^N)\|_{L^2(\Omega_1)}^2 + \frac{k_L}{\mu_U} \|\nabla \Phi_{n+1}^N\|_{L^2(\Omega_2)}^2 &\leq \left(\frac{\mathbf{K}}{\mu(C_n^N)} (\rho \mathbf{g}), \nabla \Phi_{n+1}^N \right)_{\Omega_2} + (\bar{\Psi}_{n+1}^N, \mathbf{U}_{n+1}^N)_{\Omega_1} \\ &\quad + (\bar{\Pi}_{n+1}^N, \Phi_{n+1}^N)_{\Omega_2} - (\bar{\mathcal{U}}_{n+1}^N, \Phi_{n+1}^N)_{\Gamma_2} \\ &\leq \frac{\rho}{\mu_L} \|\mathbf{K} \mathbf{g}\|_{L^2(\Omega_2)} \|\nabla \Phi_{n+1}^N\|_{L^2(\Omega_2)} + \|\bar{\Psi}_{n+1}^N\|_{L^2(\Omega_1)} \|\mathbf{U}_{n+1}^N\|_{L^2(\Omega_1)} \\ &\quad + \|\bar{\Pi}_{n+1}^N\|_{L^2(\Omega_2)} \|\Phi_{n+1}^N\|_{L^2(\Omega_2)} + \|\bar{\mathcal{U}}_{n+1}^N\|_{L^2(\Gamma_2)} \|\Phi_{n+1}^N\|_{L^2(\Gamma_2)}. \end{aligned}$$

In the following, M stands for a generic constant. From Cauchy-Schwarz's inequality, Poincaré's inequality (28), the trace inequality (26) and (15),

$$\begin{aligned} \frac{2\mu_L}{M_1^2} \|\nabla \mathbf{U}_{n+1}^N\|_{L^2(\Omega_1)}^2 + \frac{k_L}{\mu_U} \|\nabla \Phi_{n+1}^N\|_{L^2(\Omega_2)}^2 &\leq \left(\frac{\rho}{\mu_L} \|\mathbf{K} \mathbf{g}\|_{L^2(\Omega_2)} + M_P \|\bar{\Pi}_{n+1}^N\|_{L^2(\Omega_2)} + M_2 \|\bar{\mathcal{U}}_{n+1}^N\|_{L^2(\Gamma_2)} \right) \|\nabla \Phi_{n+1}^N\|_{L^2(\Omega_2)} \\ &\quad + M_P \|\bar{\Psi}_{n+1}^N\|_{L^2(\Omega_1)} \|\nabla \mathbf{U}_{n+1}^N\|_{L^2(\Omega_1)}. \end{aligned}$$

By the assumption $\mathbf{K} \in L^\infty(\Omega_2)^{2 \times 2}$ and using Young's inequality, there exists a constant $M > 0$ satisfying

$$\begin{aligned} \frac{2\mu_L}{M_1^2} \|\nabla \mathbf{U}_{n+1}^N\|_{L^2(\Omega_1)}^2 + \frac{k_L}{\mu_U} \|\nabla \Phi_{n+1}^N\|_{L^2(\Omega_2)}^2 &\leq M(1 + \|\bar{\Pi}_{n+1}^N\|_{L^2(\Omega_2)}^2 + \|\bar{\mathcal{U}}_{n+1}^N\|_{L^2(\Gamma_2)}^2 + \|\bar{\Psi}_{n+1}^N\|_{L^2(\Omega_1)}^2) \\ &\quad + \frac{\mu_L}{M_1^2} \|\nabla \mathbf{U}_{n+1}^N\|_{L^2(\Omega_1)}^2 + \frac{k_L}{2\mu_U} \|\nabla \Phi_{n+1}^N\|_{L^2(\Omega_2)}^2. \end{aligned}$$

Therefore,

$$\frac{\mu_L}{M_1^2} \|\nabla \mathbf{U}_{n+1}^N\|_{L^2(\Omega_1)}^2 + \frac{k_L}{2\mu_U} \|\nabla \Phi_{n+1}^N\|_{L^2(\Omega_2)}^2 \leq M(1 + \|\bar{\Pi}_{n+1}^N\|_{L^2(\Omega_2)}^2 + \|\bar{\mathcal{U}}_{n+1}^N\|_{L^2(\Gamma_2)}^2 + \|\bar{\Psi}_{n+1}^N\|_{L^2(\Omega_1)}^2).$$

Multiply this by Δt and sum from $n = 0$ to $n = N - 1$ to obtain

$$\begin{aligned} \frac{\mu_L}{M_1^2} \|\nabla I_0 \mathbf{U}^N\|_{L^2(0,T;L^2(\Omega_1))}^2 + \frac{k_L}{2\mu_U} \|\nabla I_0 \Phi^N\|_{L^2(0,T;L^2(\Omega_2))}^2 \\ \leq C(T + \|I_0 \bar{\Pi}^N\|_{L^2(0,T;L^2(\Omega_2))}^2 + \|I_0 \bar{\mathcal{U}}^N\|_{L^2(0,T;L^2(\Gamma_2))}^2 + \|I_0 \bar{\Psi}^N\|_{L^2(0,T;L^2(\Omega_1))}^2) \\ \leq C(T + \|\Pi\|_{L^2(0,T;L^2(\Omega_2))}^2 + \|\mathcal{U}\|_{L^2(0,T;L^2(\Gamma_2))}^2 + \|\Psi\|_{L^2(0,T;L^2(\Omega_1))}^2), \end{aligned}$$

where we have used (60) in the last step.

Therefore, we have a uniform $L^2(0, T; L^2(\Omega_2)^2)$ -estimate for $\nabla I_0 \Phi^N$ with respect to N . This will give a bound for $I_0 \mathbf{U}^N$ on Ω_2 as a result of (64) and (13). Similarly we have a uniform $L^2(0, T; L^2(\Omega_2)^{2 \times 2})$ bound for $\nabla I_0 \mathbf{U}^N$ in Ω_1 . This implies a uniform $L^2(0, T; L^2(\Omega_2)^2)$ bound for $I_0 \mathbf{U}^N$ in Ω_1 from Poincaré's inequality (28). Therefore (69) holds. \square

The following result gives bounds for the concentration terms.

Proposition 5.4. *There exists a constant \mathfrak{M} independent of N such that*

$$\|I_0 C^N\|_{L^2(0,T;H^1(\Omega))} \leq \mathfrak{M}, \quad (74)$$

$$\forall t' > 0, \quad \|I_0 C_{-t'}^N - I_0 C^N\|_{L^2((0,T-t');L^2(\Omega))}^2 \leq \mathfrak{M} t', \quad (75)$$

$$\left\| \frac{\partial}{\partial t} I_1 C^N \right\|_{L^2(0,T;(H^1(\Omega))')} \leq \mathfrak{M}, \quad (76)$$

$$\|I_1 C^N - I_0 C^N\|_{L^2(0,T;(H^1(\Omega))')}^2 \leq \mathfrak{M} \Delta t, \quad (77)$$

$$\|\bar{\mathcal{C}}_n^N\|_{L^\infty(\partial\Omega)} \leq \|\mathcal{C}\|_{L^\infty(\Sigma_T)}, \quad (78)$$

$$\|I_0 \bar{\mathcal{C}}^N\|_{L^\infty(\partial\Omega)} \leq \|\mathcal{C}\|_{L^\infty(\Sigma_T)}, \quad (79)$$

where $C_{t'}(t, x) = C(t - t', x)$ is the translation of C to $(0, T - t')$.

