Passive sensor imaging with cross correlations

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Collaboration with G. Papanicolaou.

Ref: J. Garnier and G. Papanicolaou, Passive sensor imaging using cross correlations of noisy signals in a scattering medium

`http://math.stanford.edu/~papanico/` or `http://www.proba.jussieu.fr/~garnier/`

Part I: cross correlation for travel time estimation

Part II: fourth-order cross correlation for travel time estimation

Part III: cross correlation for passive imaging

Part IV: fourth-order cross correlation for passive imaging

Part I: cross correlation for travel time estimation

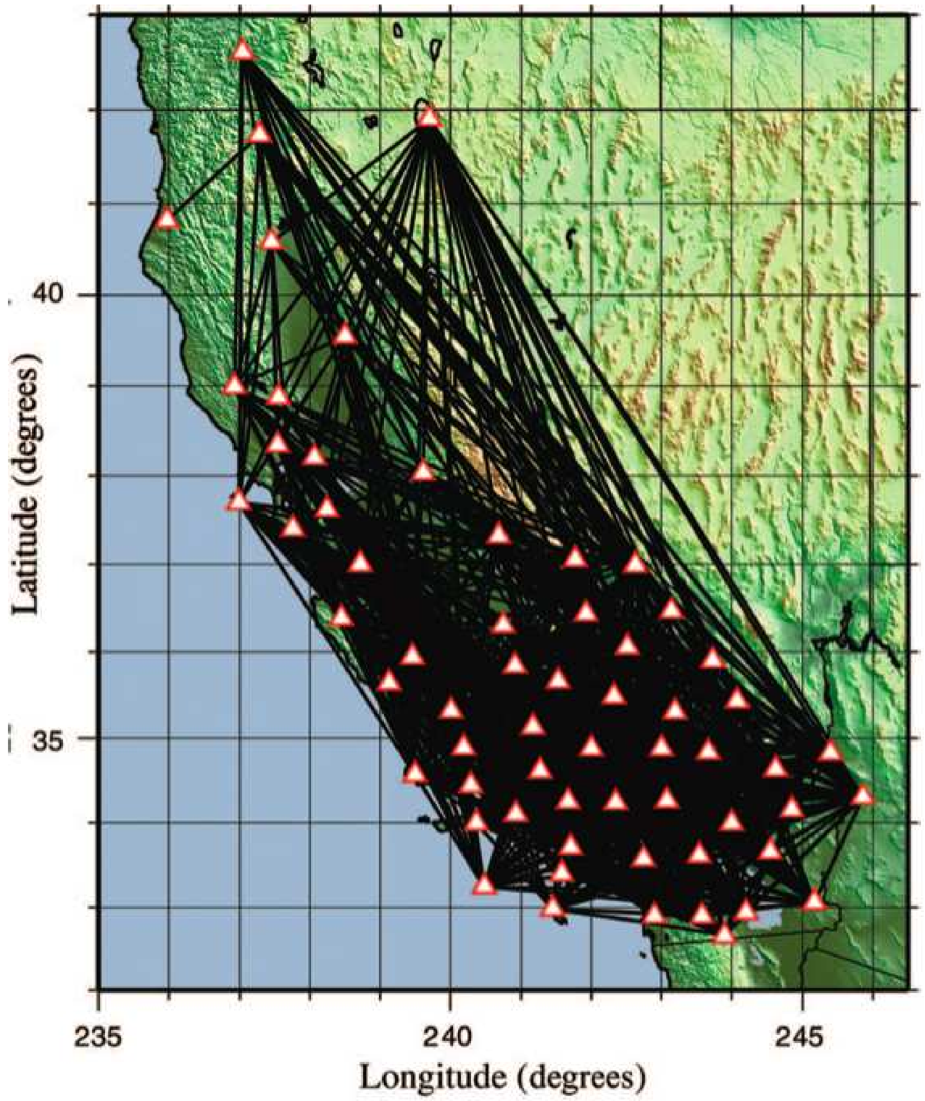
Use of ambient noise in order to estimate the travel time between two sensors $\mathbf{x}_1$ and $\mathbf{x}_2$.

- Ambient noise sources emit stationary random signals.
- Record the noisy signals $u(t,\mathbf{x}_1)$ and $u(t,\mathbf{x}_2)$ at $\mathbf{x}_1$ and $\mathbf{x}_2$.
- Compute the empirical cross correlation:

$$C_T(\tau,\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{T}\int_0^T u(t,\mathbf{x}_1)u(t+\tau,\mathbf{x}_2)dt$$

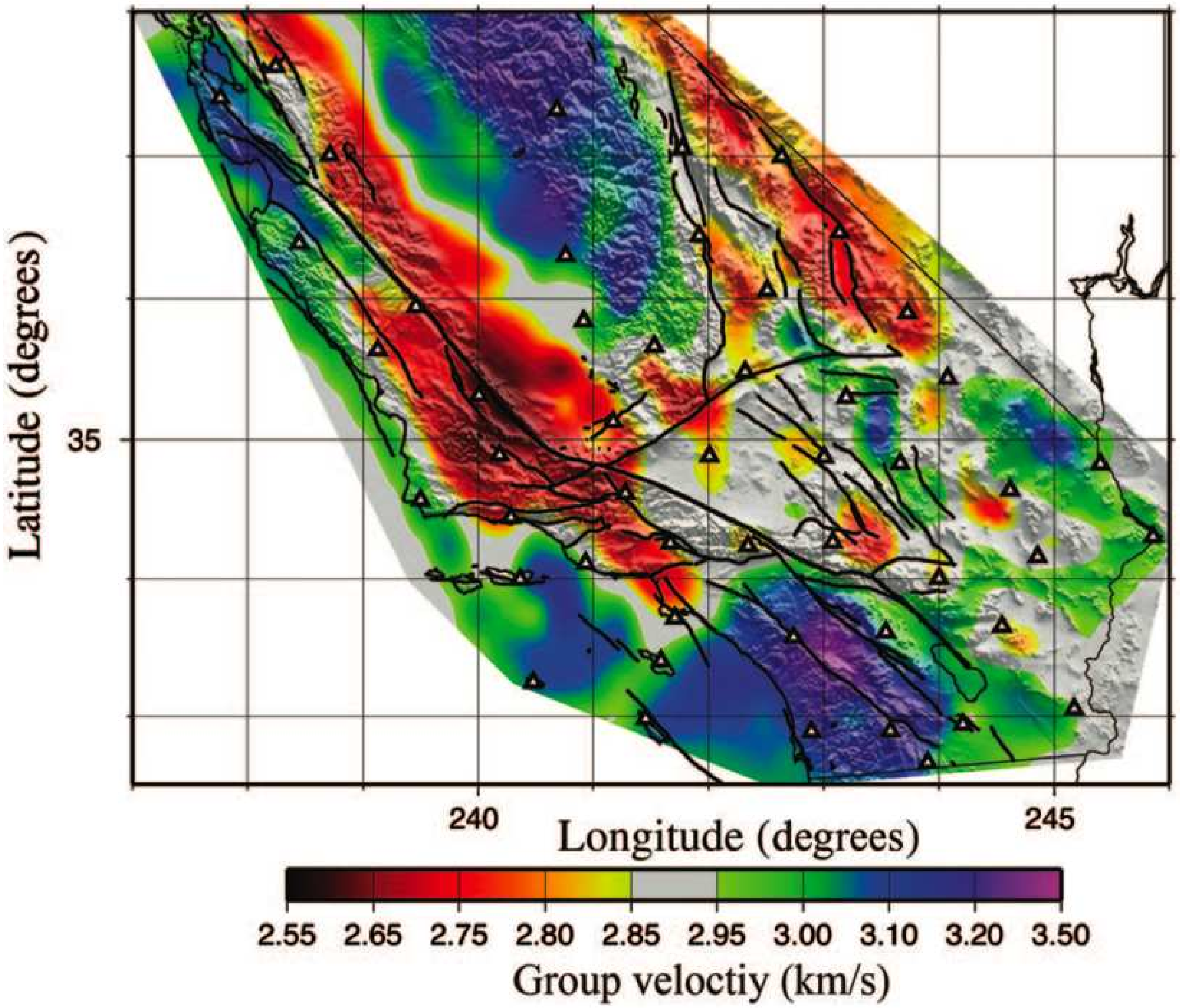
- $C_T(\tau,\mathbf{x}_1,\mathbf{x}_2)$ is related to the Green's function from $\mathbf{x}_1$ to $\mathbf{x}_2$!
- The singular component of the Green's function from $\mathbf{x}_1$ to $\mathbf{x}_2$ gives the travel time from $\mathbf{x}_1$ to $\mathbf{x}_2$.

Estimations of travel times between pairs of sensors



Surface (Rayleigh) waves [from Larose et al, Geophysics 71, 2006, SI11-SI21]

Background velocity estimation from travel time estimations



[from Larose et al, Geophysics 71, 2006, SI11-SI21]

Wave equation

$$\frac{1}{c^2(\mathbf{x})}\frac{\partial^2 u}{\partial t^2} - \Delta_{\mathbf{x}} u = n(t,\mathbf{x})$$

Sources $n(t,\mathbf{x})$: Gaussian process, with mean zero, stationary in time, with covariance function

$$\langle n(t_1,\mathbf{y}_1)n(t_2,\mathbf{y}_2)\rangle = F^\varepsilon(t_2-t_1)\Gamma(\mathbf{y}_1,\mathbf{y}_2)$$

Assume

$$\Gamma(\mathbf{y}_1,\mathbf{y}_2) = \theta(\mathbf{y}_1)\delta(\mathbf{y}_1-\mathbf{y}_2)$$

$$F^\varepsilon(t_2-t_1) = F\left(\frac{t_2-t_1}{\varepsilon}\right)$$

$\varepsilon =$ratio of the decoherence time of the noise sources over the typical travel times between sensors.

Solution u:

$$u(t,\mathbf{x}) = \int\int n(s,\mathbf{y})G(t-s,\mathbf{x},\mathbf{y})dsd\mathbf{y}$$

Green's function:

$$\frac{1}{c^2(\mathbf{x})}\frac{\partial^2 G}{\partial t^2} - \Delta_{\mathbf{x}} G = \delta(t)\delta(\mathbf{x}-\mathbf{y})$$

starting from $G(0,\mathbf{x},\mathbf{y}) = \partial_t G(0,\mathbf{x},\mathbf{y}) = 0$.

Cross correlation

Empirical cross correlation:

$$C_T(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{T} \int_0^T u(t, \mathbf{x}_1) u(t+\tau, \mathbf{x}_2) dt$$

1. The expectation of C_T (with respect to the distribution of the sources) is independent of the integration time T:

$$\langle C_T(\tau, \mathbf{x}_1, \mathbf{x}_2) \rangle = C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2)$$

where $C^{(1)}$ is given by

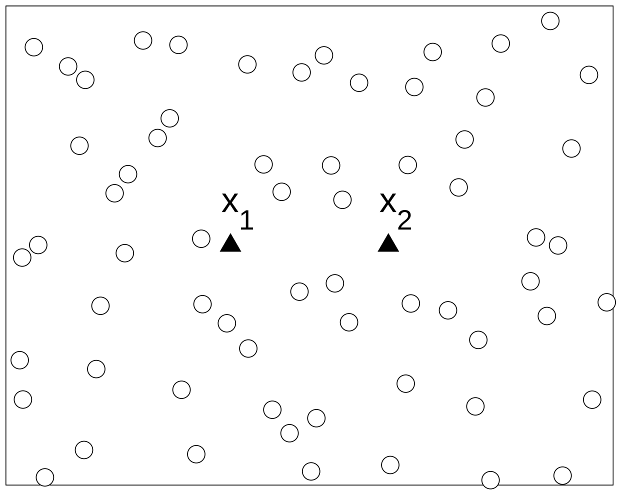
$$C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \int d\mathbf{y} \theta(\mathbf{y}) \int d\omega \overline{\hat{G}}(\omega, \mathbf{x}_1, \mathbf{y}) \hat{G}(\omega, \mathbf{x}_2, \mathbf{y}) \hat{F}^{\varepsilon}(\omega) e^{-i\omega\tau}$$

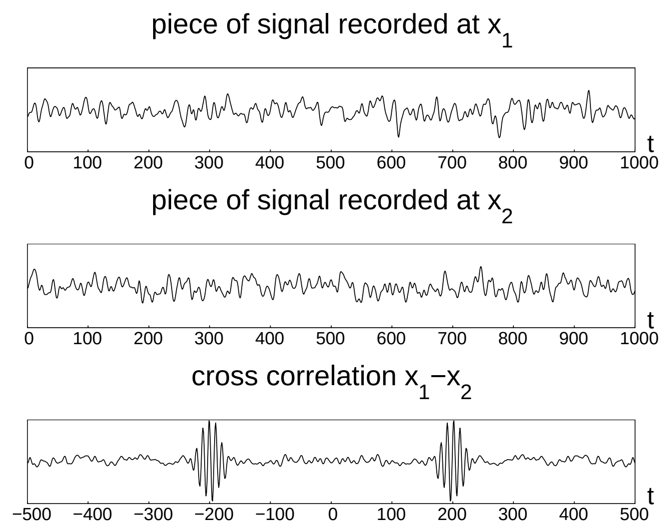
2. The empirical cross correlation is a self-averaging quantity:

$$C_T(\tau, \mathbf{x}_1, \mathbf{x}_2) \stackrel{T \to \infty}{\longrightarrow} C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2)$$

in probability. More precisely, the fluctuations of C_T around its expectation $C^{(1)}$ are of order $T^{-1/2}$.

