Imaging in Random Media

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Outline

- Monday I. Introduction and overview

  II. Time reversal, migration and least squares imaging. Basic resolution theory

  Monday afternoon: C. Tsogka. An overview of computational results with migration imaging and introduction to computational issues

- Tuesday III. Resolution theory, use of the Kirchhoff-Helmholtz identities

  IV. Noise sources and correlations. Open media and cavities. Velocity estimation and imaging with distributed sensors (to be continued on Thursday afternoon by J. Garnier).

  Tuesday afternoon: C. Tsogka. Computational wave propagation and array imaging

Outline continued

- Wednesday V. The singular value decomposition, in detection and imaging
  VI. Edge illumination, the Fraunhofer regime and inverse filters
  Wednesday afternoon: L. Borcea. Imaging with layer annihilation
- Thursday VII. Waves in random media: Layered media, the paraxial approximation, radiative transport
  VIII. Time reversal in random media, super-resolution, statistical stability
  Thursday afternoon: J. Garnier. Passive sensor imaging with cross correlations
- Friday IX. Coherent interferometry for imaging in random media
  X. Discussion of research problems: Time reversal, imaging, random media, simulations, communications, optimization and adaptivity

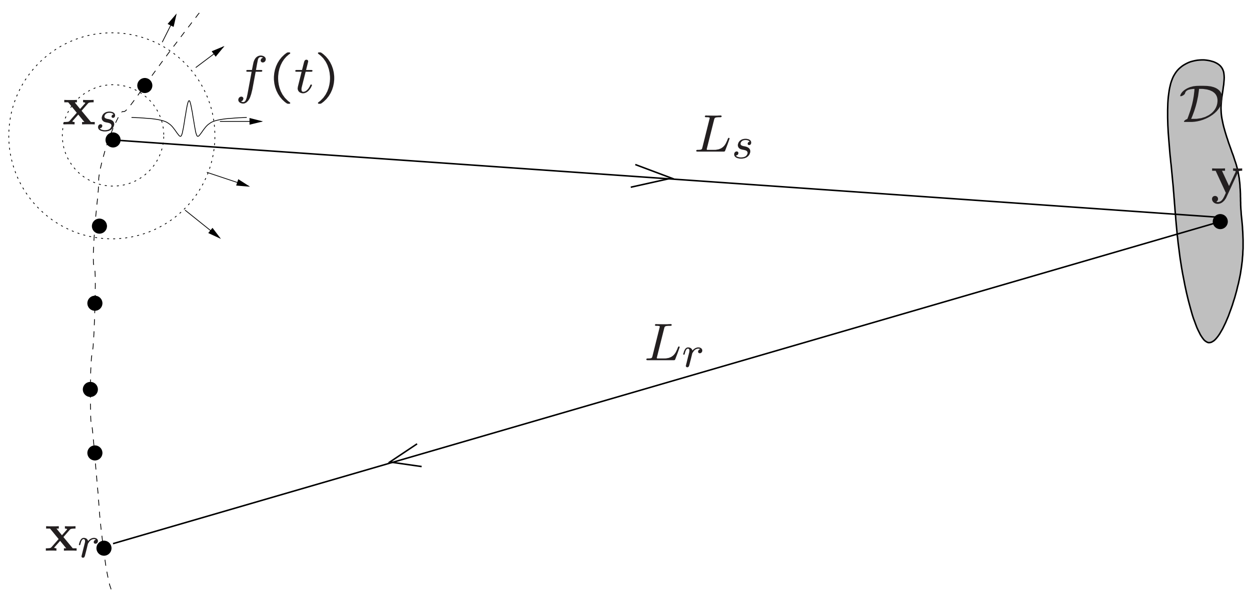
Introduction and overview

Inverse problems, detection, imaging

- Inverse problem: Given data as part of the solution of a problem (PDE ...), find the unknown parameters (structure) in it. Overly general, too inclusive for imaging

- Detection: Given two (or more) sets of data and an underlying model, find if they are consistent with it. Notion of detectability threshold. Too special and restrictive for imaging

- Imaging: From imperfect information (rough forward models, limited and noisy data) estimate parts of the unknown (parameter) structure that is of interest. In particular, quantify and understand the trade-offs between data size, computational complexity, and resolution.



Active sensor data: $P(\mathbf{x}_r, \mathbf{x}_s, t)$ for $(\mathbf{x}_r, \mathbf{x}_s, t)$ a set of source-receiver locations. Can be up to a function in $R^2 \times R^2$ plus time in R_+ for planar arrays.