Proof. (78) follows from (39) and the last estimate (79) is a direct consequence of (60). We will prove the first four bounds. In (48), omitting the superscript N and letting $\psi = C_{n+1}$, we have

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi(C_{n+1} - C_n) C_{n+1} dx + \int_{\Omega} \mathbf{F}(\mathbf{U}_{n+1}) \nabla C_{n+1} \cdot \nabla C_{n+1} dx - \int_{\Omega} C_{n+1} \mathbf{U}_{n+1} \cdot \nabla C_{n+1} dx \\ + \int_{\partial\Omega} (C_{n+1} (\bar{\mathcal{U}}_{n+1})^+ - \bar{\mathcal{C}}_{n+1} (\bar{\mathcal{U}}_{n+1})^-) C_{n+1} d\sigma = \int_{\Omega} \bar{\Lambda}_{n+1} C_{n+1} dx. \end{aligned}$$

By Stokes' theorem and (43) we have

$$\begin{aligned} \int_{\Omega} C_{n+1} \mathbf{U}_{n+1} \cdot \nabla C_{n+1} dx &= - \int_{\Omega} \nabla (C_{n+1} \mathbf{U}_{n+1}) C_{n+1} dx + \int_{\partial\Omega} \bar{\mathcal{U}}_{n+1} C_{n+1}^2 d\sigma \\ &= - \int_{\Omega} (C_{n+1} \nabla \cdot \mathbf{U}_{n+1} + \nabla C_{n+1} \cdot \mathbf{U}_{n+1}) C_{n+1} dx + \int_{\partial\Omega} \bar{\mathcal{U}}_{n+1} C_{n+1}^2 d\sigma. \end{aligned}$$

Since we have (42), this implies

$$2 \int_{\Omega} C_{n+1} \mathbf{U}_{n+1} \cdot \nabla C_{n+1} dx = - \int_{\Omega} \nabla \cdot \mathbf{U}_{n+1} C_{n+1}^2 dx + \int_{\partial\Omega} \bar{\mathbf{U}}_{n+1} C_{n+1}^2 d\sigma = - \int_{\Omega_2} \bar{\Pi}_{n+1} C_{n+1}^2 dx + \int_{\partial\Omega} \bar{\mathbf{U}}_{n+1} C_{n+1}^2 d\sigma.$$

Then

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi(C_{n+1} - C_n) C_{n+1} dx + \int_{\Omega} \mathbf{F}(\mathbf{U}_{n+1}) \nabla C_{n+1} \cdot \nabla C_{n+1} dx + \frac{1}{2} \left(\int_{\Omega_2} \bar{\Pi}_{n+1} C_{n+1}^2 dx - \int_{\partial\Omega} \bar{\mathbf{U}}_{n+1} C_{n+1}^2 d\sigma \right) \\ + \int_{\partial\Omega} (C_{n+1} (\bar{\mathbf{U}}_{n+1})^+ - \bar{\mathcal{C}}_{n+1} (\bar{\mathbf{U}}_{n+1})^-) C_{n+1} d\sigma = \int_{\Omega} \bar{\Lambda}_{n+1} C_{n+1} dx. \end{aligned}$$

Note that $(\bar{\mathbf{U}}_{n+1})^+ - \frac{1}{2} \bar{\mathbf{U}}_{n+1} = \frac{1}{2} |\bar{\mathbf{U}}_{n+1}|$. So,

$$\begin{aligned} \frac{1}{\Delta t} \int_{\Omega} \phi(C_{n+1} - C_n) C_{n+1} dx + \int_{\Omega} \mathbf{F}(\mathbf{U}_{n+1}) \nabla C_{n+1} \cdot \nabla C_{n+1} dx + \frac{1}{2} \int_{\Omega_2} \bar{\Pi}_{n+1} C_{n+1}^2 dx + \frac{1}{2} \int_{\partial\Omega} |\bar{\mathbf{U}}_{n+1}| C_{n+1}^2 d\sigma \\ = \int_{\partial\Omega} \bar{\mathcal{C}}_{n+1} (\bar{\mathbf{U}}_{n+1})^- C_{n+1} d\sigma + \int_{\Omega} \bar{\Lambda}_{n+1} C_{n+1} dx. \end{aligned}$$

Using the assumption $\Pi \geq 0$ and (20),

$$\frac{1}{\Delta t} \int_{\Omega} \phi(C_{n+1} - C_n) C_{n+1} dx + \alpha \int_{\Omega} |\nabla C_{n+1}|^2 dx \leq \int_{\partial\Omega} \bar{\mathcal{C}}_{n+1} (\bar{\mathbf{U}}_{n+1})^- C_{n+1} d\sigma + \int_{\Omega} \bar{\Lambda}_{n+1} C_{n+1} dx.$$

Finally noting that $\frac{1}{2}(C_{n+1}^2 - C_n^2) \leq (C_{n+1} - C_n)C_{n+1}$, we further obtain

$$\frac{1}{2\Delta t} \int_{\Omega} \phi(C_{n+1}^2 - C_n^2) dx + \alpha \|\nabla C_{n+1}\|_{L^2(\Omega)}^2 \leq \int_{\Omega} \bar{\Lambda}_{n+1} C_{n+1} dx + \int_{\partial\Omega} (\bar{\mathbf{U}}_{n+1})^- \bar{\mathcal{C}}_{n+1} C_{n+1} d\sigma.$$

This, (67) and (39) implies

$$\frac{1}{2\Delta t} \int_{\Omega} \phi(C_{n+1}^2 - C_n^2) dx + \alpha \|\nabla C_{n+1}\|_{L^2(\Omega)}^2 \leq \mathcal{N} \|\bar{\Lambda}_{n+1}\|_{L^1(\Omega)} + \mathcal{N} \|\mathcal{C}\|_{L^\infty(\Sigma_T)} \|(\bar{\mathbf{U}}_{n+1})^-\|_{L^1(\partial\Omega)}.$$

Multiplying by $2\Delta t$, summing from 0 to $m-1$, for any $1 \leq m \leq N$, and using (58), (60) and (72) we get

$$\begin{aligned} \int_{\Omega} \phi C_m^2 dx + 2\alpha \sum_{n=0}^{m-1} \Delta t \|\nabla C_{n+1}\|_{L^2(\Omega)}^2 &\leq \int_{\Omega} \phi C_0^2 dx + 2\mathcal{N} \sum_{n=0}^{m-1} \Delta t \|\bar{\Lambda}_{n+1}\|_{L^1(\Omega)} + 2\mathcal{N}^2 \sum_{n=0}^{m-1} \Delta t \|\bar{\mathbf{U}}_{n+1}\|_{L^1(\partial\Omega)} \\ &= \int_{\Omega} \phi C_0^2 dx + 2\mathcal{N} \|I_0 \bar{\Lambda}\|_{L^1(Q_T)} + 2\mathcal{N}^2 \|I_0 \bar{\mathbf{U}}\|_{L^1(\Sigma_T)} \\ &\leq \int_{\Omega} \phi C_0^2 dx + 2\mathcal{N} \|\Lambda\|_{L^1(Q_T)} + 2\mathcal{N}^2 \|\mathcal{U}\|_{L^1(\Sigma_T)}. \end{aligned}$$