Source distribution over all space, in a homogeneous medium





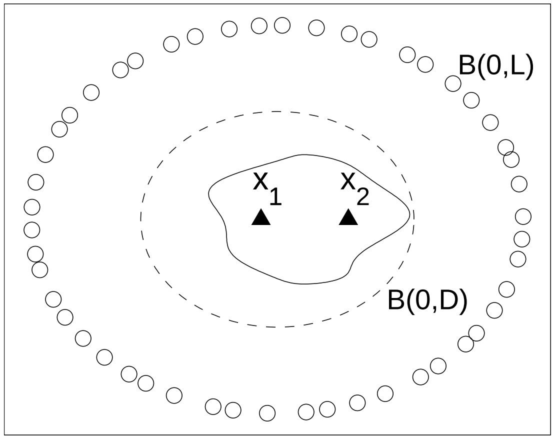
$$\frac{1}{c_0^2}\left(\frac{1}{T_a}+\frac{\partial}{\partial t}\right)^2 u - \Delta_{\mathbf{x}} u = n(t,\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d$$

$$\langle n(t_1,\mathbf{y}_1) n(t_2,\mathbf{y}_2)\rangle = F^{\varepsilon}(t_2-t_1)\delta(\mathbf{y}_2-\mathbf{y}_1), \quad \mathbf{y}_1,\mathbf{y}_2\in\mathbb{R}^d$$

We have (up to a multiplicative constant):

$$\frac{\partial}{\partial\tau}C^{(1)}(\tau,\mathbf{x}_1,\mathbf{x}_2) = e^{-\frac{|\mathbf{x}_1-\mathbf{x}_2|}{c_0 T_a}}\left[F^{\varepsilon} * G(\tau,\mathbf{x}_1,\mathbf{x}_2) - F^{\varepsilon} * G(-\tau,\mathbf{x}_1,\mathbf{x}_2)\right]$$

Sources distributed on a closed surface



$$C^{(1)}(\tau,\mathbf{x}_1,\mathbf{x}_2) = \int d\omega \int_{\partial B(\mathbf{0},L)} dS(\mathbf{y}) \overline{\hat{G}}(\omega,\mathbf{x}_1,\mathbf{y}) \hat{G}(\omega,\mathbf{x}_2,\mathbf{y}) \hat{F}^{\varepsilon}(\omega) e^{-i\omega\tau}$$

Helmholtz-Kirchhoff theorem: If the medium is homogeneous (velocity c_e) outside $B(\mathbf{0},D)$, then $\forall \mathbf{x}_1,\mathbf{x}_2 \in B(\mathbf{0},D)$ we have for $L \gg D$:

$$\hat{G}(\omega,\mathbf{x}_1,\mathbf{x}_2) - \overline{\hat{G}}(\omega,\mathbf{x}_1,\mathbf{x}_2) = \frac{2i\omega}{c_e} \int_{\partial B(\mathbf{0},L)} dS(\mathbf{y}) \overline{\hat{G}}(\omega,\mathbf{x}_1,\mathbf{y}) \hat{G}(\omega,\mathbf{x}_2,\mathbf{y})$$

If 1) the medium is homogeneous outside $B(\mathbf{0},D)$,

2) the sources are distributed uniformly on the sphere $\partial B(\mathbf{0},L)$, with $L \gg D$.

Then for any $\mathbf{x}_1,\mathbf{x}_2 \in B(\mathbf{0},D)$ we have (up to a multiplicative constant):

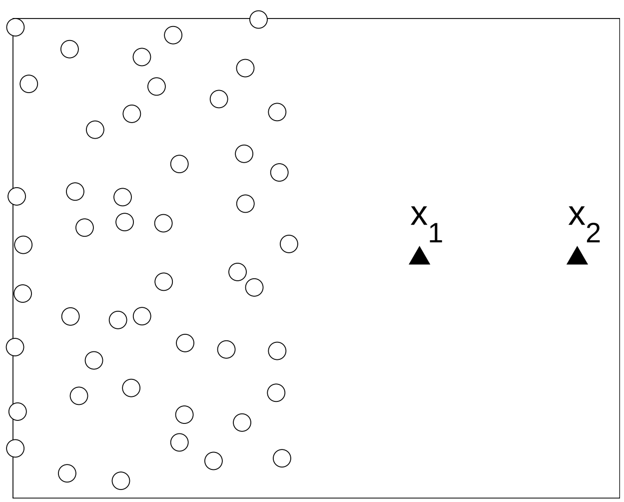
$$\frac{\partial}{\partial\tau} C^{(1)}(\tau,\mathbf{x}_1,\mathbf{x}_2) = F^{\varepsilon} * G(\tau,\mathbf{x}_1,\mathbf{x}_2) - F^{\varepsilon} * G(-\tau,\mathbf{x}_1,\mathbf{x}_2)$$

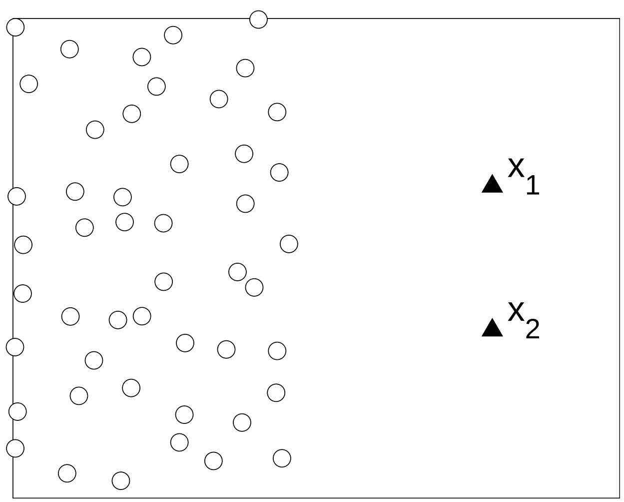
Summary: if the fields at the sensors are incoherent superpositions of waves coming from all directions, then

$$\frac{\partial}{\partial \tau} C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) \simeq F^{\varepsilon} * G(\tau, \mathbf{x}_1, \mathbf{x}_2) - F^{\varepsilon} * G(-\tau, \mathbf{x}_1, \mathbf{x}_2)$$

Keyword: ”Directional diversity”

What about situations such as





?

Simple geometry in a smoothly varying background

Simple geometry hypothesis: $c(\mathbf{x})$ is smooth and there is a unique ray between any pair of points (in the region of interest).

We use the WKB (geometric optics) approximation:

$$\hat{G}\Big(\frac{\omega}{\varepsilon},\mathbf{x},\mathbf{y}\Big) \sim a(\mathbf{x},\mathbf{y})e^{i\frac{\omega}{\varepsilon}\tau(\mathbf{x},\mathbf{y})}$$

valid when $\varepsilon \ll 1$, where the travel time is

$$\tau(\mathbf{x},\mathbf{y}) = \inf\Big\{T \text{ s.t. } \exists(\mathbf{X}_t)_{t\in[0,T]} \in \mathcal{C}^1, \mathbf{X}_0=\mathbf{x}, \mathbf{X}_T=\mathbf{y}, \Big|\frac{d\mathbf{X}_t}{dt}\Big| = c(\mathbf{X}_t)\Big\}$$

Localized sources: stationary phase analysis

$F^\varepsilon(t) = F(\frac{t}{\varepsilon}) \Longrightarrow \hat{F}^\varepsilon(\omega) = \varepsilon \hat{F}(\varepsilon\omega)$.

$$\begin{aligned} C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) &= \frac{1}{2\pi} \int d\mathbf{y} \theta(\mathbf{y}) \int d\omega \overline{\hat{G}}(\omega, \mathbf{x}_1, \mathbf{y}) \hat{G}(\omega, \mathbf{x}_2, \mathbf{y}) e^{-i\omega\tau} \varepsilon \hat{F}(\varepsilon\omega) \\ &= \frac{1}{2\pi} \int d\mathbf{y} \theta(\mathbf{y}) \int d\omega \overline{\hat{G}}\Big(\frac{\omega}{\varepsilon}, \mathbf{x}_1, \mathbf{y}\Big) \hat{G}\Big(\frac{\omega}{\varepsilon}, \mathbf{x}_2, \mathbf{y}\Big) e^{-i\frac{\omega}{\varepsilon}\tau} \hat{F}(\omega) \end{aligned}$$

WKB approximation for $\hat{G}$:

$$C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2\pi} \int d\mathbf{y} \theta(\mathbf{y}) \int d\omega \hat{F}(\omega) \overline{a}(\mathbf{x}_1, \mathbf{y}) a(\mathbf{x}_2, \mathbf{y}) e^{i\frac{\omega}{\varepsilon}\mathcal{T}(\mathbf{y})}$$

with the rapid phase

$$\omega\mathcal{T}(\mathbf{y}) = \omega[\tau(\mathbf{x}_2, \mathbf{y}) - \tau(\mathbf{x}_1, \mathbf{y}) - \tau]$$

Use of the stationary phase theorem. The dominant contribution comes from the stationary points $(\omega, \mathbf{y})$ satisfying:

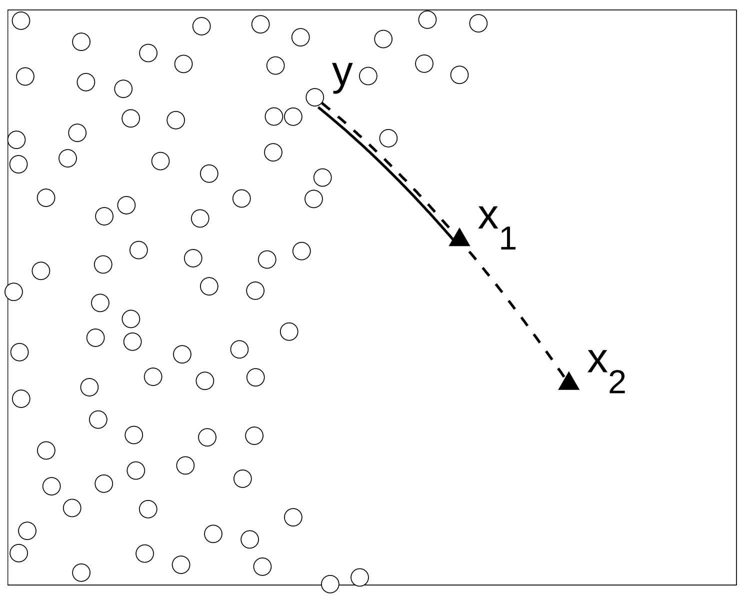
$$\partial_\omega\Big(\omega\mathcal{T}(\mathbf{y})\Big) = 0, \quad \nabla_{\mathbf{y}}\Big(\omega\mathcal{T}(\mathbf{y})\Big) = \mathbf{0}$$

$\hookrightarrow$ two conditions:

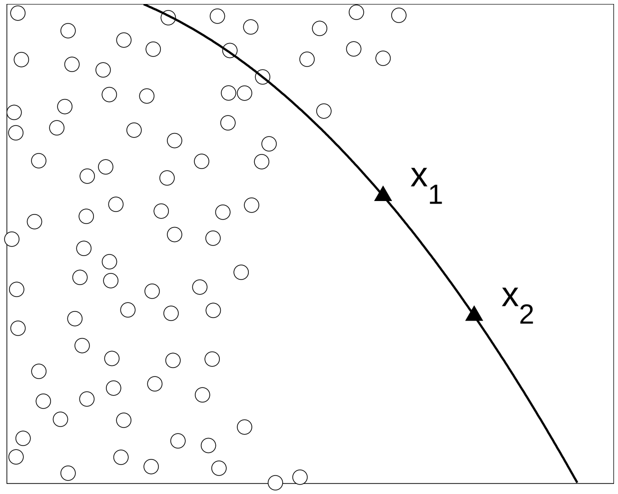
$$\tau(\mathbf{x}_2, \mathbf{y}) - \tau(\mathbf{x}_1, \mathbf{y}) = \tau, \quad \nabla_{\mathbf{y}}\tau(\mathbf{y}, \mathbf{x}_2) = \nabla_{\mathbf{y}}\tau(\mathbf{y}, \mathbf{x}_1)$$

$$\tau(\mathbf{x}_2,\mathbf{y})-\tau(\mathbf{x}_1,\mathbf{y})=\tau, \quad \nabla_{\mathbf{y}}\tau(\mathbf{y},\mathbf{x}_2)=\nabla_{\mathbf{y}}\tau(\mathbf{y},\mathbf{x}_1)$$

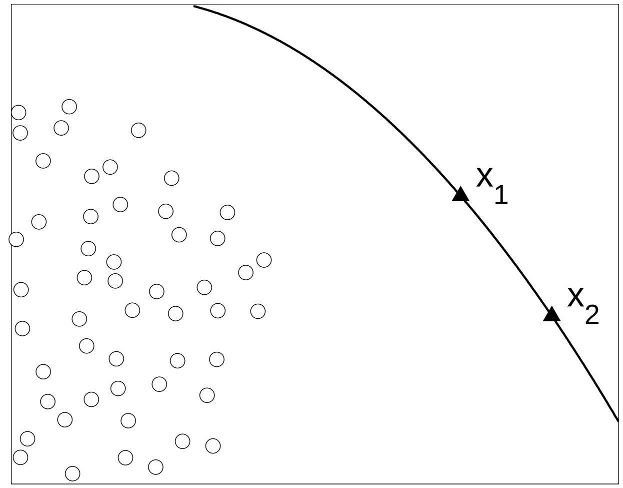
$\Longrightarrow \mathbf{x}_1$ and $\mathbf{x}_2$ are on the same ray issuing from $\mathbf{y}$ and $\tau = \pm\tau(\mathbf{x}_1,\mathbf{x}_2)$.