Passive sensor data: $P(\mathbf{x}_r, t)$. can be up to a three-dimensional dataset for planar arrays.

Acquisition geometries, data structure

Different data acquisition geometries: Arrays, synthetic aperture arrays (zero-offset), distributed sensors, full aperture imaging (medical).

Narrowband (microwaves), broadband (ultrasound) and noise probing signals.

Coherent (radar, sonar, seismic ...) and incoherent (infrared, optical tomography, X-ray tomography) imaging.

Signal-to-Noise ratio (SNR) issues.

Array data model and the nonlinear inverse problem

The time Fourier transform of the data, $\widehat{P}(\mathbf{x}_r, \mathbf{x}_s, \omega)$, is modeled by $\widehat{U} = \widehat{f}_B(\omega - \omega_0)\widehat{G}_F(\mathbf{x}_r, \mathbf{x}_s, \omega; c)$ where $\widehat{f}_B(\omega - \omega_0)$ is the Fourier transform of the pulse $f(t) = e^{-i\omega_0 t} f_B(t)$ and $\widehat{G}_F$ is the Green's function that solves the Helmholtz equation

$$(\Delta + k^2 n^2(\mathbf{x}))\widehat{G}_F = -\delta(\mathbf{x} - \mathbf{y}), \quad k = \frac{\omega}{c_0}, \quad n(\mathbf{x}) = \frac{c_0}{c(\mathbf{x})}$$

with a radiation condition. The index of refraction is $n(\mathbf{x})$.

The inverse problem, the array least squares problem, is:
Minimize $J[c] + \alpha ||c||_{REG}$ where

$$J[c] = \int d\omega \sum_{\mathbf{x}_s, \mathbf{x}_r} \left| \widehat{P}(\mathbf{x}_r, \mathbf{x}_s, \omega) - \widehat{U}(\mathbf{x}_r, \mathbf{x}_s, \omega; c) \right|^2$$

and α is a strength of regularization parameter.

Note that this is a NONLINEAR problem for the unknown index of refraction $n(x)$ or propagation speed $c(x)$.

Structure of velocity (refractive index) and linearization

We need a model that distinguishes between
(a) a background velocity that is known or can be estimated,
(b) the reflectors or targets that we wish to image, and
(c) the clutter that is part of the background that we do not know, and can only estimate its overall influence statistically.

This motivates writing

$$n^2(\mathbf{x}) = n_{BG}^2(\mathbf{x}) + \rho(\mathbf{x}) + \mu(\mathbf{x})$$

where the background index of refraction is $n(\mathbf{x}) = c_0/c_{BG}(\mathbf{x})$, the target reflectivity is $\rho(\mathbf{x})$, and the clutter is modeled by the mean zero, stationary random function $\mu(\mathbf{x})$.

We next linearize in the reflectivity by writing $\widehat{G}_F = \widehat{G} + \delta\widehat{G}$ where

$$(\Delta + k^2(n_{BG}^2(\mathbf{x}) + \mu(\mathbf{x})))\widehat{G} = -\delta(\mathbf{x} - \mathbf{y})$$

is the (random) background Green's function and

$$\delta\widehat{G}(\mathbf{x}, \mathbf{y}, \omega) = k^2 \int d\mathbf{z} \rho(\mathbf{z}) \widehat{G}(\mathbf{x}, \mathbf{z}, \omega) \widehat{G}(\mathbf{z}, \mathbf{y}, \omega)$$

We model the data by

$$\widehat{P}(\mathbf{x}_r, \mathbf{x}_s, \omega) \sim \widehat{f}_B(\mathbf{x}_s, \omega - \omega_0)\delta\widehat{G}(\mathbf{x}_r, \mathbf{x}_s, \omega) = (\widehat{A}(\omega)\rho)(\mathbf{x}_r, \mathbf{x}_s)$$

where $\widehat{A}(\omega)$ is the random, frequency dependent, linear operator that maps reflectivities to array data. The least squares linearized inverse problem is to minimize $J_L[\rho]$ where

$$J_L[\rho] = \int d\omega \sum_{\mathbf{x}_r, \mathbf{x}_s} |\widehat{P}(\mathbf{x}_r, \mathbf{x}_s, \omega) - (\widehat{A}(\omega)\rho)(\mathbf{x}_r, \mathbf{x}_s)|^2$$

When ρ is a sum of M functions with small support (compared to the wavelength) with M smaller than the array size N, then the normal solution for this problem is

$$\rho \sim \int d\omega (\widehat{A}^H(\omega)\widehat{A}(\omega))^{-1}\widehat{A}^H(\omega)\widehat{P}(\omega)$$