Therefore from (18), for all $1 \leq m \leq N$,

$$\phi_L \|C_m\|_{L^2(\Omega)}^2 + 2\alpha \sum_{n=0}^{m-1} \Delta t \|\nabla C_{n+1}\|_{L^2(\Omega)}^2 \leq A. \quad (80)$$

where $A = \int_{\Omega} \phi C_0^2 dx + 2\mathcal{N} \|\Lambda\|_{L^1(Q_T)} + 2\mathcal{N}^2 \|\mathcal{U}\|_{L^1(\Sigma_T)}$. This implies (74) as

$$\|I_0 C\|_{L^2(0,T;H^1(\Omega))} = \left(\sum_{n=0}^{N-1} \Delta t \|C_{n+1}\|_{H^1(\Omega)}^2 \right)^{\frac{1}{2}} = \left(\sum_{n=0}^{N-1} \Delta t (\|C_{n+1}\|_{L^2(\Omega)}^2 + \|\nabla C_{n+1}\|_{L^2(\Omega)}^2) \right)^{\frac{1}{2}}$$

$$\leq \left(\left(\sum_{n=0}^{N-1} \Delta t \frac{A}{\phi_L} \right) + \frac{A}{2\alpha} \right)^{\frac{1}{2}} = \left(\frac{AT}{\phi_L} + \frac{A}{2\alpha} \right)^{\frac{1}{2}}.$$

To prove (75) fix $t' > 0$. As before define $\lceil t \rceil = \min\{n \in \mathbb{Z} : n \geq t\}$. Then

$$\begin{aligned} \int_{\Omega} \phi(I_0 C_{-t'} - I_0 C)^2 dx &= \int_{\Omega} \phi \left(C_{\lceil \frac{t+t'}{\Delta t} \rceil} - C_{\lceil \frac{t}{\Delta t} \rceil} \right)^2 dx = \int_{\Omega} \phi \left(C_{\lceil \frac{t+t'}{\Delta t} \rceil} - C_{\lceil \frac{t}{\Delta t} \rceil} \right) \left(C_{\lceil \frac{t+t'}{\Delta t} \rceil} - C_{\lceil \frac{t}{\Delta t} \rceil} \right) dx \\ &= \int_{\Omega} \phi \sum_{n=\lceil \frac{t}{\Delta t} \rceil}^{\lceil \frac{t+t'}{\Delta t} \rceil - 1} (C_{n+1} - C_n) \left(C_{\lceil \frac{t+t'}{\Delta t} \rceil} - C_{\lceil \frac{t}{\Delta t} \rceil} \right) dx = \int_{\Omega} \phi \sum_{n=n_0(t)}^{n_1(t)-1} (C_{n+1} - C_n) (C_{n_1(t)} - C_{n_0(t)}) dx, \end{aligned}$$

where $n_0(t) = \lceil \frac{t}{\Delta t} \rceil$ and $n_1(t) = \lceil \frac{t+t'}{\Delta t} \rceil$. Multiplying (48) by Δt , summing from $n_0(t)$ to $n_1(t) - 1$ and choosing $\psi = C_{n_1(t)} - C_{n_0(t)}$ we have

$$\begin{aligned} \int_{\Omega} \sum_{n=n_0(t)}^{n_1(t)-1} \phi (C_{n+1} - C_n) (C_{n_1(t)} - C_{n_0(t)}) dx &= -\Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} \mathbf{F}(\mathbf{U}_{n+1}) \nabla C_{n+1} \cdot \nabla (C_{n_1(t)} - C_{n_0(t)}) dx \\ + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} C_{n+1} \mathbf{U}_{n+1} \cdot \nabla (C_{n_1(t)} - C_{n_0(t)}) dx &- \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\partial\Omega} (C_{n+1} \bar{\mathbf{U}}_{n+1}^+ - \bar{\mathbf{C}}_{n+1} \bar{\mathbf{U}}_{n+1}^-) (C_{n_1(t)} - C_{n_0(t)}) d\sigma \\ &+ \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} \bar{\Lambda}_{n+1} (C_{n_1(t)} - C_{n_0(t)}) dx. \end{aligned}$$

Then from (19) and (67), we obtain

$$\begin{aligned} \int_{\Omega} \sum_{n=n_0(t)}^{n_1(t)-1} \phi (C_{n+1} - C_n) (C_{n_1(t)} - C_{n_0(t)}) dx &\leq \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} F_B |\nabla C_{n+1}| (|\nabla C_{n_1(t)}| + |\nabla C_{n_0(t)}|) dx \\ &+ \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} \mathcal{N} |\mathbf{U}_{n+1}| (|\nabla C_{n_1(t)}| + |\nabla C_{n_0(t)}|) dx \\ &+ \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\partial\Omega} 2\mathcal{N} (\mathcal{N} + |\bar{\mathbf{C}}_{n+1}|) |\bar{\mathbf{U}}_{n+1}| d\sigma + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} 2\mathcal{N} |\bar{\Lambda}_{n+1}| dx \\ \leq \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} (F_B^2 |\nabla C_{n+1}|^2 + \frac{|\nabla C_{n_1(t)}|^2}{2} + \frac{|\nabla C_{n_0(t)}|^2}{2}) dx &+ \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} (\mathcal{N}^2 |\mathbf{U}_{n+1}|^2 + \frac{|\nabla C_{n_1(t)}|^2}{2} + \frac{|\nabla C_{n_0(t)}|^2}{2}) dx \\ &+ \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\partial\Omega} 2\mathcal{N} (\mathcal{N} + |\bar{\mathbf{C}}_{n+1}|) |\bar{\mathbf{U}}_{n+1}| d\sigma + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} 2\mathcal{N} |\bar{\Lambda}_{n+1}| dx \\ \leq \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \left(\int_{\Omega} (F_B^2 |\nabla C_{n+1}|^2 + \mathcal{N}^2 |\mathbf{U}_{n+1}|^2 + 2\mathcal{N} |\bar{\Lambda}_{n+1}|) dx &+ \int_{\partial\Omega} 2\mathcal{N} (\mathcal{N} + |\bar{\mathbf{C}}_{n+1}|) |\bar{\mathbf{U}}_{n+1}| d\sigma \right) \\ &+ \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} |\nabla C_{n_1(t)}|^2 + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\Omega} |\nabla C_{n_0(t)}|^2. \end{aligned}$$

Define

$$p_n := \int_{\Omega} (F_B^2 |\nabla C_n|^2 + \mathcal{N}^2 |\mathbf{U}_n|^2 + 2\mathcal{N} |\bar{\Lambda}_n|) dx + \int_{\partial\Omega} 2\mathcal{N} (\mathcal{N} + |\bar{\mathbf{C}}_n|) |\bar{\mathbf{U}}_n| d\sigma, \text{ and } q_n := \int_{\Omega} |\nabla C_n|^2.$$

Therefore, we can rewrite

$$\int_{\Omega} \phi(I_0 C_{-t'} - I_0 C)^2 dx \leq \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} p_{n+1} + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} q_{n_1(t)} + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} q_{n_0(t)}.$$

Now let

$$\chi_n(t, t+t') = \begin{cases} 1, & \text{if } n\Delta t \in [t, t+t') \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \int_0^{T-t'} \sum_{n=n_0(t)}^{n_1(t)-1} p_{n+1} dt &= \int_0^{T-t'} \sum_{n=0}^{N-1} p_{n+1} \chi_n(t, t+t') dt = \sum_{n=0}^{N-1} p_{n+1} \int_0^{T-t'} \chi_n(t, t+t') dt \\ &\leq \sum_{n=0}^{N-1} p_{n+1} \int_{\mathbb{R}} \chi_n(t, t+t') dt = \sum_{n=0}^{N-1} p_{n+1} \int_{n\Delta t-t'}^{n\Delta t} dt = t' \sum_{n=0}^{N-1} p_{n+1}. \end{aligned}$$