Also: $\mathbf{y}$ should be in the support of θ



Singular component at $\tau(\mathbf{x}_1, \mathbf{x}_2)$



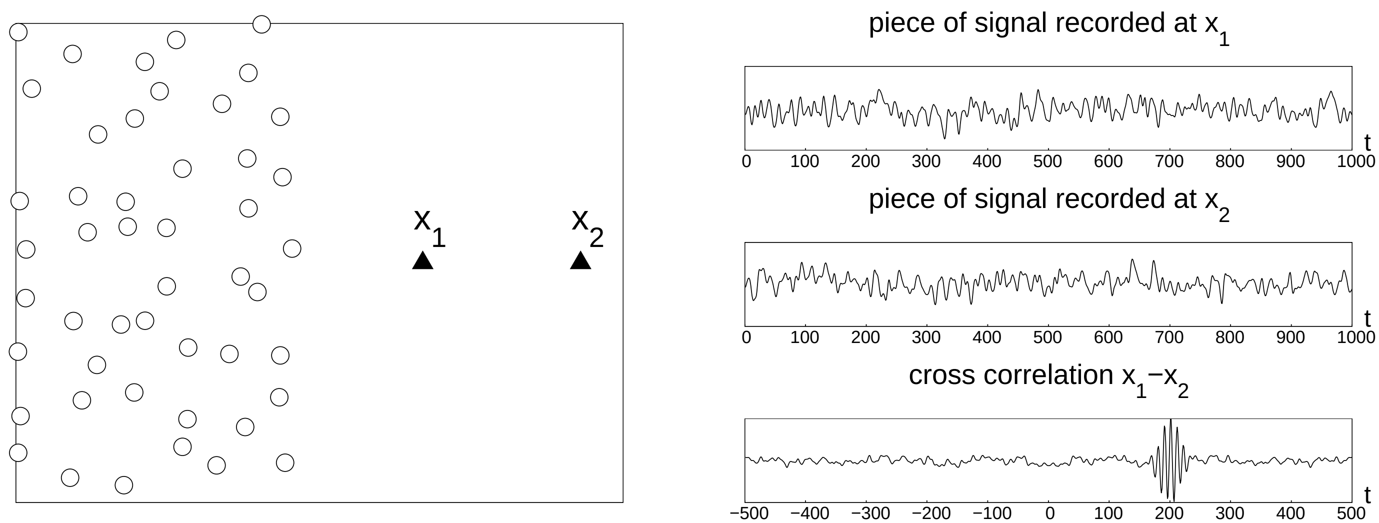
No singular component

Conclusion: The cross correlation $C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2)$ has singular components iff the ray joining $\mathbf{x}_1$ and $\mathbf{x}_2$ reaches into the source region (i.e. the support of θ). Then there are one or two singular components at $\tau = \pm\tau(\mathbf{x}_1, \mathbf{x}_2)$.

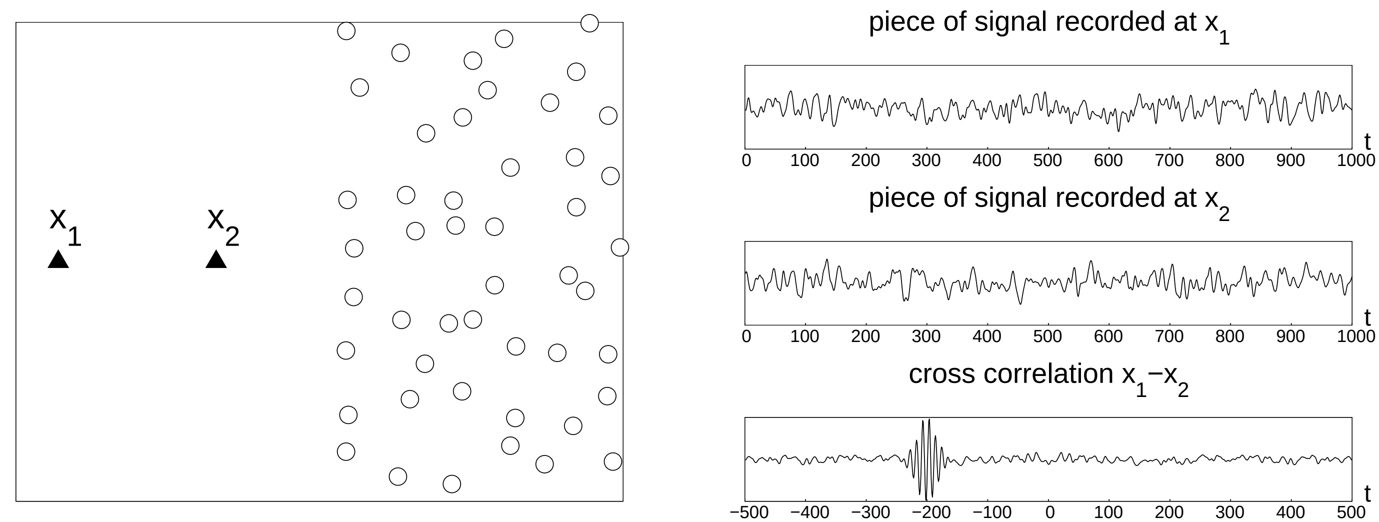
More exactly:

the rays $\mathbf{y} \to \mathbf{x}_1 \to \mathbf{x}_2$ contribute to the singular component at $\tau = \tau(\mathbf{x}_1, \mathbf{x}_2)$,

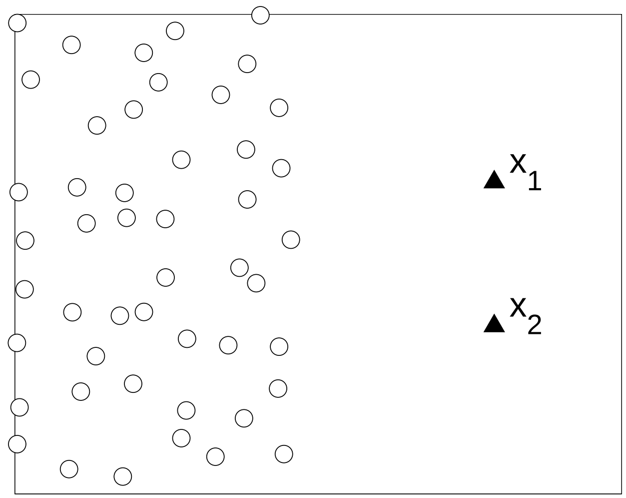
the rays $\mathbf{y} \to \mathbf{x}_2 \to \mathbf{x}_1$ contribute to the singular component at $\tau = -\tau(\mathbf{x}_1, \mathbf{x}_2)$.

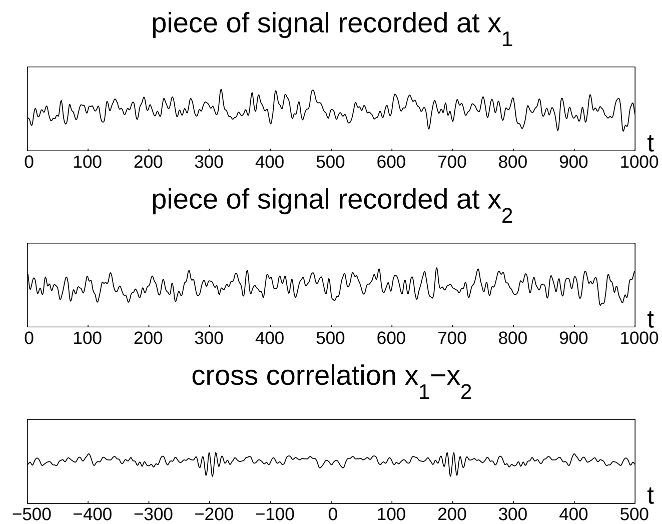


x1
x2
piece of signal recorded at x1
t
0 100 200 300 400 500 600 700 800 900 1000
piece of signal recorded at x2
t
0 100 200 300 400 500 600 700 800 900 1000
cross correlation x1−x2
t
−500 −400 −300 −200 −100 0 100 200 300 400 500



x1
x2
piece of signal recorded at x1
t
0 100 200 300 400 500 600 700 800 900 1000
piece of signal recorded at x2
t
0 100 200 300 400 500 600 700 800 900 1000
cross correlation x1−x2
t
−500 −400 −300 −200 −100 0 100 200 300 400 500



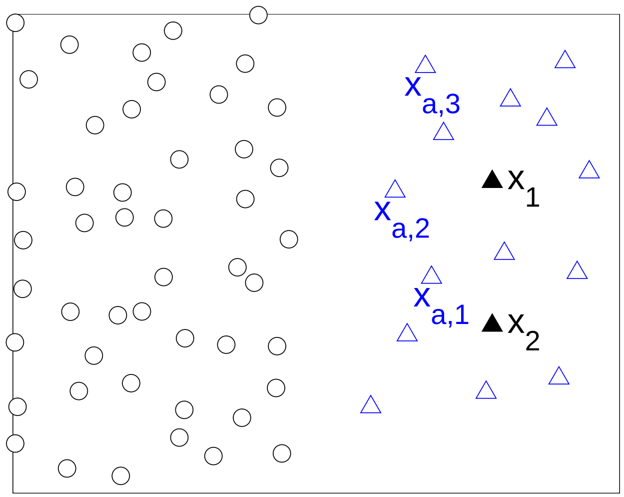


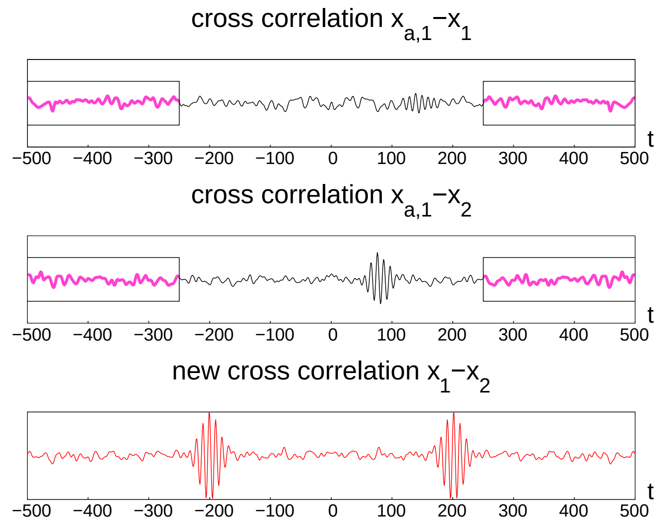
Here, the cross correlation method does not allow for travel time estimation, because there is not enough ”directional diversity”.

Idea: exploit the scattering properties of the medium.

- The scattered waves have more directional diversity than the direct waves from the noise sources.
- The contributions of the scattered waves are in the tails of the cross correlations.
- By cross correlating the tails of the cross correlations, it is possible to exploit scattered waves and their enhanced directional diversity (first suggested by M. Campillo).