In the general case

$$\rho \sim \int d\omega \widehat{A}^H(\omega)(\widehat{A}(\omega)\widehat{A}^H(\omega))^{-1}\widehat{P}(\omega)$$

The adjoint operator $\widehat{A}^H(\omega)$ maps array data to reflectivities and is given by

$$(\widehat{A}^H(\omega)\widehat{P}(\omega))(\mathbf{z}) = \frac{\omega^2}{c_0^2} \sum_{\mathbf{x}_s,\mathbf{x}_r} \widehat{f}_B(\mathbf{x}_s, \omega - \omega_0)\widehat{G}(\mathbf{z}, \mathbf{x}_s, \omega)\widehat{G}(\mathbf{z}, \mathbf{x}_r, \omega) \cdot \overline{\widehat{P}(\mathbf{x}_r, \mathbf{x}_s, \omega)}$$

Problem: $\widehat{A}^H(\omega)$ is random and is not known, so this least squares solution cannot be implemented to give an image!
However: The operator $\int d\omega \widehat{A}^H(\omega)\widehat{A}(\omega)$ is close to the identity operator, with high probability. It is related to the time reversal operator for source imaging. A similar result holds for $\int d\omega \widehat{A}(\omega)\widehat{A}^H(\omega)$. This motivates dropping the normalizing factors (the inverses) in the least squares solution

$$\rho(\mathbf{z}) \sim \int d\omega (\widehat{A}^H(\omega)\widehat{P}(\omega))(\mathbf{z}),$$

but resolution is affected, especially in narrowband cases and in random media!

Assume that the background velocity is known and that there is no clutter. Denote the deterministic background Green's function by $\widehat{G}_0(\mathbf{x}, \mathbf{y}, \omega) = \frac{e^{i\omega\tau(\mathbf{x},\mathbf{y})}}{4\pi|\mathbf{x}-\mathbf{y}|}$. We can then use the following imaging functional for the reflectivity $\rho(\mathbf{y}^S)$:

$$I^{KM}(\mathbf{y}^S) = \sum_{\mathbf{x}_s, \mathbf{x}_r} P(\mathbf{x}_r, \mathbf{x}_s, \tau(\mathbf{x}_s, \mathbf{y}^S) + \tau(\mathbf{y}^S, \mathbf{x}_r))$$

Here $\tau(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|/c_0$ is the travel time from $\mathbf{x}$ to $\mathbf{y}$ when the speed of propagation is c_0.

Travel time migration (1970's) is an elegant way to 'triangulate' the location of a scatterer using array (or distributed sensor) data without having to estimate travel times.

Mathematical theory for travel time migration has been developed by Beylkin, Burridge, Symes, Bleistein, ...

Denote the deterministic background Green's function by $\widehat{G}_0(\mathbf{x}, \mathbf{y}, \omega)$. Then the migration functional for imaging the reflectivity $\rho(\mathbf{y}^S)$ can be written as

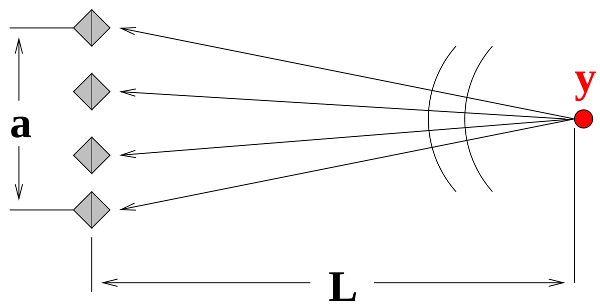
$$I(\mathbf{y}^S) = \int d\omega \sum_{\mathbf{x}_s, \mathbf{x}_r} \overline{\widehat{P}(\mathbf{x}_r, \mathbf{x}_s, \omega)} \widehat{G}_0(\mathbf{y}^S, \mathbf{x}_s, \omega) \widehat{G}_0(\mathbf{y}^S, \mathbf{x}_r, \omega)$$