Observe that $n_0(t) = m$ for some $m \in \mathbb{N}$ if and only if $t \in ((m-1)\Delta t, m\Delta t]$. Then

$$\begin{aligned} \int_0^{T-t'} \sum_{n=n_0(t)}^{n_1(t)-1} q_{n_0(t)} dt &= \int_0^{T-t'} \sum_{n=0}^{N-1} q_{n_0(t)} \chi_n(t, t+t') dt = \int_0^{T-t'} q_{n_0(t)} \sum_{n=0}^{N-1} \chi_n(t, t+t') dt \\ &\leq \sum_{m=1}^N \int_{(m-1)\Delta t}^{m\Delta t} q_m \sum_{n=0}^{N-1} \chi_n(t, t+t') dt \leq \sum_{m=1}^N q_m \sum_{n=0}^{N-1} \int_{(m-1)\Delta t}^{m\Delta t} \chi_n(t, t+t') dt \\ &= \sum_{m=1}^N q_m \sum_{n=0}^{N-1} \int_{(2m-n-1)\Delta t}^{(2m-n)\Delta t} \chi_n(s + (n-m)\Delta t, s + (n-m)\Delta t + t') ds \\ &= \sum_{m=1}^N q_m \sum_{n=0}^{N-1} \int_{(2m-n-1)\Delta t}^{(2m-n)\Delta t} \chi_m(s, s+t') ds \\ &\leq \sum_{m=1}^N q_m \int_{\mathbb{R}} \chi_m(s, s+t') ds = \sum_{m=1}^N q_m \int_{m\Delta t-t'}^{m\Delta t} ds = t' \sum_{m=1}^N q_m. \end{aligned}$$

Similarly,

$$\int_0^{T-t'} \sum_{n=n_0(t)}^{n_1(t)+1} q_{n_1(t)} dt = t' \sum_{m=1}^N q_m.$$

Therefore, from (18),

$$\|I_0 C_{-t'} - I_0 C\|_{L^2((0, T-t'); L^2(\Omega))}^2 = \int_0^{T-t'} \int_{\Omega} (I_0 C_{-t'} - I_0 C)^2 dx \leq t' \frac{\Delta t}{\phi_L} \sum_{n=1}^N (p_n + 2q_n).$$

Let us see that $\Delta t \sum_{n=1}^N p_n$ and $\Delta t \sum_{n=1}^N q_n$ are bounded uniformly in N . From Cauchy-Schwarz's inequality,

$$\begin{aligned} \Delta t \sum_{n=1}^N p_n &\leq F_B^2 \Delta t \sum_{n=1}^N \|\nabla C_n\|_{L^2(\Omega)}^2 + \Delta t \sum_{n=1}^N \mathcal{N}^2 \|U_n\|_{L^2(\Omega)}^2 + 2\mathcal{N} \Delta t \sum_{n=1}^N \|\bar{\Lambda}_n\|_{L^1(\Omega)} \\ &\quad + 2\mathcal{N}^2 \Delta t \sum_{n=1}^N \|\bar{U}_n\|_{L^1(\partial\Omega)} + 2\mathcal{N} \Delta t \sum_{n=1}^N \|\bar{C}_n\|_{L^2(\partial\Omega)} \|\bar{U}_n\|_{L^2(\partial\Omega)}. \end{aligned}$$

Then, (58), (69), (70), (72), (73) and (79) implies

$$\Delta t \sum_{n=1}^N p_n \leq F_B^2 \frac{A}{2\alpha} + \mathcal{N}^2 \mathcal{M}^2 + 2\mathcal{N} \|\Lambda\|_{L^1(Q_T)} + 2\mathcal{N}^2 \|\mathcal{U}\|_{L^1(\Sigma_T)} + 2\mathcal{N} \|\mathcal{C}\|_{L^2(\Sigma_T)} \|\mathcal{U}\|_{L^2(\Sigma_T)}.$$

Also from (80),

$$\Delta t \sum_{n=1}^N q_n = \Delta t \sum_{n=1}^N \|\nabla C_n\|_{L^2(\Omega)}^2 \leq \frac{A}{2\alpha}.$$

Therefore, $\Delta t \sum_{n=1}^N p_n$ and $\Delta t \sum_{n=1}^N q_n$ are bounded uniformly in N implying

$$\|I_0 C_{-t'} - I_0 C\|_{L^2((0, T-t'); L^2(\Omega))}^2 \leq \mathfrak{M} t',$$

where \mathfrak{M} is a constant independent of N . Let us prove (76). From (57)

$$\begin{aligned} \left\| \frac{\partial}{\partial t} I_1 C \right\|_{L^2(0, T; H^1(\Omega)')}^2 &= \int_0^T \left\| \frac{\partial}{\partial t} I_1 C \right\|_{H^1(\Omega)}^2 dt = \int_0^T \left\| \sum_{n=0}^{N-1} \frac{1}{\Delta t} (C_{n+1} - C_n) \chi_{]n\Delta t, (n+1)\Delta t[}(t) \right\|_{H^1(\Omega)'}^2 dt \\ &= \sum_{m=0}^{N-1} \int_{m\Delta t}^{(m+1)\Delta t} \frac{1}{(\Delta t)^2} \|C_{m+1} - C_m\|_{H^1(\Omega)'}^2 dt = \frac{1}{\Delta t} \sum_{m=0}^{N-1} \|C_{m+1} - C_m\|_{H^1(\Omega)'}^2. \end{aligned}$$

To bound this let $\psi \in H^1(\Omega)$. Cauchy-Schwarz's inequality, (19), (48), (78) and (39) give

$$\begin{aligned} \frac{1}{\Delta t} |\langle \phi(C_{n+1} - C_n), \psi \rangle_{(H^1(\Omega))', H^1(\Omega)}| &\leq (F_B \|\nabla C_{n+1}\|_{L^2(\Omega)} + \|C_{n+1}\|_{L^\infty(\Omega)} \|\mathbf{U}_{n+1}\|_{L^2(\Omega)}) \|\nabla \psi\|_{L^2(\Omega)} \\ &\quad + (\|C_{n+1}\|_{L^4(\partial\Omega)} \|\psi\|_{L^4(\partial\Omega)} + \|\bar{\mathcal{C}}_{n+1}\|_{L^\infty(\partial\Omega)} \|\psi\|_{L^2(\partial\Omega)}) \|\bar{\mathbf{U}}_{n+1}\|_{L^2(\partial\Omega)} \\ &\quad + \|\bar{\Lambda}_{n+1}\|_{H^1(\Omega)'} \|\psi\|_{H^1(\Omega)}. \end{aligned}$$

Then by (27), (67) and (39) we have

$$\begin{aligned} \frac{1}{\Delta t} |\langle \phi(C_{n+1} - C_n), \psi \rangle_{(H^1(\Omega))', H^1(\Omega)}| &\leq (F_B \|C_{n+1}\|_{H^1(\Omega)} + \mathcal{N} \|\mathbf{U}_{n+1}\|_{L^2(\Omega)} \\ &\quad + (M_4^2 \|C_{n+1}\|_{H^1(\Omega)} + M_2 \|\mathcal{C}\|_{L^\infty(\Sigma_T)}) \|\bar{\mathbf{U}}_{n+1}\|_{L^2(\partial\Omega)} + \|\bar{\Lambda}_{n+1}\|_{H^1(\Omega)'} \|\psi\|_{H^1(\Omega)}. \end{aligned}$$

Taking supremum over all $\psi \in H^1(\Omega)$ such that $\|\psi\|_{H^1(\Omega)} = 1$ and using (18), we see that there exists a constant M independent of N such that

$$\frac{1}{\Delta t^2} \|C_{n+1} - C_n\|_{H^1(\Omega)'}^2 \leq M (\|C_{n+1}\|_{H^1(\Omega)}^2 + \|\mathbf{U}_{n+1}\|_{L^2(\Omega)}^2 + \|\bar{\mathbf{U}}_{n+1}\|_{L^2(\partial\Omega)}^2 + \|\bar{\Lambda}_{n+1}\|_{H^1(\Omega)'}^2).$$