Part II: Fourth-order cross correlations for travel time estimation





Use of auxiliary sensors $\mathbf{x}_{a,j}$, $j = 1, \dots, N$. Algorithm:

1) compute the cross correlations $C_T(\tau, \mathbf{x}_{a,j}, \mathbf{x}_1)$ and $C_T(\tau, \mathbf{x}_{a,j}, \mathbf{x}_2)$, for each j:

$$C_T(\tau, \mathbf{x}_{a,j}, \mathbf{x}_1) = \frac{1}{T}\int_0^T u(t, \mathbf{x}_{a,j}) u(t+\tau, \mathbf{x}_1) dt \,, \quad C_T(\tau, \mathbf{x}_{a,j}, \mathbf{x}_2) = \frac{1}{T}\int_0^T u(t, \mathbf{x}_{a,j}) u(t+\tau, \mathbf{x}_2) dt$$

2) consider the tails of $C_T(\tau, \mathbf{x}_{a,j}, \mathbf{x}_1)$ and $C_T(\tau, \mathbf{x}_{a,j}, \mathbf{x}_2)$:

$$C_{T,\mathrm{coda}}(\tau, \mathbf{x}_{a,j}, \mathbf{x}_l) = C_T(\tau, \mathbf{x}_{a,j}, \mathbf{x}_l) \left[\mathbf{1}_{(-\infty, -T_{\mathrm{coda}})}(\tau) + \mathbf{1}_{(T_{\mathrm{coda}}, \infty)}(\tau) \right]$$

3) compute the cross correlations between the tails and sum over j:

$$C^{(3)}_{T',T}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \sum_{j=1}^{N} \int_{-T'}^{T'} C_{T,\mathrm{coda}}(\tau', \mathbf{x}_{a,j}, \mathbf{x}_1) C_{T,\mathrm{coda}}(\tau' + \tau, \mathbf{x}_{a,j}, \mathbf{x}_2) d\tau'$$

Model for the scattering medium

Medium characterized by a smoothly varying background velocity $c_0(\mathbf{x})$ and a clutter, modeled by a collection of localized scatterers around $(\mathbf{z}_j)_{j\geq 1}$.

Full Green's function:

$$\hat{G}(\omega,\mathbf{x},\mathbf{y}) = \hat{G}_0(\omega,\mathbf{x},\mathbf{y}) + \hat{G}_1(\omega,\mathbf{x},\mathbf{y})$$

with $\hat{G}_0$ the Green's function of the background medium c_0:

$$\frac{\omega^2}{c_0^2(\mathbf{x})}\hat{G}_0 + \Delta_{\mathbf{x}}\hat{G}_0 = -\delta(\mathbf{x}-\mathbf{y})$$

and (Born approximation)

$$\hat{G}_1(\omega,\mathbf{x},\mathbf{y}) = \omega^2 \sum_j \hat{G}_0(\omega,\mathbf{x},\mathbf{z}_j)\sigma_j \hat{G}_0(\omega,\mathbf{z}_j,\mathbf{y})$$

We look for an estimate of c_0, or an estimate of

$$\tau(\mathbf{x},\mathbf{y}) = \inf\left\{T \text{ s.t. } \exists(\mathbf{X}_t)_{t\in[0,T]} \in \mathcal{C}^1, \mathbf{X}_0 = \mathbf{x}, \mathbf{X}_T = \mathbf{y}, \left|\frac{d\mathbf{X}_t}{dt}\right| = c_0(\mathbf{X}_t)\right\}$$

Simplifications:

1) The distribution of the auxiliary sensors $(\mathbf{x}_{a,j})_{j=1,\ldots,N}$ is dense
$\rightarrow$ continuous approximation with the density $\chi_a(\mathbf{x}_a)$.

$$\sum_j \psi(\mathbf{x}_{a,j}) \simeq \int d\mathbf{x}_a \chi_a(\mathbf{x}_a)\psi(\mathbf{x}_a)$$

for any test function ψ.

2) The distribution of the scatterers $(\mathbf{z}_j)_{j\geq 1}$ is dense
$\rightarrow$ continuous approximation with the density $\chi_s(\mathbf{z}_s)$:

$$\sum_j \psi(\mathbf{z}_j) \simeq \int d\mathbf{z}_s \chi_s(\mathbf{z}_s)\psi(\mathbf{z}_s)$$

3) Scattering amplitudes $(\sigma_j)_{j\geq 1}$ of the scatterers are independent and identically distributed with $\mathbb{E}[\sigma_j^2] = \sigma^2$.

Stationary phase analysis of the cross correlation $C^{(3)}$

Using the Born approximation for $\hat{G} = \hat{G}_0 + \hat{G}_1$ and the WKB approximation for $\hat{G}_0$:

$$\begin{aligned} C_1^{(3)}(\tau, \mathbf{x}, \mathbf{y}) &= \frac{\sigma^2}{2\pi} \int d\mathbf{x}_a d\mathbf{y}_1 d\mathbf{y}_2 d\mathbf{z}_s \chi_a(\mathbf{x}_a) \theta(\mathbf{y}_1) \theta(\mathbf{y}_2) \chi_s(\mathbf{z}_s) \int d\omega \omega^4 \hat{F}(\omega)^2 \\ &\times a(\mathbf{x}_a, \mathbf{y}_1) \overline{a}(\mathbf{x}_1, \mathbf{z}_s) \overline{a}(\mathbf{z}_s, \mathbf{y}_1) \overline{a}(\mathbf{x}_a, \mathbf{y}_2) a(\mathbf{x}_2, \mathbf{z}_s) a(\mathbf{z}_s, \mathbf{y}_2) e^{i \frac{\omega}{\varepsilon} \mathcal{T}(\mathbf{x}_a, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}_s)} \end{aligned}$$

with the rapid phase

$$\begin{aligned} \omega \mathcal{T}(\mathbf{x}_a, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}_s) &= \omega \big[\tau(\mathbf{x}_a, \mathbf{y}_1) - \tau(\mathbf{x}_1, \mathbf{z}_s) - \tau(\mathbf{z}_s, \mathbf{y}_1) \\ &\quad - \tau(\mathbf{x}_a, \mathbf{y}_2) + \tau(\mathbf{x}_2, \mathbf{z}_s) + \tau(\mathbf{z}_s, \mathbf{y}_2) - \tau \big] \end{aligned}$$

Stationary points:

$$\big(\partial_\omega, \nabla_{\mathbf{x}_a}, \nabla_{\mathbf{y}_1}, \nabla_{\mathbf{y}_2}, \nabla_{\mathbf{z}_s}\big) \big(\omega \mathcal{T}(\mathbf{x}_a, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}_s)\big) = \big(0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\big)$$

$\hookrightarrow$ five conditions:

$$\tau(\mathbf{x}_a, \mathbf{y}_1) - \tau(\mathbf{x}_1, \mathbf{z}_s) - \tau(\mathbf{z}_s, \mathbf{y}_1) - \tau(\mathbf{x}_a, \mathbf{y}_2) + \tau(\mathbf{x}_2, \mathbf{z}_s) + \tau(\mathbf{z}_s, \mathbf{y}_2) = \tau$$

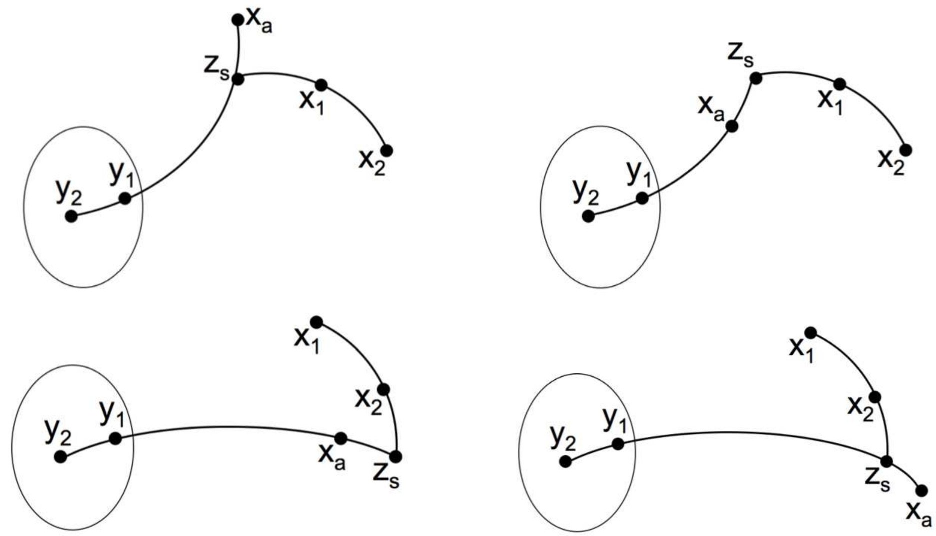
$$\nabla_{\mathbf{x}_a} \tau(\mathbf{x}_a, \mathbf{y}_1) = \nabla_{\mathbf{x}_a} \tau(\mathbf{x}_a, \mathbf{y}_2)$$

$$\nabla_{\mathbf{y}_1} \tau(\mathbf{y}_1, \mathbf{x}_a) = \nabla_{\mathbf{y}_1} \tau(\mathbf{y}_1, \mathbf{z}_s)$$

$$\nabla_{\mathbf{y}_2} \tau(\mathbf{y}_2, \mathbf{x}_a) = \nabla_{\mathbf{y}_2} \tau(\mathbf{y}_2, \mathbf{z}_s)$$

$$\nabla_{\mathbf{z}_s} \tau(\mathbf{z}_s, \mathbf{y}_1) + \nabla_{\mathbf{z}_s} \tau(\mathbf{z}_s, \mathbf{x}_1) = \nabla_{\mathbf{z}_s} \tau(\mathbf{z}_s, \mathbf{y}_2) + \nabla_{\mathbf{z}_s} \tau(\mathbf{z}_s, \mathbf{x}_2)$$

There are stationary points:



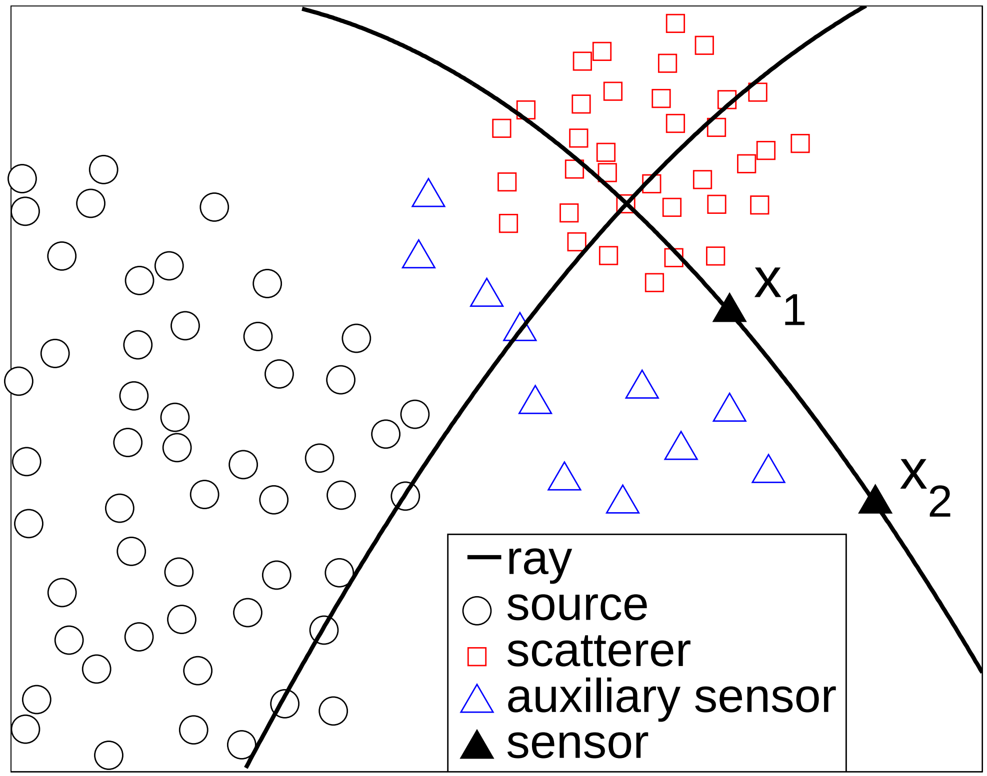
Conclusion: $C^{(3)}$ has singular components if:

1) there are scatterers $\mathbf{z}_s$ along the ray joining $\mathbf{x}_1$ and $\mathbf{x}_2$ (but not between $\mathbf{x}_1$ and $\mathbf{x}_2$).

2) there are auxiliary sensors $\mathbf{x}_a$ along rays joining sources $\mathbf{y}_1, \mathbf{y}_2$ and scatterers $\mathbf{z}_s$.

These singular components are at $\tau = \pm\tau(\mathbf{x}_1, \mathbf{x}_2)$.

It is not required that the ray joining $\mathbf{x}_1$ and $\mathbf{x}_2$ reaches into the source region !