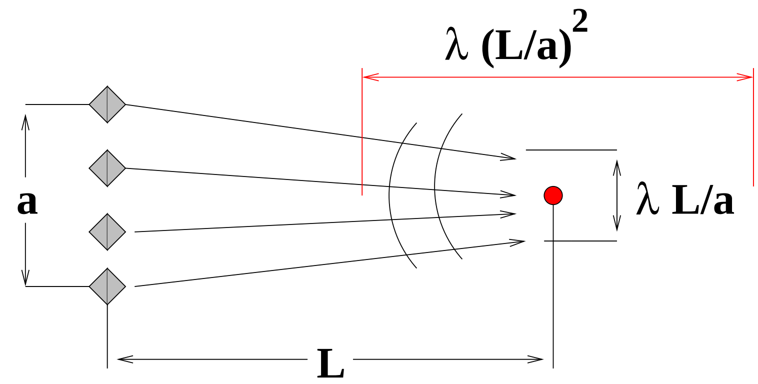
If we take $\widehat{G}_0 \sim e^{i\omega\tau(\mathbf{x},\mathbf{y})}$, where $\tau(\mathbf{x}, \mathbf{y})$ is the travel time from $\mathbf{x}$ to $\mathbf{y}$, then in the time domain we get the Kirchhoff migration imaging functional.

For active sources (passive arrays) the imaging functional is

$$I(\mathbf{y}^S) = \int d\omega \sum_{\mathbf{x}_r} \overline{\widehat{P}(\mathbf{x}_r, \omega)} \widehat{G}_0(\mathbf{y}^S, \mathbf{x}_r, \omega),$$

which is the time reversed field in homogeneous (unphysical) medium.





Cross-range (Rayleigh) resolution: $\frac{\lambda_0 L}{a}$

Range resolution (narrowband): $\frac{\lambda_0 L^2}{a^2}$

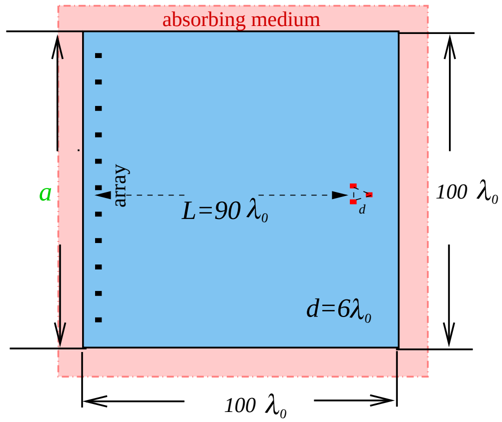
Range resolution (broadband): $\frac{c_0}{B}$

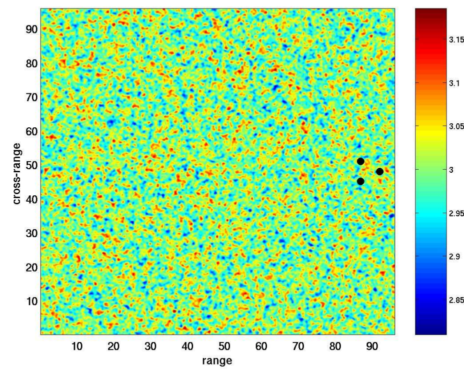
Ultrasonic nondestructive testing: $\lambda_0 = 3cm,\ a = 1m,\ L = 3-5m$ (propagation speed $3Km/sec.$)

Cross-range resolution: $9-15cm$

•What happens in a randomly inhomogeneous medium?

It depends on whether it is known exactly, as in TIME-REVERSAL, or only its large scale features are known, as in IMAGING.



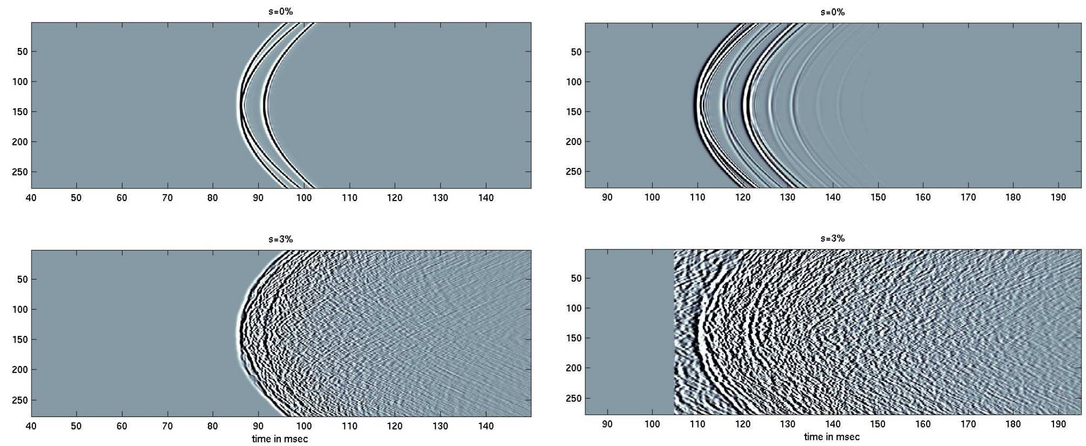


Computational domain $100\lambda_0 \times 100\lambda_0$ with central wavelength $\lambda_0 = 3cm$ (at central frequency $f_0 = 100KHz$ and with $c_0 = 3km/sec$), surrounded by a perfectly matched layer (pink).