Multiplying by Δt , summing from 0 to $N-1$ and using (74), (58) and (60) we obtain (76). (77) follows from (76) as

$$\begin{aligned} \|I_1 C - I_0 C\|_{L^2(0, T; H^1(\Omega)')}^2 &= \int_0^T \left\| \sum_{n=0}^{N-1} \left((1+n - \frac{t}{\Delta t})(C_n - C_{n+1}) \right) \chi_{]n\Delta t, (n+1)\Delta t[}(t) \right\|_{H^1(\Omega)'}^2 dt \\ &= \sum_{m=0}^{N-1} \int_{m\Delta t}^{(m+1)\Delta t} \left\| (1+m - \frac{t}{\Delta t})(C_m - C_{m+1}) \right\|_{H^1(\Omega)'}^2 dt = \sum_{m=0}^{N-1} \|C_m - C_{m+1}\|_{H^1(\Omega)'}^2 \int_{m\Delta t}^{(m+1)\Delta t} (1+m - \frac{t}{\Delta t})^2 dt \\ &= \frac{\Delta t}{3} \sum_{m=0}^{N-1} \|C_m - C_{m+1}\|_{H^1(\Omega)'}^2. \end{aligned}$$

□

6 Passing to the limit

6.1 Passing to the limit

Passing to the limit in (63)-(65) requires certain convergence properties that we now state and prove.

Proposition 6.1. *There exists a subsequence of $\{C^N\}_{N \geq 1}$ still denoted by $\{C^N\}_{N \geq 1}$ and a function $c \in L^\infty(Q_T) \cap L^2(0, T; H^1(\Omega))$ such that $t \rightarrow c(t, \cdot) \in \mathcal{C}([0, T]; (H^1(\Omega))')$ satisfying*

$$I_0 C^N \rightarrow c \text{ weakly-}\star \text{ in } L^\infty(Q_T), \quad (81)$$

$$I_0 C^N \rightarrow c \text{ weakly in } L^2(0, T; H^1(\Omega)), \quad (82)$$

$$I_0 C^N \rightarrow c \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \quad (83)$$

$$I_0 C^N \rightarrow c \text{ strongly in } L^2(\Sigma_T), \quad (84)$$

$$\frac{\partial}{\partial t} I_1 C^N \rightarrow \frac{\partial}{\partial t} c \text{ weakly in } L^2(0, T; H^1(\Omega)'), \quad (85)$$

$$I_1 C^N \rightarrow c \text{ strongly in } \mathcal{C}^0([0, T]; H^1(\Omega)'), \quad (86)$$

$$I_0 \bar{\Lambda}^N \rightarrow \Lambda \text{ strongly in } L^2(0, T; H^1(\Omega)'), \quad (87)$$

$$I_0 \bar{\mathcal{C}}^N \rightarrow \mathcal{C} \text{ strongly in } L^2(\Sigma_T), \quad (88)$$

as $N \rightarrow \infty$.

Proof. The last two convergence results follows trivially from (61). To prove the rest we will use the estimates from the previous section. From remark (68) and (74), we know that $\{I_0 C^N\}_N$ is bounded in $L^\infty(Q_T)$ and in $L^2(0, T; H^1(\Omega))$. Because $L^\infty(Q_T) = (L^1(Q_T))'$, by a corollary to Banach-Alaoglu Theorem [23, p.230], we can extract a subsequence still denoted by $\{C^N\}_N$ (from now on we will denote each extracted subsequence by $\{C^N\}_N$) and find a function $c \in L^\infty(Q_T)$ such that (81) holds. Next the reflexivity of the space $L^2(0, T; H^1(\Omega))$ implies that there exists a subsequence $\{C^N\}_N$ and a function $c_1 \in L^2(0, T; H^1(\Omega))$ such that $I_0 C^N \rightarrow c_1$ weakly in $L^2(0, T; H^1(\Omega))$. This also implies that $I_0 C^N \rightarrow c_1$ weakly- \star in $L^\infty(Q_T)$. Therefore, $c_1 = c$ by uniqueness of the weak- \star limits. Hence (82) holds. From (75), $\|I_0 C_{-t'}^N - I_0 C^N\|_{L^2((0, T-t'); L^2(\Omega))} \rightarrow 0$ as $t' \rightarrow 0$ uniformly for all N . Theorem 2.1 states that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. So applying [24, Theorem 5 p.84] we can find a subsequence $\{C^N\}_N$ and a function $c_2 \in L^2(Q_T)$ such that $I_0 C^N \rightarrow c_2$ strongly in $L^2(Q_T)$. This further implies the weak convergence in $L^2(Q_T)$. But (82) gives weak convergence in $L^2(Q_T)$ as well. Therefore, $c_2 = c$ by the uniqueness of weak limits and hence (83) holds. Similarly, as $H^1(\Omega)$ is compactly embedded in $H^{\frac{1}{2}}(\Omega)$ (see Theorem 2.1), we can find a subsequence $\{C^N\}_N$ such that $I_0 C^N \rightarrow c$ strongly in $L^2(0, T; H^{\frac{1}{2}}(\Omega))$. Then the continuity of the trace operator gives (84).

Recall from (68) that $I_1 C^N$ is uniformly bounded. So again by Banach-Alaoglu theorem, up to a subsequence, there exists $c_3 \in L^\infty(Q_T)$ such that

$$I_1 C^N \rightarrow c_3 \text{ weakly-}\star \text{ in } L^\infty(Q_T).$$

The bound (76) and the reflexivity of $L^2(0, T; H^1(\Omega))'$ gives a subsequence for which we have (again by uniqueness of weak- \star limits $c_3 = c$)

$$\frac{\partial}{\partial t} I_1 C^N \rightarrow \frac{\partial}{\partial t} c \text{ weakly in } L^2(0, T; H^1(\Omega)').$$

We know that $\{I_1 C^N\}_N$ is bounded in $L^\infty(Q_T)$ by (68), $\{\frac{\partial}{\partial t} I_1 C^N\}_N$ is bounded in $L^2(0, T; H^1(\Omega))'$ by (76). Also from Theorem 2.1 and Schauder's theorem [25], $L^\infty(\Omega)$ is compactly embedded in $H^1(\Omega)'$. Then (86) is a consequence of [24, Corollary 4, p.85] which implies that there exists a subsequence $\{C^N\}_N$ and a function $\tilde{c} \in \mathcal{C}^0([0, T]; H^1(\Omega)')$ such that

$$I_1 C^N \rightarrow \tilde{c} \text{ strongly in } \mathcal{C}^0([0, T]; H^1(\Omega)').$$

The bound (77) implies

$$I_1 C^N - I_0 C^N \rightarrow 0 \text{ strongly in } L^2(0, T; H^1(\Omega)').$$

This together with (83) yields

$$I_1 C^N \rightarrow c \text{ strongly in } L^2(0, T; H^1(\Omega)')$$

and thus $\tilde{c} = c$. □

Proposition 6.2. *The following convergence results hold.*

$$I_0 \bar{\Pi}^N \rightarrow \Pi \text{ strongly in } L^2(0, T; L^2(\Omega_2)), \quad (89)$$

$$I_0 \bar{\Psi}^N \rightarrow \Psi \text{ strongly in } L^2(0, T; L^2(\Omega_1)^2), \quad (90)$$

$$I_0 \bar{\mathcal{U}}^N \rightarrow \mathcal{U} \text{ strongly in } L^2(\Sigma_T), \quad (91)$$

and there exists $\mathbf{u} \in L^2(Q_T)^2$ such that

$$I_0 \mathbf{U}^N \rightarrow \mathbf{u} \text{ strongly in } L^2(Q_T)^2. \quad (92)$$