Here:

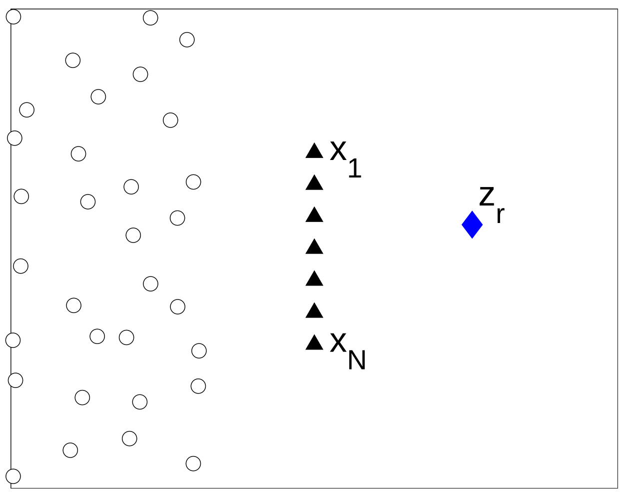
It is not possible to extract the travel time $\tau(\mathbf{x}_1, \mathbf{x}_2)$ from $C^{(1)}$

It is possible to extract the travel time $\tau(\mathbf{x}_1, \mathbf{x}_2)$ from $C^{(3)}$

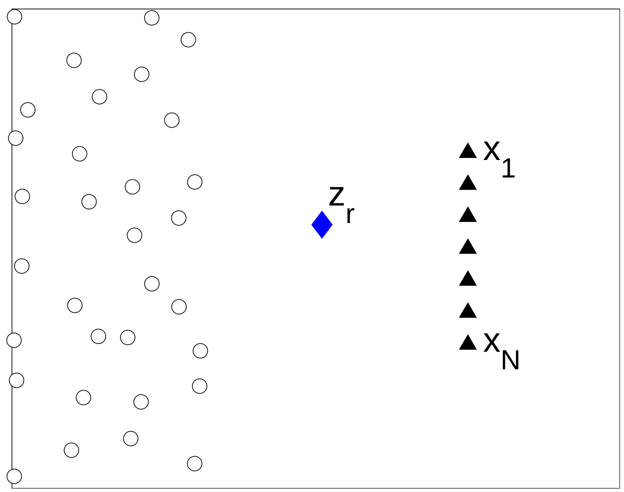
Part III: Passive imaging by cross correlation of noisy signals

- Network or array of passive sensors $\mathbf{x}_j,\ j = 1, \ldots, N$
- Ambient noise sources emitting stationary random signals
- Target (small reflector) at $\mathbf{z}_\mathrm{r}$

- Different light configurations



Daylight configuration



Backlight configuration

- Two types of situations:
- Data in the absence (C_0) and in the presence (C) of the reflector
- Data only in the presence of the reflector

- We know the background medium: the travel times between the sensors and points in the region around $\mathbf{z}_\mathrm{r}$ are known.

Identification of the singular components of the cross correlations

Using Born and WKB approximations

$$\hat{G}_0\left(\frac{\omega}{\varepsilon},\mathbf{x}_1,\mathbf{x}_2\right) \sim a(\mathbf{x}_1,\mathbf{x}_2)e^{i\frac{\omega}{\varepsilon}\tau(\mathbf{x}_1,\mathbf{x}_2)}$$

$$\hat{G}\left(\frac{\omega}{\varepsilon},\mathbf{x}_1,\mathbf{x}_2\right) \sim a(\mathbf{x}_1,\mathbf{x}_2)e^{i\frac{\omega}{\varepsilon}\tau(\mathbf{x}_1,\mathbf{x}_2)}+\omega^2 a(\mathbf{x}_1,\mathbf{x}_\mathrm{r})\sigma_\mathrm{r} a(\mathbf{z}_\mathrm{r},\mathbf{x}_2)e^{i\frac{\omega}{\varepsilon}[\tau(\mathbf{x}_1,\mathbf{z}_\mathrm{r})+\tau(\mathbf{z}_\mathrm{r},\mathbf{x}_2)]}$$

Differential cross correlation

$$C^{(1)}(\tau,\mathbf{x}_1,\mathbf{x}_2)-C_0^{(1)}(\tau,\mathbf{x}_1,\mathbf{x}_2)=\frac{\sigma_\mathrm{r}}{2\pi}\int d\mathbf{y}\theta(\mathbf{y})\int d\omega\,\omega^2\hat{F}(\omega)\overline{a}(\mathbf{x}_1,\mathbf{y})a(\mathbf{x}_2,\mathbf{z}_\mathrm{r})a(\mathbf{z}_\mathrm{r},\mathbf{y})e^{i\frac{\omega}{\varepsilon}\mathcal{T}(\mathbf{y})}+\dots$$

with the rapid phase

$$\omega\mathcal{T}(\mathbf{y})=\omega[\tau(\mathbf{y},\mathbf{x}_2)-\tau(\mathbf{y},\mathbf{z}_\mathrm{r})-\tau(\mathbf{z}_\mathrm{r},\mathbf{x}_1)-\tau]$$

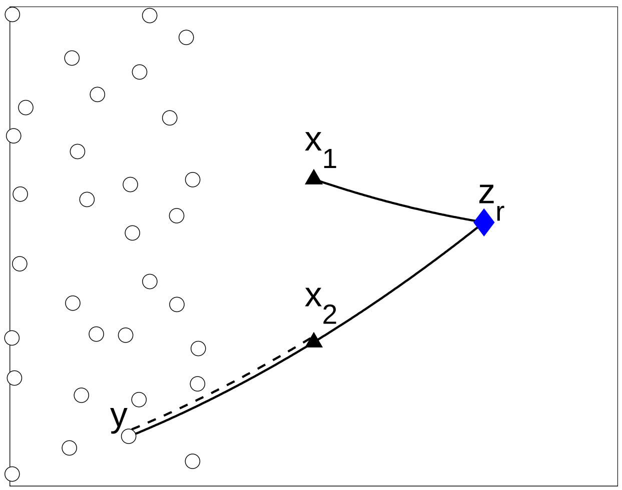
The dominant contribution comes from the stationary points $(\omega,\mathbf{y})$ satisfying

$$\partial_\omega\Big(\omega\mathcal{T}(\mathbf{y})\Big)=0\,,\quad \nabla_\mathbf{y}\Big(\omega\mathcal{T}(\mathbf{y})\Big)=\mathbf{0}$$

which gives the conditions

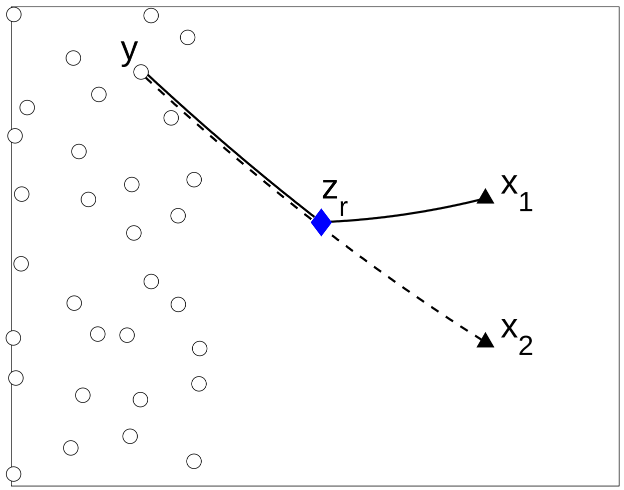
$$\tau(\mathbf{y},\mathbf{x}_2)-\tau(\mathbf{y},\mathbf{z}_\mathrm{r})-\tau(\mathbf{z}_\mathrm{r},\mathbf{x}_1)=\tau\,,\quad \nabla_\mathbf{y}\tau(\mathbf{y},\mathbf{x}_2)=\nabla_\mathbf{y}\tau(\mathbf{y},\mathbf{z}_\mathrm{r})$$

$$\tau(\mathbf{y},\mathbf{x}_2)-\tau(\mathbf{y},\mathbf{z}_\mathrm{r})-\tau(\mathbf{z}_\mathrm{r},\mathbf{x}_1)=\tau, \quad \nabla_\mathbf{y}\tau(\mathbf{y},\mathbf{x}_2)=\nabla_\mathbf{y}\tau(\mathbf{y},\mathbf{z}_\mathrm{r})$$



Daylight

$$-\tau(\mathbf{x}_2,\mathbf{z}_\mathrm{r})-\tau(\mathbf{z}_\mathrm{r},\mathbf{x}_1)=\tau$$



Backlight

$$\tau(\mathbf{x}_2,\mathbf{z}_\mathrm{r})-\tau(\mathbf{z}_\mathrm{r},\mathbf{x}_1)=\tau$$

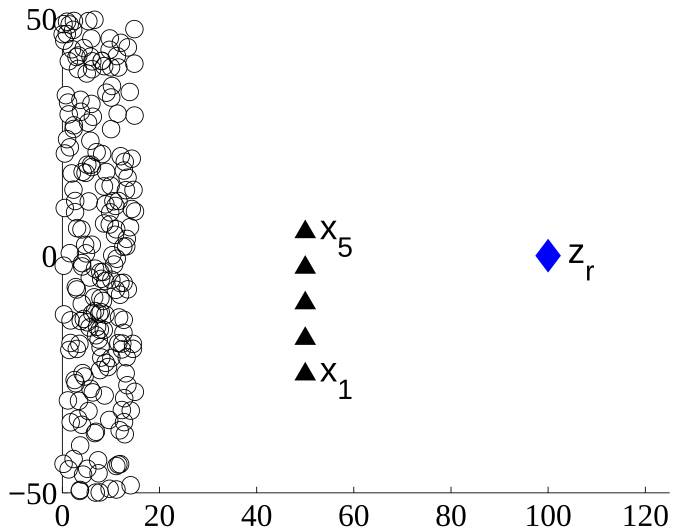
Conclusion:

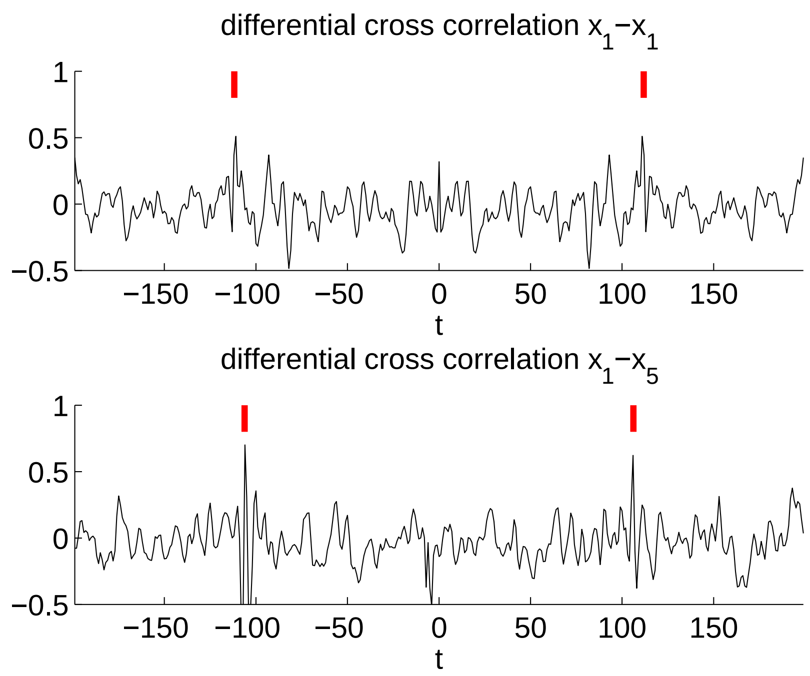
1) In a daylight imaging configuration, the singular components of $C^{(1)}-C_0^{(1)}$ are at $\tau=\pm(\tau(\mathbf{x}_1,\mathbf{z}_\mathrm{r})+\tau(\mathbf{x}_2,\mathbf{z}_\mathrm{r}))$.

2) In a backlight imaging configuration, the singular component of $C^{(1)}-C_0^{(1)}$ is at $\tau=\tau(\mathbf{x}_2,\mathbf{z}_\mathrm{r})-\tau(\mathbf{x}_1,\mathbf{z}_\mathrm{r})$.

Daylight configuration

- Data in the absence ($C_0(\tau,\mathbf{x}_j,\mathbf{x}_l),\ j,l=1,\ldots,5$) and in the presence ($C(\tau,\mathbf{x}_j,\mathbf{x}_l)$, $j,l=1,\ldots,5$) of the reflector





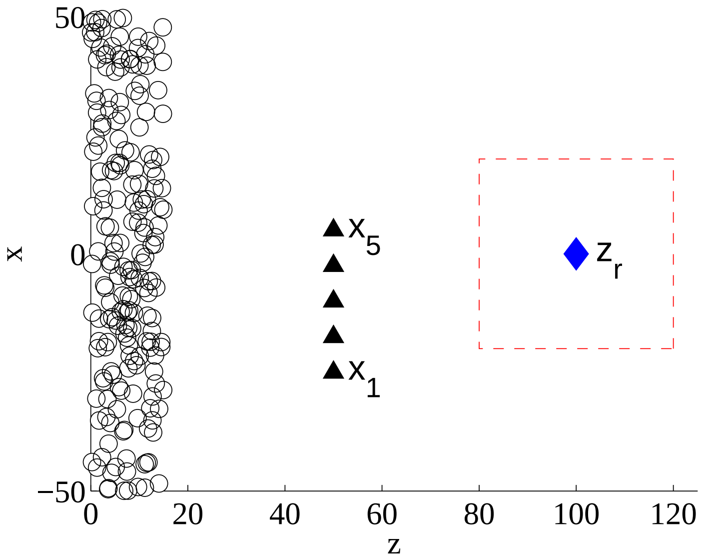
Differential cross correlations:

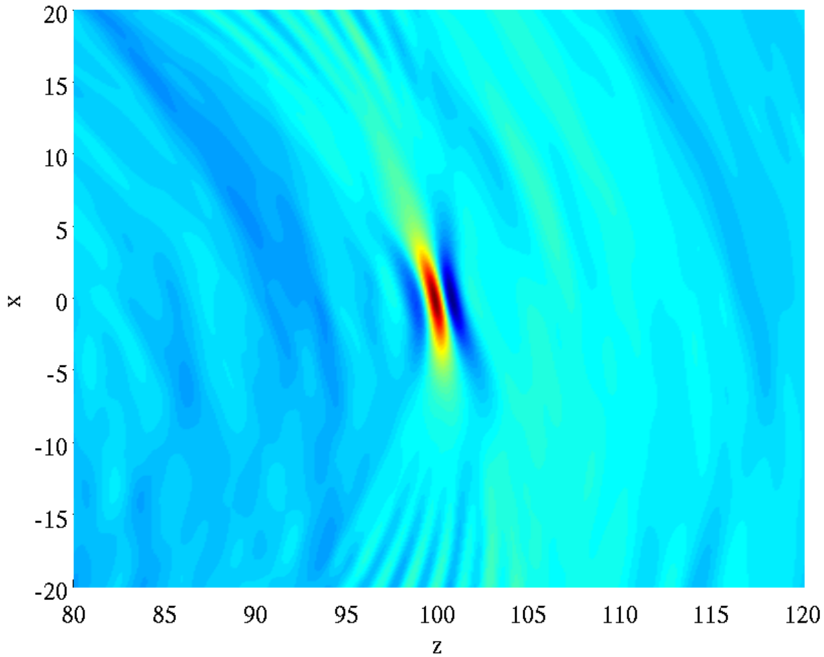
$$\Delta C(\tau,\mathbf{x}_j,\mathbf{x}_l) = (C - C_0)(\tau,\mathbf{x}_j,\mathbf{x}_l)$$

Daylight configuration - migration

Theory: ΔC has singular components at $\tau = \pm[\tau(\mathbf{x}_j, \mathbf{z}_r) + \tau(\mathbf{x}_l, \mathbf{z}_r)]$.

- Migration of the differential cross correlations $\Delta C = C - C_0$





Kirchhoff Migration (KM) functional (image) for the search point $\mathbf{z}^S$:

$$I(\mathbf{z}^S) = \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]}\widehat{\Delta C^+}(\omega,\mathbf{x}_j,\mathbf{x}_l) + e^{i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]}\widehat{\Delta C^-}(\omega,\mathbf{x}_j,\mathbf{x}_l)$$

$$\Delta C^-(\tau,\mathbf{x}_j,\mathbf{x}_l) = (C-C_0)(\tau,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(-\infty,0)}(\tau), \quad \Delta C^+(\tau,\mathbf{x}_j,\mathbf{x}_l) = (C-C_0)(\tau,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(0,\infty)}(\tau)$$

Daylight configuration - resolution

KM functional (image):

$$I(\mathbf{z}^S) \simeq \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]} \widehat{\Delta C}(\omega,\mathbf{x}_j,\mathbf{x}_l)$$

Equivalent to KM for array imaging using an array of active sensors $(\mathbf{x}_j)_{j=1,\dots,N}$ emitting broadband pulses. The data is then the impulse response matrix $(P(t,\mathbf{x}_j,\mathbf{x}_l))_{j,l=1,\dots,N}$ and the KM functional is

$$I^{\mathrm{KM}}(\mathbf{z}^S) = \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]} \widehat{P}(\omega,\mathbf{x}_j,\mathbf{x}_l)$$

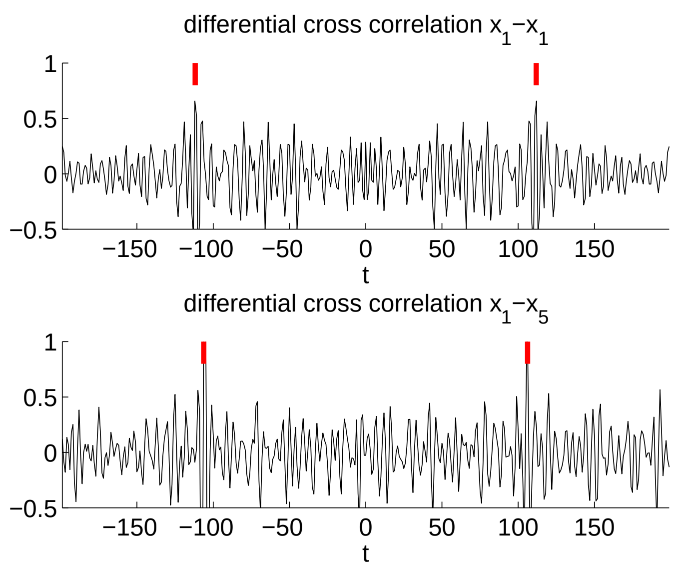
Range resolution $\simeq c_0/B$, where B is the bandwidth.

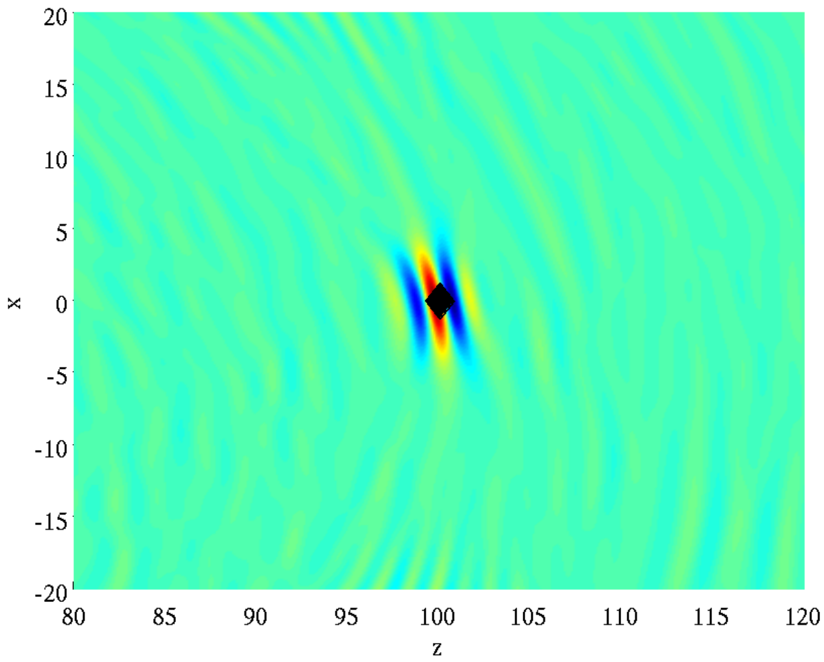
Cross range resolution (for a linear array) $\simeq \lambda_0 L/a$, where λ_0 is the carrier frequency, L is the distance from the array to the reflector, a the diameter of the array.

Cross range resolution (for a network) $\simeq c_0/B$ (triangulation).

Daylight configuration - bandwidth 50%

- Migration of the differential cross correlations $\Delta C = C - C_0$



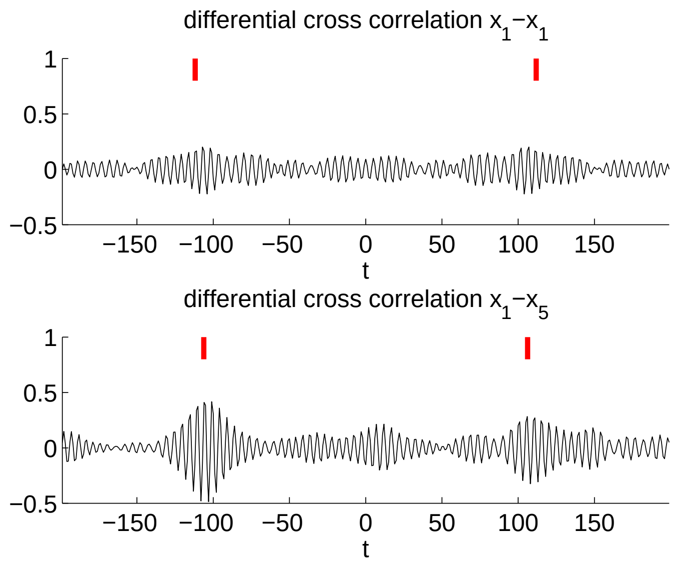


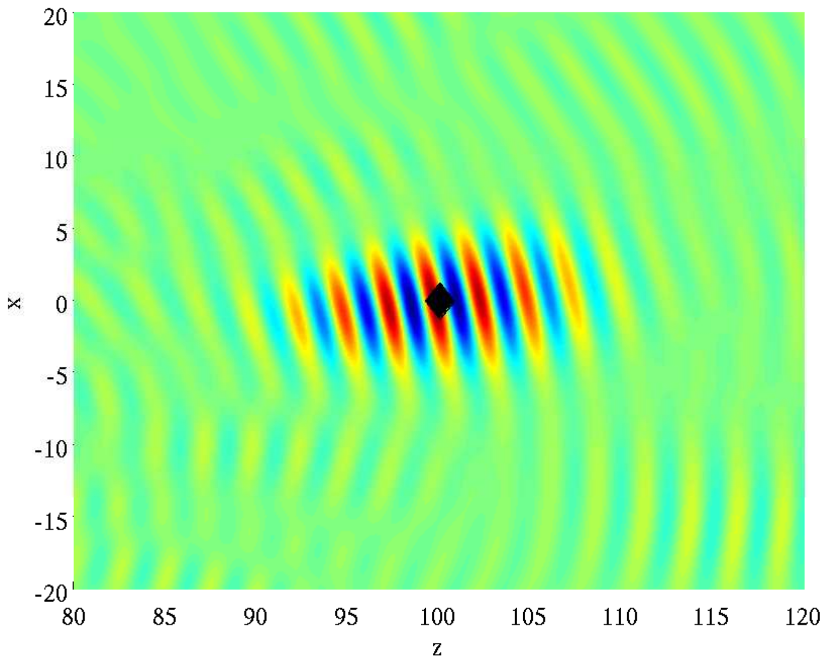
KM functional (image):

$$I(\mathbf{z}^S) = \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]} \widehat{\Delta C^+}(\omega, \mathbf{x}_j, \mathbf{x}_l) + e^{i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]} \widehat{\Delta C^-}(\omega, \mathbf{x}_j, \mathbf{x}_l)$$

Daylight configuration - bandwidth 10%

- Migration of the differential cross correlations $\Delta C = C - C_0$



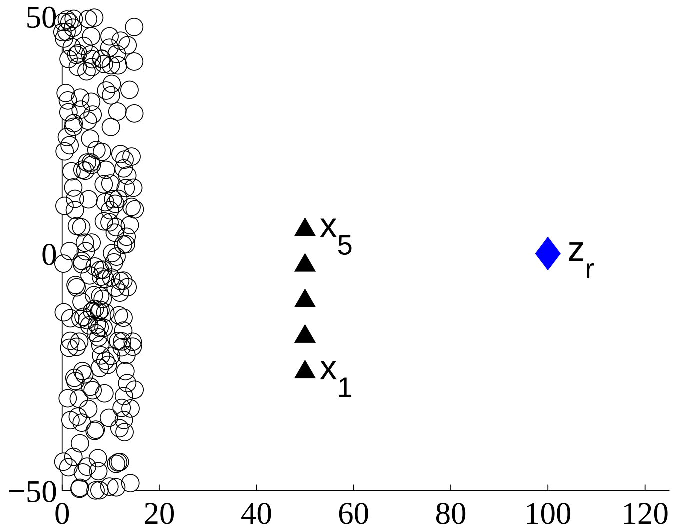


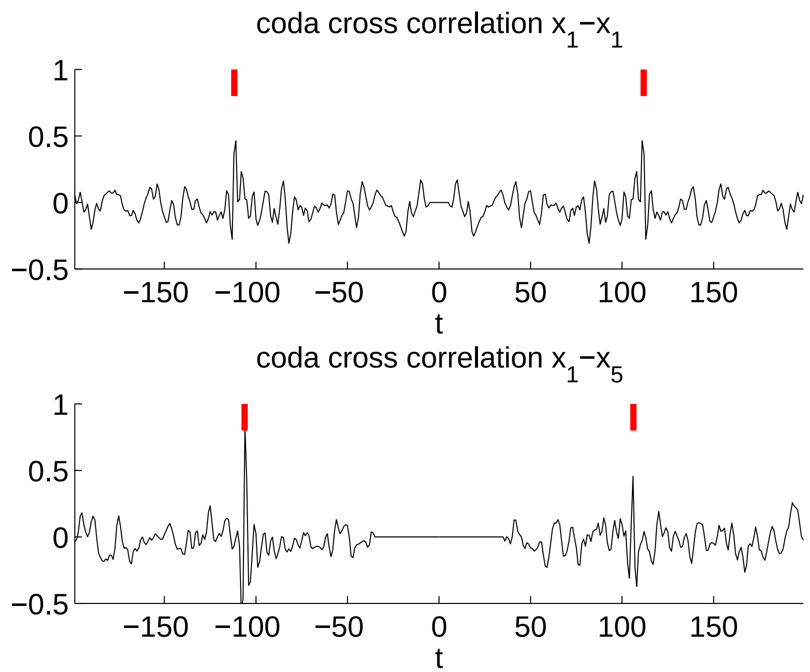
KM functional (image):

$$I(\mathbf{z}^S) = \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]}\widehat{\Delta C^{+}}(\omega,\mathbf{x}_j,\mathbf{x}_l) + e^{i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]}\widehat{\Delta C^{-}}(\omega,\mathbf{x}_j,\mathbf{x}_l)$$

Daylight configuration

- Data *only* in the presence of the reflector: $C(\tau, \mathbf{x}_j, \mathbf{x}_l)$, $j, l = 1, \ldots, 5$.