Proof. The results (89), (90) and (91) are direct consequences of (61). For (92), we will use the equation (P). Consider the following problem where c is the limit found in Proposition 6.1. Find $(\mathbf{u}|_{\Omega_1}, \varphi) \in L^2(0, T; \mathbf{V}_1) \times L^\infty(0, T; R_2)$ satisfying

$$\begin{aligned} \int_0^T \left(2(\mu(c)\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\Omega_1} + \left(\frac{\mathbf{K}}{\mu(c)}(\nabla\varphi - \rho\mathbf{g}), \nabla q\right)_{\Omega_2} + (\varphi, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} + G(\mathbf{K}^{-1/2}\mathbf{u} \cdot \boldsymbol{\tau}_{12}, \mathbf{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \right. \\ \left. - (\mathbf{u} \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} \right) dt = \int_0^T \left((\Psi, \mathbf{v})_{\Omega_1} + (\Pi, q)_{\Omega_2} - (\mathcal{U}, q)_{\Gamma_2} \right) dt, \end{aligned} \quad (93)$$

for all $\mathbf{v} \in L^2(0, T; \mathbf{V}_1)$ and for all $q \in L^2(0, T; R_2)$. It is known that there exists a unique solution (\mathbf{u}, φ) to this problem [18, 17]. Next define $\mathbf{u}|_{\Omega_2} \in L^2(0, T; L^2(\Omega_2)^2)$ as

$$\mathbf{u} = -\frac{\mathbf{K}}{\mu(c)}(\nabla\varphi - \rho\mathbf{g}) \text{ a.e. in } \Omega_2 \times (0, T).$$

The difference of (63) and (93) yields

$$\begin{aligned} \int_0^T \left(2(\mu(\widetilde{I_0 C_{\Delta t}^N})\mathbf{D}(I_0 \mathbf{U}^N) - \mu(c)\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\Omega_1} + \left(\frac{\mathbf{K}}{\mu(\widetilde{I_0 C_{\Delta t}^N})}(\nabla I_0 \Phi^N - \rho\mathbf{g}) - \frac{\mathbf{K}}{\mu(c)}(\nabla\varphi - \rho\mathbf{g}), \nabla q\right)_{\Omega_2} \right. \\ \left. + (I_0 \Phi^N - \varphi, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} + G(\mathbf{K}^{-1/2}(I_0 \mathbf{U}^N - \mathbf{u}) \cdot \boldsymbol{\tau}_{12}, \mathbf{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - ((I_0 \mathbf{U}^N - \mathbf{u}) \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} \right) dt \\ = \int_0^T \left((I_0 \bar{\Psi}^N - \Psi, \mathbf{v})_{\Omega_1} + (I_0 \bar{\Pi}^N - \Pi, q)_{\Omega_2} - (I_0 \bar{\mathcal{U}}^N - \mathcal{U}, q)_{\Gamma_2} \right) dt. \end{aligned} \quad (94)$$

Observe that the first and the second terms can be written as

$$\begin{aligned} \int_0^T 2(\mu(\widetilde{I_0 C_{\Delta t}^N})\mathbf{D}(I_0 \mathbf{U}^N) - \mu(c)\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\Omega_1} dt \\ = \int_0^T 2\left(\mu(\widetilde{I_0 C_{\Delta t}^N})(\mathbf{D}(I_0 \mathbf{U}^N) - \mathbf{D}(\mathbf{u})), \mathbf{D}(\mathbf{v})\right)_{\Omega_1} dt + \int_0^T 2\left((\mu(\widetilde{I_0 C_{\Delta t}^N}) - \mu(c))\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v})\right)_{\Omega_1} dt, \end{aligned}$$

and

$$\begin{aligned}
& \int_0^T \left(\frac{\mathbf{K}}{\mu(\widetilde{I_0 C_{\Delta t}^N})} (\nabla I_0 \Phi^N - \rho \mathbf{g}) - \frac{\mathbf{K}}{\mu(c)} (\nabla \varphi - \rho \mathbf{g}), \nabla q \right)_{\Omega_2} dt \\
&= \int_0^T \left(\frac{\mathbf{K}}{\mu(\widetilde{I_0 C_{\Delta t}^N})} (\nabla I_0 \Phi^N - \nabla \varphi), \nabla q \right)_{\Omega_2} dt + \int_0^T \left(\left(\frac{1}{\mu(\widetilde{I_0 C_{\Delta t}^N})} - \frac{1}{\mu(c)} \right) \mathbf{K} \nabla \varphi, \nabla q \right)_{\Omega_2} dt \\
&\quad - \int_0^T \left(\left(\frac{\mathbf{K}}{\mu(\widetilde{I_0 C_{\Delta t}^N})} - \frac{\mathbf{K}}{\mu(c)} \right) \rho \mathbf{g}, \nabla q \right)_{\Omega_2} dt.
\end{aligned}$$

Then letting $\mathbf{v} = I_0 \mathbf{U}^N - \mathbf{u}$, $q = I_0 \Phi^N - \varphi$ in (94), we have

$$\begin{aligned}
& \int_0^T \left(2(\mu(\widetilde{I_0 C_{\Delta t}^N}) \mathbf{D}(I_0 \mathbf{U}^N - \mathbf{u}), \mathbf{D}(I_0 \mathbf{U}^N - \mathbf{u}))_{\Omega_1} + \left(\frac{\mathbf{K}}{\mu(\widetilde{I_0 C_{\Delta t}^N})} \nabla(I_0 \Phi^N - \varphi), \nabla(I_0 \Phi^N - \varphi) \right)_{\Omega_2} \right. \\
&\quad \left. + G(\mathbf{K}^{-1/2}(I_0 \mathbf{U}^N - \mathbf{u}) \cdot \boldsymbol{\tau}_{12}, (I_0 \mathbf{U}^N - \mathbf{u}) \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \right) dt \\
&= \int_0^T \left((I_0 \bar{\Psi}^N - \Psi, I_0 \mathbf{U}^N - \mathbf{u})_{\Omega_1} + (I_0 \bar{\Pi}^N - \Pi, I_0 \Phi^N - \varphi)_{\Omega_2} - (I_0 \bar{\mathcal{U}}^N - \mathcal{U}, I_0 \Phi^N - \varphi)_{\Gamma_2} \right. \\
&\quad \left. - 2((\mu(\widetilde{I_0 C_{\Delta t}^N}) - \mu(c)) \mathbf{D}(\mathbf{u}), \mathbf{D}(I_0 \mathbf{U}^N - \mathbf{u}))_{\Omega_1} - \left(\left(\frac{1}{\mu(\widetilde{I_0 C_{\Delta t}^N})} - \frac{1}{\mu(c)} \right) \mathbf{K} \nabla \varphi, \nabla(I_0 \Phi^N - \varphi) \right)_{\Omega_2} \right. \\
&\quad \left. + \rho \left(\left(\frac{1}{\mu(\widetilde{I_0 C_{\Delta t}^N})} - \frac{1}{\mu(c)} \right) \mathbf{K} \mathbf{g}, \nabla(I_0 \Phi^N - \varphi) \right)_{\Omega_2} \right) dt.
\end{aligned}$$