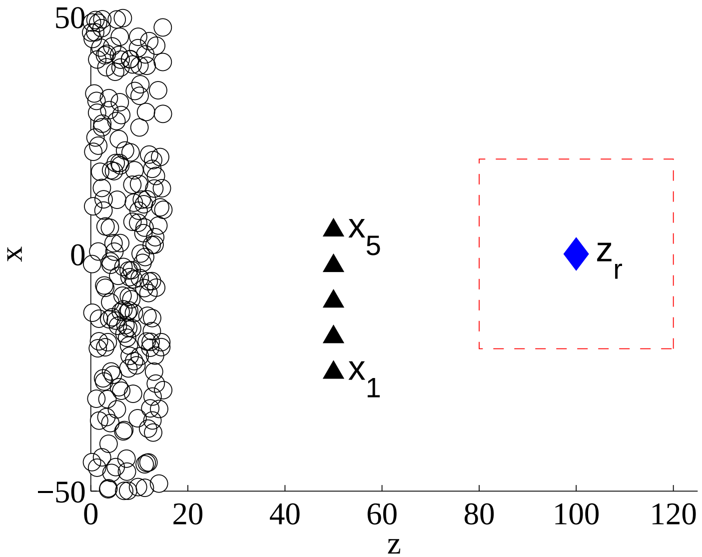


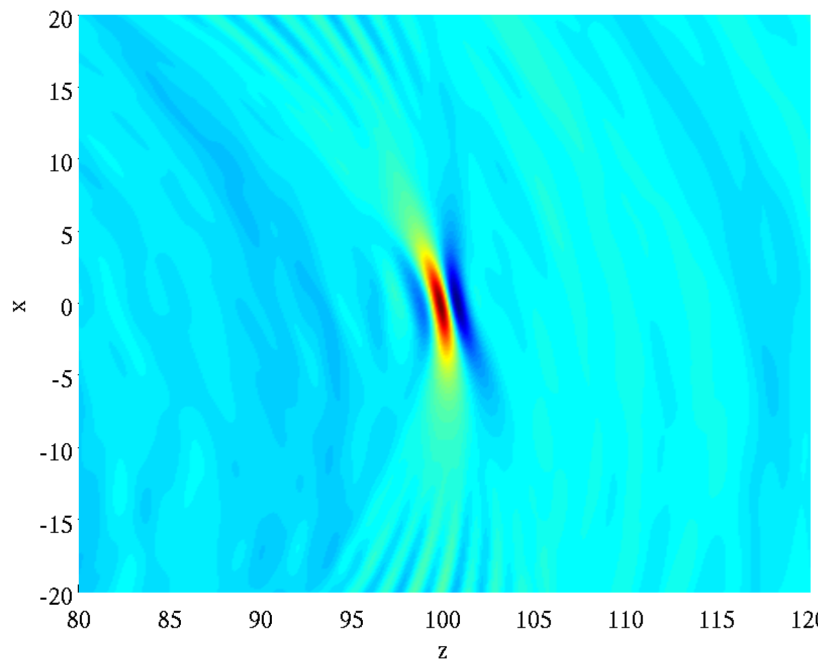
$$C_{\text{coda}}(\tau, \mathbf{x}_j, \mathbf{x}_l) = C(\tau, \mathbf{x}_j, \mathbf{x}_l)\left[\mathbf{1}_{(-\infty, -\tau(\mathbf{x}_j, \mathbf{x}_l))}(\tau) + \mathbf{1}_{(\tau(\mathbf{x}_j, \mathbf{x}_l), \infty)}(\tau)\right]$$

Daylight configuration - migration

Theory: $C_{\rm coda}(\tau,\mathbf{x}_j,\mathbf{x}_l)$ has singular components at $\tau = \pm[\tau(\mathbf{x}_j,\mathbf{z}_{\rm r}) + \tau(\mathbf{x}_l,\mathbf{z}_{\rm r})]$.
Triangular inequality: $|\tau(\mathbf{x}_j,\mathbf{z}_{\rm r}) + \tau(\mathbf{x}_l,\mathbf{z}_{\rm r})| \geq \tau(\mathbf{x}_j,\mathbf{x}_l) \Longrightarrow$ singular components in $C_{\rm coda}$.

- Migration of the coda cross correlations





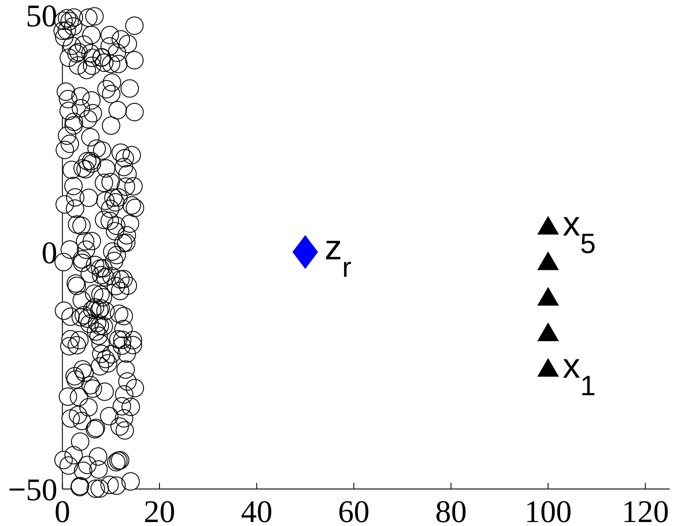
KM functional (image):

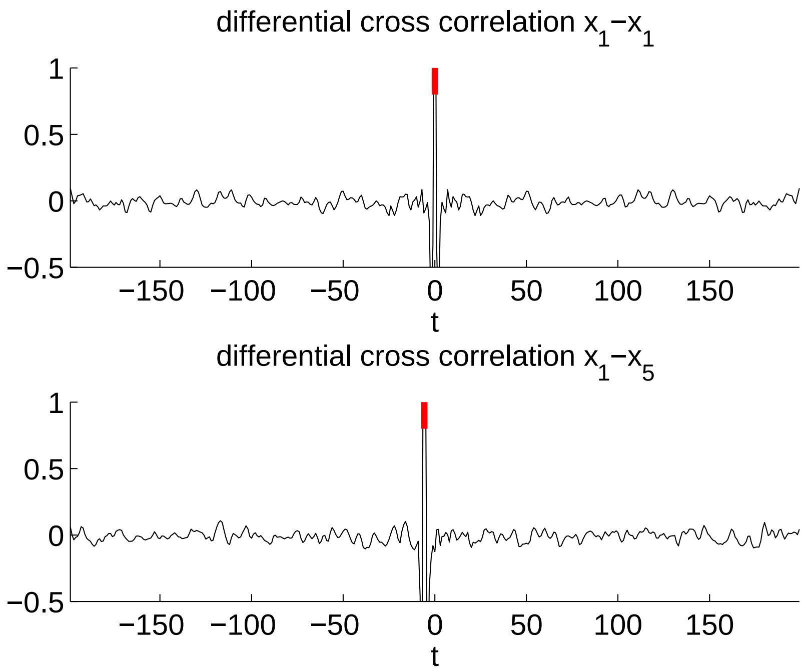
$$I(\mathbf{z}^S) = \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]}\widehat{C_{\rm coda}^{+}}(\omega,\mathbf{x}_j,\mathbf{x}_l) + e^{i\omega[\tau(\mathbf{z}^S,\mathbf{x}_j)+\tau(\mathbf{z}^S,\mathbf{x}_l)]}\widehat{C_{\rm coda}^{-}}(\omega,\mathbf{x}_j,\mathbf{x}_l)$$

$$C_{\rm coda}^{-}(\tau,\mathbf{x}_j,\mathbf{x}_l) = C(\tau,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(-\infty,-\tau(\mathbf{x}_j,\mathbf{x}_l))}(\tau)\,, \quad C_{\rm coda}^{+}(\tau,\mathbf{x}_j,\mathbf{x}_l) = C(\tau,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(\tau(\mathbf{x}_j,\mathbf{x}_l),\infty)}(\tau)$$

Backlight configuration

- Data in the absence (C_0) and in the presence (C) of the reflector



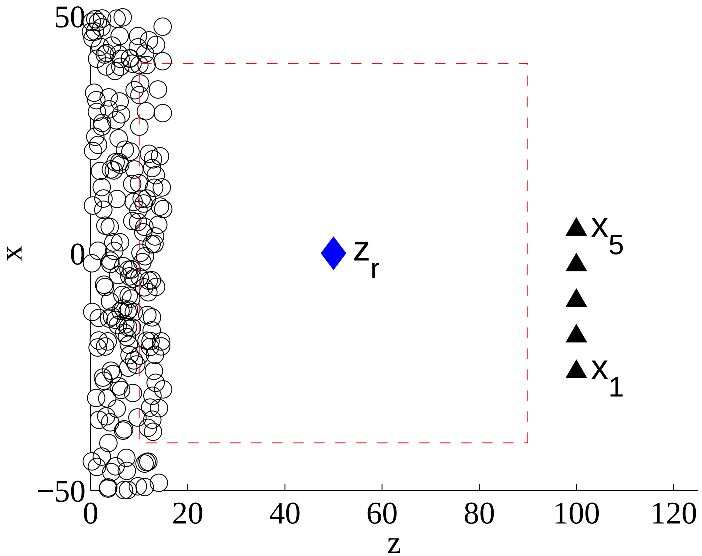


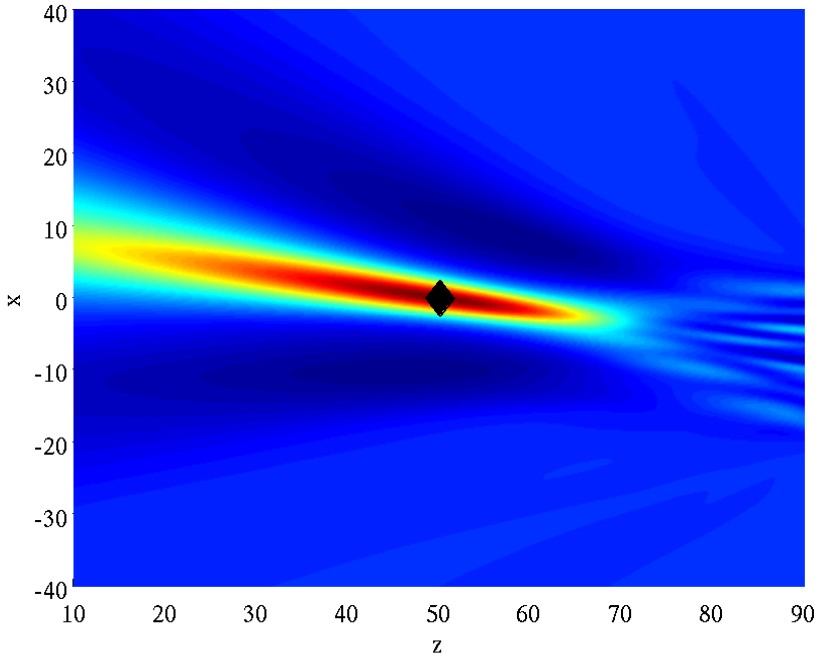
$$\Delta C(\tau, \mathbf{x}_j, \mathbf{x}_l) = (C - C_0)(\tau, \mathbf{x}_j, \mathbf{x}_l)$$

Backlight configuration - migration

Theory: $\Delta C(\tau,\mathbf{x}_j,\mathbf{x}_l)$ has singular components at $\tau = \tau(\mathbf{x}_l,\mathbf{z}_\mathrm{r}) - \tau(\mathbf{x}_j,\mathbf{z}_\mathrm{r})$.

- Migration of the differential cross correlations $\Delta C = C - C_0$





KM functional (image):

$$I(\mathbf{z}^S) = \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j)]}\widehat{\Delta C^+}(\omega,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(0,\infty)}(\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j))$$
$$+e^{i\omega[\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j)]}\widehat{\Delta C^-}(\omega,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(-\infty,0)}(\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j))$$

$$\Delta C^-(\tau,\mathbf{x}_j,\mathbf{x}_l) = (C-C_0)(\tau,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(-\infty,0)}(\tau), \quad \Delta C^+(\tau,\mathbf{x}_j,\mathbf{x}_l) = (C-C_0)(\tau,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(0,\infty)}(\tau)$$

Backlightlight configuration - resolution

KM functional

$$\mathcal{I}(\mathbf{z}^S) \simeq \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j)]}\widehat{C}(\omega,\mathbf{x}_j,\mathbf{x}_l)$$

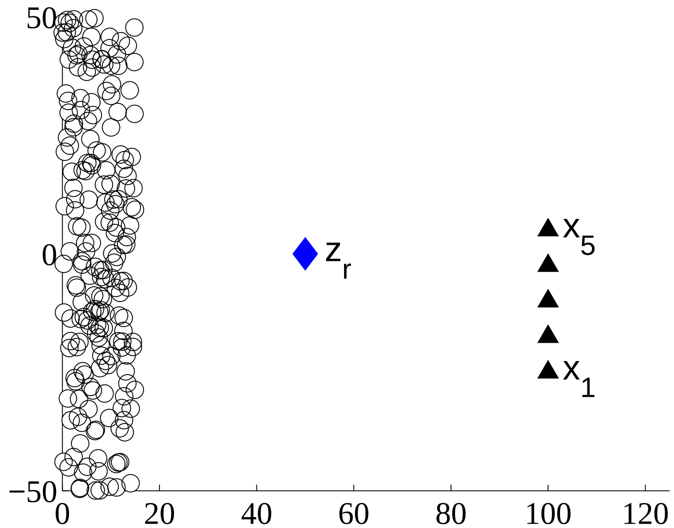
Same form as the Matched Filter (MF) imaging functional used when $\mathbf{z}_\mathrm{r}$ is a source emitting an impulse that is recorded by passive sensors at $(\mathbf{x}_j)_{j=1,\dots,N}$ and the data is the vector $P(t,\mathbf{x}_j)$. The MF functional is

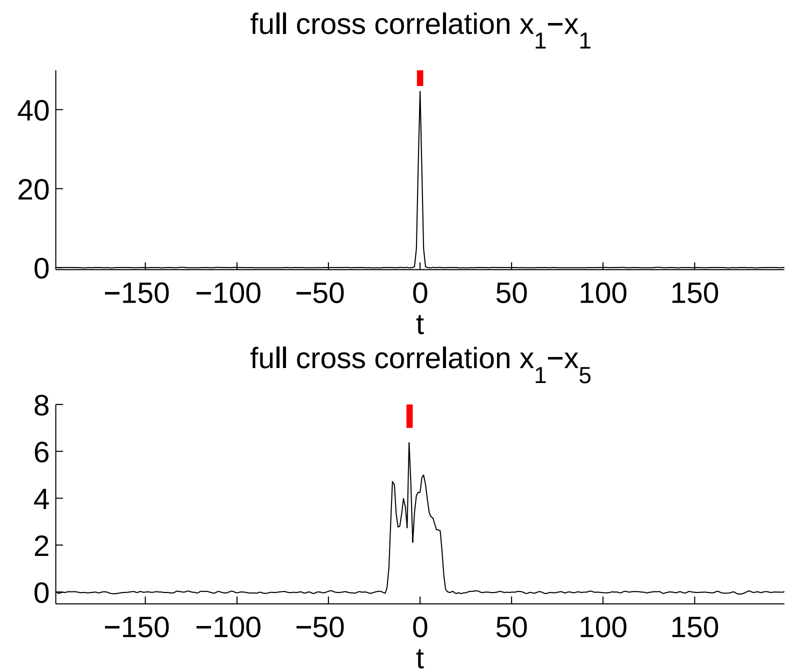
$$\begin{aligned}
\mathcal{I}^{\mathrm{MF}}(\mathbf{z}^S) &= \frac{1}{2\pi}\int d\omega \Big|\sum_{l=1}^{N} e^{-i\omega\tau(\mathbf{z}^S,\mathbf{x}_l)}\hat{P}(\omega,\mathbf{x}_l)\Big|^2 \\
&= \frac{1}{2\pi}\int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j)]}\hat{P}(\omega,\mathbf{x}_l)\overline{\hat{P}(\omega,\mathbf{x}_j)},
\end{aligned}$$

Backlight cross correlation imaging with passive sensor arrays provides poor range resolution, as in MF imaging.

Backlight configuration

- Data only in the presence (C) of the reflector
- Migration of the full cross correlations



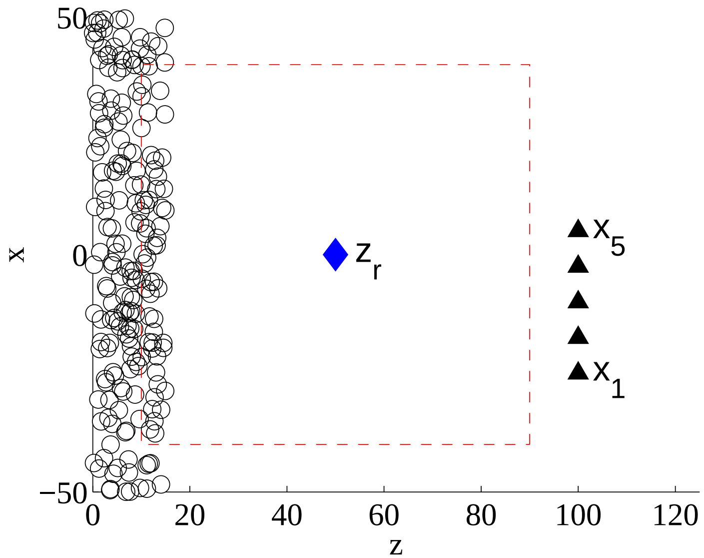


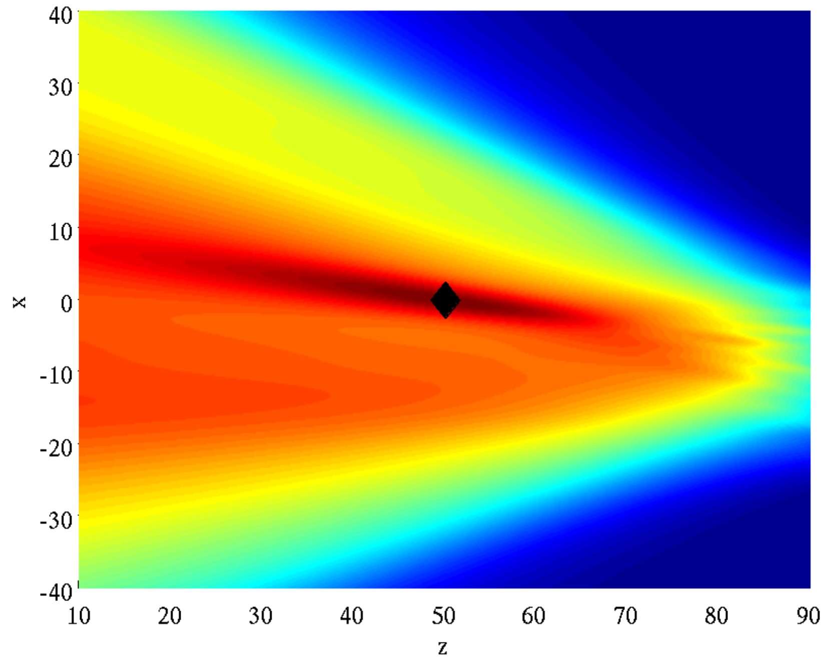
Singular components at $\tau = \tau(\mathbf{x}_l, \mathbf{z}_r) - \tau(\mathbf{x}_j, \mathbf{z}_r)$.

Triangular inequality $|\tau(\mathbf{x}_l, \mathbf{z}_r) - \tau(\mathbf{x}_j, \mathbf{z}_r)| \leq \tau(\mathbf{x}_j, \mathbf{x}_l) \Longrightarrow$ singular components buried in the main (singular and non-sigular) components, the coda cross correlation technique cannot be applied.

Backlight configuration

- Migration of the full cross correlations





KM functional (image):

$$I(\mathbf{z}^S) = \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j)]}\widehat{C^+}(\omega,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(0,\infty)}(\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j))$$
$$+e^{i\omega[\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j)]}\widehat{C^-}(\omega,\mathbf{x}_j,\mathbf{x}_l)\mathbf{1}_{(-\infty,0)}(\tau(\mathbf{z}^S,\mathbf{x}_l)-\tau(\mathbf{z}^S,\mathbf{x}_j))$$

Perspectives:

- Use the sensor network for travel time estimation and background velocity estimation (to be used in migration)
- Exploit the scattering properties of the medium and apply Coherent Interferometry (CINT), which amounts to cross correlate the coda cross correlations

CINT

- Key idea in CINT with arrays of sensors in active imaging: migrate cross correlations of the impulse response matrix, rather than the impulse response itself.

- Here: migrate cross correlations of the coda/differential cross correlations $C(\tau, \mathbf{x}_j, \mathbf{x}_l)$, which means that we use fourth-order cross correlations.

- It is important to compute these fourth-order cross correlations *locally* in time and space, and not over the whole time interval and the whole set of pairs of sensors.

CINT 1

Consider the square of the KM functional:

$$\left| I^{\text{KM}}(\mathbf{z}^S) \right|^2 = \sum_{j,j',l,l'=1}^{N} \iint \hat{C}(\omega, \mathbf{x}_j, \mathbf{x}_l) \overline{\hat{C}}(\omega', \mathbf{x}_{j'}, \mathbf{x}_{l'}) \times e^{-i\omega[\tau(\mathbf{x}_j,\mathbf{z}^S)+\tau(\mathbf{x}_l,\mathbf{z}^S)]} e^{i\omega'[\tau(\mathbf{x}_{j'},\mathbf{z}^S)+\tau(\mathbf{x}_{l'},\mathbf{z}^S)]} d\omega d\omega'$$

Decoherence frequency Ω_d: frequency gap beyond which the coda are not correlated.
Remark: The reciprocal of the decoherence frequency is the delay spread (duration of the coda).

$$I^{\text{CINT}}(\mathbf{z}^S, \Omega_d) = \sum_{j,j',l,l'=1}^{N} \iint_{|\omega-\omega'|\leq\Omega_d} \hat{C}(\omega, \mathbf{x}_j, \mathbf{x}_l) \overline{\hat{C}}(\omega', \mathbf{x}_{j'}, \mathbf{x}_{l'}) \times e^{-i\omega[\tau(\mathbf{x}_j,\mathbf{z}^S)+\tau(\mathbf{x}_l,\mathbf{z}^S)]} e^{i\omega'[\tau(\mathbf{x}_{j'},\mathbf{z}^S)+\tau(\mathbf{x}_{l'},\mathbf{z}^S)]} d\omega d\omega'$$

Compare with the square of the KM functional: The CINT functional and the square of the KM functional differ only in that the frequencies $|\omega - \omega'| > \Omega_d$ are eliminated in CINT.

CINT 2

Decoherence length X_d: distance between sensors beyond which the coda that can be recorded at them are not correlated.

$$I^{\rm CINT}(\mathbf{z}^S,\Omega_d,X_d) = \sum_{\substack{j,j',l,l'=1\\ |\mathbf{x}_j-\mathbf{x}_{j'}|\leq X_d,|\mathbf{x}_l-\mathbf{x}_{l'}|\leq X_d}}^{N} \iint_{|\omega-\omega'|\leq\Omega_d} d\omega d\omega' \hat{C}(\omega,\mathbf{x}_j,\mathbf{x}_l)\overline{\hat{C}}(\omega',\mathbf{x}_{j'},\mathbf{x}_{l'})$$
$$\times e^{-i\omega[\tau(\mathbf{x}_j,\mathbf{z}^S)+\tau(\mathbf{x}_l,\mathbf{z}^S)]} e^{i\omega'[\tau(\mathbf{x}_{j'},\mathbf{z}^S)+\tau(\mathbf{x}_{l'},\mathbf{z}^S)]}$$

- The range resolution of CINT is of the order of c_0/Ω_d and the cross range resolution of the order of $\lambda_0 L/X_d$.
- An adaptive procedure for estimating optimally the unknown parameters Ω_d and X_d is based on minimizing a suitable norm of the image to improve its quality, both in terms of resolution and signal-to-noise ratio.