Then (17) and (13) and the positivity of the third term gives

$$\begin{aligned}
& 2\mu_L \|\mathbf{D}(I_0 \mathbf{U}^N - \mathbf{u})\|_{L^2(\Omega_1)}^2 + \frac{k_L}{\mu_U} \|\nabla(I_0 \Phi^N - \varphi)\|_{L^2(\Omega_2)}^2 \\
&\leq \int_0^T \left((I_0 \bar{\Psi}^N - \Psi, I_0 \mathbf{U}^N - \mathbf{u})_{\Omega_1} + (I_0 \bar{\Pi}^N - \Pi, I_0 \Phi^N - \varphi)_{\Omega_2} - (I_0 \bar{\mathcal{U}}^N - \mathcal{U}, I_0 \Phi^N - \varphi)_{\Gamma_2} \right. \\
&\quad \left. - 2((\mu(\widetilde{I_0 C_{\Delta t}^N}) - \mu(c)) \mathbf{D}(\mathbf{u}), \mathbf{D}(I_0 \mathbf{U}^N - \mathbf{u}))_{\Omega_1} - \left(\left(\frac{1}{\mu(\widetilde{I_0 C_{\Delta t}^N})} - \frac{1}{\mu(c)} \right) \mathbf{K} \nabla \varphi, \nabla(I_0 \Phi^N - \varphi) \right)_{\Omega_2} \right) dt \\
&\quad + \int_0^T \left(\rho \left(\frac{1}{\mu(\widetilde{I_0 C_{\Delta t}^N})} - \frac{1}{\mu(c)} \right) \mathbf{K} \mathbf{g}, \nabla(I_0 \Phi^N - \varphi) \right)_{\Omega_2} dt.
\end{aligned}$$

Using Cauchy-Schwarz's, Poincaré's (28) and Young's inequalities together with (15) and (26)

$$\begin{aligned}
& \frac{\mu_L}{M_1^2} \|\nabla(I_0 \mathbf{U}^N - \mathbf{u})\|_{L^2(0,T;L^2(\Omega_1))}^2 + \frac{k_L}{2\mu_U} \|\nabla(I_0 \Phi^N - \varphi)\|_{L^2(0,T;L^2(\Omega_2))}^2 \\
&\leq M \left(\|I_0 \bar{\Psi}^N - \Psi\|_{L^2(0,T;L^2(\Omega_1))}^2 + \|I_0 \bar{\Pi}^N - \Pi\|_{L^2(0,T;L^2(\Omega_2))}^2 + \|I_0 \bar{\mathcal{U}}^N - \mathcal{U}\|_{L^2(0,T;L^2(\Gamma_2))}^2 \right. \\
&\quad \left. + \|(\mu(\widetilde{I_0 C_{\Delta t}^N}) - \mu(c)) \mathbf{D}(\mathbf{u})\|_{L^2(0,T;L^2(\Omega_1))}^2 + \left\| \left(\frac{1}{\mu(\widetilde{I_0 C_{\Delta t}^N})} - \frac{1}{\mu(c)} \right) \mathbf{K} \nabla \varphi \right\|_{L^2(0,T;L^2(\Omega_2))}^2 \right. \\
&\quad \left. + \left\| \rho \left(\frac{1}{\mu(\widetilde{I_0 C_{\Delta t}^N})} - \frac{1}{\mu(c)} \right) \mathbf{K} \mathbf{g} \right\|_{L^2(0,T;L^2(\Omega_2))}^2 \right),
\end{aligned}$$

where M is a generic constant independent of N . Then by uniform boundedness (13) and continuity of μ , (76), (89), (90) and (91) together with the Lebesgue dominated convergence theorem imply

$$\frac{\mu_L}{M_1^2} \|\nabla(I_0 \mathbf{U}^N - \mathbf{u})\|_{L^2(0,T;L^2(\Omega_1))}^2 + \frac{k_L}{2\mu_U} \|\nabla(I_0 \Phi^N - \varphi)\|_{L^2(0,T;L^2(\Omega_2))}^2 \rightarrow 0, \text{ as } N \rightarrow \infty.$$

Thus as $N \rightarrow \infty$,

$$\nabla I_0 \mathbf{U}^N \rightarrow \nabla \mathbf{u} \text{ strongly in } L^2(0, T; L^2(\Omega_1)^{2 \times 2}), \quad (95)$$

$$\nabla I_0 \Phi^N \rightarrow \nabla \varphi \text{ strongly in } L^2(0, T; L^2(\Omega_2)^2). \quad (96)$$

Then (92) follows from (64), the continuity of μ , (28), proposition 6.1, (95) and (96). \square

6.2 Proof of Theorem 4.2

We are now ready to prove the existence result for the weak solution of the restricted problem. Recall that in order to obtain a weak solution we need to pass to the limit in the approximate solution equations (63)-(66). Passing to the limit in the flow equations (63) and (64) and the bound (66) is easy due to the continuity and the bound (13) of μ , (95) and (96). The convergence result (85) implies that

$$\frac{\partial}{\partial t} I_1 C^N \rightarrow \frac{\partial}{\partial t} c \text{ weakly-} \star \text{ in } L^2(0, T; H^1(\Omega)').$$

Thus,

$$\lim_{N \rightarrow \infty} \int_0^T \left\langle \frac{\partial}{\partial t} I_1 C^N, \psi \right\rangle dt = \int_0^T \left\langle \frac{\partial}{\partial t} c, \psi \right\rangle dt, \quad \forall \psi \in L^2(0, T; H^1(\Omega)). \quad (97)$$

Note that

$$(I_0 C^N I_0 \mathbf{U}^N, \nabla \psi)_{Q_T} - (c \mathbf{u}, \nabla \psi)_{Q_T} = ((I_0 C^N - c) I_0 \mathbf{U}^N, \nabla \psi)_{Q_T} + (c(I_0 \mathbf{U}^N - \mathbf{u}), \nabla \psi)_{Q_T} := I_1 + I_2.$$

I_1 converges to zero by (81) and (69) and I_2 converges to zero by (92) and the result from Proposition 6.1 that $c \in L^\infty(Q_T)$. Thus

$$\lim_{N \rightarrow \infty} (I_0 C^N I_0 \mathbf{U}^N, \nabla \psi)_{Q_T} = (c \mathbf{u}, \nabla \psi)_{Q_T}. \quad (98)$$

To pass to the limit in the third term in (65), we write:

$$\begin{aligned} & \int_{Q_T} \mathbf{F}(I_0 \mathbf{U}^N) \nabla I_0 C^N \cdot \nabla \psi dx dt - \int_{Q_T} \mathbf{F}(\mathbf{u}) \nabla c \cdot \nabla \psi dx dt \\ &= \int_{Q_T} (\mathbf{F}(I_0 \mathbf{U}^N) - \mathbf{F}(\mathbf{u})) \nabla I_0 C^N \cdot \nabla \psi dx dt + \int_{Q_T} \mathbf{F}(I_0 \mathbf{U}^N) (\nabla I_0 C^N - \nabla c) \cdot \nabla \psi dx dt := J_1 + J_2. \end{aligned}$$

The strong convergence $I_0 \mathbf{U}^N - \mathbf{u} \rightarrow 0$ in $L^2(Q_T)^2$ implies $I_0 \mathbf{U}^N - \mathbf{u} \rightarrow 0$ a.e. in Q_T up to a subsequence. So as \mathbf{F} is continuous,

$$\mathbf{F}(I_0 \mathbf{U}^N) - \mathbf{F}(\mathbf{u}) \rightarrow 0 \text{ a.e. in } Q_T.$$

Therefore J_1 converges to zero. By (82) and the bound (19) on \mathbf{F} , J_2 converges to zero as well. Hence

$$\lim_{N \rightarrow \infty} \int_{Q_T} \mathbf{F}(I_0 \mathbf{U}^N) \nabla I_0 C^N \cdot \nabla \psi dx dt = \int_{Q_T} \mathbf{F}(\mathbf{u}) \nabla c \cdot \nabla \psi dx dt. \quad (99)$$

The boundary terms in (65) are handled as follows

$$\begin{aligned} & \int_{\Sigma_T} (I_0 C^N (I_0 \bar{\mathcal{U}}^N)^+ - I_0 \bar{C}^N (I_0 \bar{\mathcal{U}}^N)^-) \psi d\sigma dt - \int_{\Sigma_T} (c \mathcal{U}^+ - c \mathcal{U}^-) \psi d\sigma dt = \int_{\Sigma_T} (I_0 C^N - c) (I_0 \bar{\mathcal{U}}^N)^+ \psi d\sigma dt \\ & + \int_{\Sigma_T} c ((I_0 \bar{\mathcal{U}}^N)^+ - \mathcal{U}^+) \psi d\sigma dt - \int_{\Sigma_T} (I_0 \bar{C}^N - c) (I_0 \bar{\mathcal{U}}^N)^- \psi d\sigma dt - \int_{\Sigma_T} c ((I_0 \bar{\mathcal{U}}^N)^- - \mathcal{U}^-) \psi d\sigma dt. \end{aligned} \quad (100)$$

By (84), (88) and (91) the terms on the right hand side of (100) converge to zero. Hence,

$$\lim_{N \rightarrow \infty} \int_{\Sigma_T} I_0 C^N (I_0 \bar{\mathcal{U}}^N)^+ \psi d\sigma dt - \int_{\Sigma_T} I_0 \bar{C}^N (I_0 \bar{\mathcal{U}}^N)^- \psi d\sigma dt = \int_{\Sigma_T} c \mathcal{U}^+ \psi d\sigma dt - \int_{\Sigma_T} c \mathcal{U}^- \psi d\sigma dt. \quad (101)$$

Finally from (87),

$$\lim_{N \rightarrow \infty} \int_0^T \langle I_0 \bar{\Lambda}^N, \phi \rangle_{(H^1(\Omega)', H^1(\Omega))} dt = \int_0^T \langle \Lambda, \phi \rangle_{(H^1(\Omega)', H^1(\Omega))} dt. \quad (102)$$

Combining (97), (98), (99), (101) and (102), we obtain (65). We also need to prove the following to complete the proof of Theorem 4.1:

$$c(x, 0) = c_0(x), \quad (103)$$

$$0 \leq c(x, t) \leq \mathcal{N} \text{ a.e. } (x, t) \in Q_T, \quad (104)$$

where \mathcal{N} is defined in Proposition 5.1. To prove (103), we observe from (86) that $I_1 C^N(0, \cdot) \rightarrow c(0, \cdot)$ strongly in $H^1(\Omega)'$. But $I_1 C^N(0) = c_0$ for all N . So $c(\cdot, 0) = c_0(\cdot)$ a.e. in Ω . For (104), recall that we have a uniform bound (68) on $I_0 C^N$. Then letting $N \rightarrow \infty$ and using the Lebesgue dominated convergence theorem, we finally get

$$0 \leq c(x, t) \leq \mathcal{N} \text{ a.e in } Q_T.$$

7 Proof of Theorem 4.1

This section completes the proof of the main result of this paper. The previous section proves existence of a weak solution $(\mathbf{u}, \varphi) \in L^2((0, T); \mathbf{V}_1) \times L^2((0, T); R_2)$. Hence as the final step, we recover the Stokes pressure p using an inf-sup condition.

Lemma 7.1. *For any $q \in L^2(0, T; R_1)$, there exists $\mathbf{v} \in L^2(0, T; X_1)$ such that $\nabla \cdot \mathbf{v} = q$ in $(0, T) \times \Omega_1$ and*

$$\|\mathbf{v}\|_{L^2(0, T; X_1)} \leq M \|q\|_{L^2(0, T; R_1)},$$

for some positive constant M independent of \mathbf{v}, q .

Proof. Let $q \in L^2(0, T; R_1)$. For any $t \in [0, T]$, define $q^t(x) = q(x, t)$, for all $x \in \Omega_1$. Then $q^t \in R_1$. From the inf-sup condition [17, Lemma 1.2], there exists $\mathbf{v}^t \in X_1$ and $\beta > 0$ independent of q_t , such that

$$\nabla \cdot \mathbf{v}^t = q^t \text{ in } \Omega_1, \quad \|\nabla \mathbf{v}^t\|_{L^2(\Omega_1)} \leq \beta \|q^t\|_{R_1}.$$

Now, set $\mathbf{v}(x, t) = \mathbf{v}^t(x)$, for any $(x, t) \in (0, T) \times X$. Then $\nabla \cdot \mathbf{v} = q$ and as $q \in L^2(0, T; R_1)$, $\nabla \cdot \mathbf{v} \in L^2(0, T; X_1)$. Integrating the above inequality from 0 to T in time, we also have

$$\|\mathbf{v}\|_{L^2(0, T; X_1)} \leq \beta \|q\|_{L^2(0, T; R_1)}.$$

□

Equivalently, we have the following inf-sup condition: there exists a constant $\beta > 0$ such that

$$\inf_{q \in L^2(0, T; R_1)} \sup_{\mathbf{v} \in L^2(0, T; X_1)} \frac{\int_0^T (q, \nabla \cdot \mathbf{v})_{\Omega_1}}{\|q\|_{L^2(0, T; R_1)} \|\mathbf{v}\|_{L^2(0, T; X_1)}} \geq \beta.$$

This trivially implies that

$$\inf_{q \in L^2(0, T; R_1)} \sup_{(\mathbf{v}, r) \in L^2(0, T; X_1 \times R_2)} \frac{\int_0^T (q, \nabla \cdot \mathbf{v})_{\Omega_1}}{\|q\|_{L^2(0, T; R_1)} \|(\mathbf{v}, r)\|_{L^2(0, T; X_1 \times R_2)}} \geq \beta.$$

From (32), we have for any $\mathbf{v} \in L^2(0, T; X_1)$ and $q \in L^2(0, T; R_2)$:

$$\int_0^T (\nabla \cdot \mathbf{v}, p)_{\Omega_1} dt = L(\mathbf{v}, q), \quad (105)$$

where L is a continuous linear functional on $L^2(0, T; \mathbf{X}_1) \times L^2(0, T; R_2)$:

$$L(\mathbf{v}, q) = \int_0^T \left(2(\mu(c)\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\Omega_1} + \left(\frac{\mathbf{K}}{\mu(c)}(\nabla\varphi - \rho\mathbf{g}), \nabla q\right)_{\Omega_2} + (\varphi, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} + G(\mathbf{K}^{-1/2}\mathbf{u} \cdot \boldsymbol{\tau}_{12}, \mathbf{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{u} \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} - (\Psi, \mathbf{v})_{\Omega_1} - (\Pi, q)_{\Omega_2} + (\mathcal{U}, q)_{\Gamma_2} \right) dt. \quad (106)$$

As (\mathbf{u}, φ) solves (37), L vanishes on the space $L^2(0, T; \mathbf{V}_1) \times L^2(0, T; R_2)$. Thus again from [26, Lemma 4.1], there exists a unique $p \in L^2(0, T; R_1)$ such that for all $(\mathbf{v}, q) \in L^2(0, T; \mathbf{X}_1) \times L^2(0, T; R_2)$ such that (105) holds, which completes the proof of Theorem 4.1.

Remark 7.2. *This inf-sup condition also shows that the weak problems (32) and (37) are equivalent.*

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