The array has 185 receiving elements $\lambda_0/2$ apart, for an aperture of $92\lambda_0$.

Passive and active array data

-Passive array -Active array



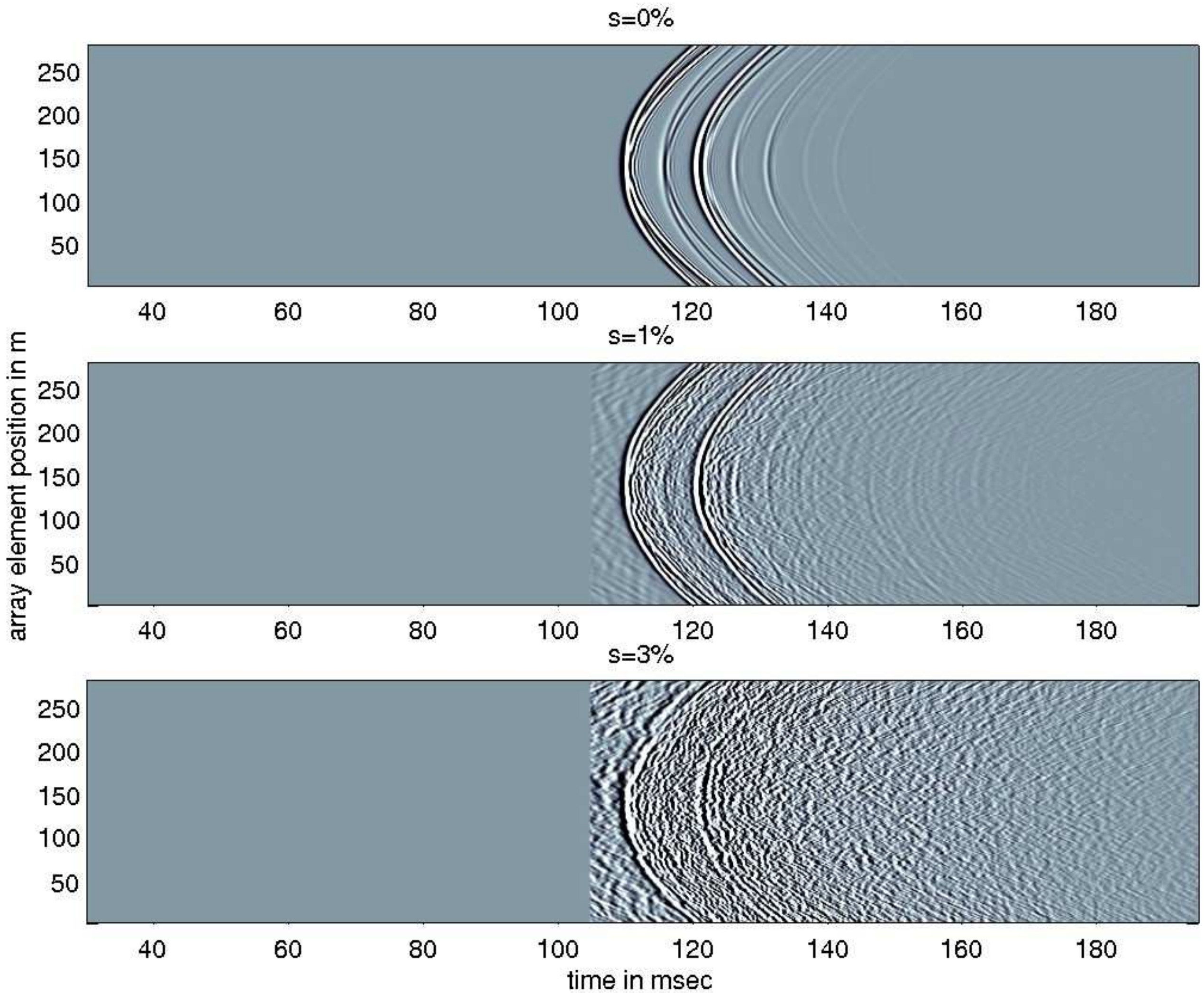
Left figures: Passive array

Right figures: Active array with central illumination

Top figures: Homogeneous medium.

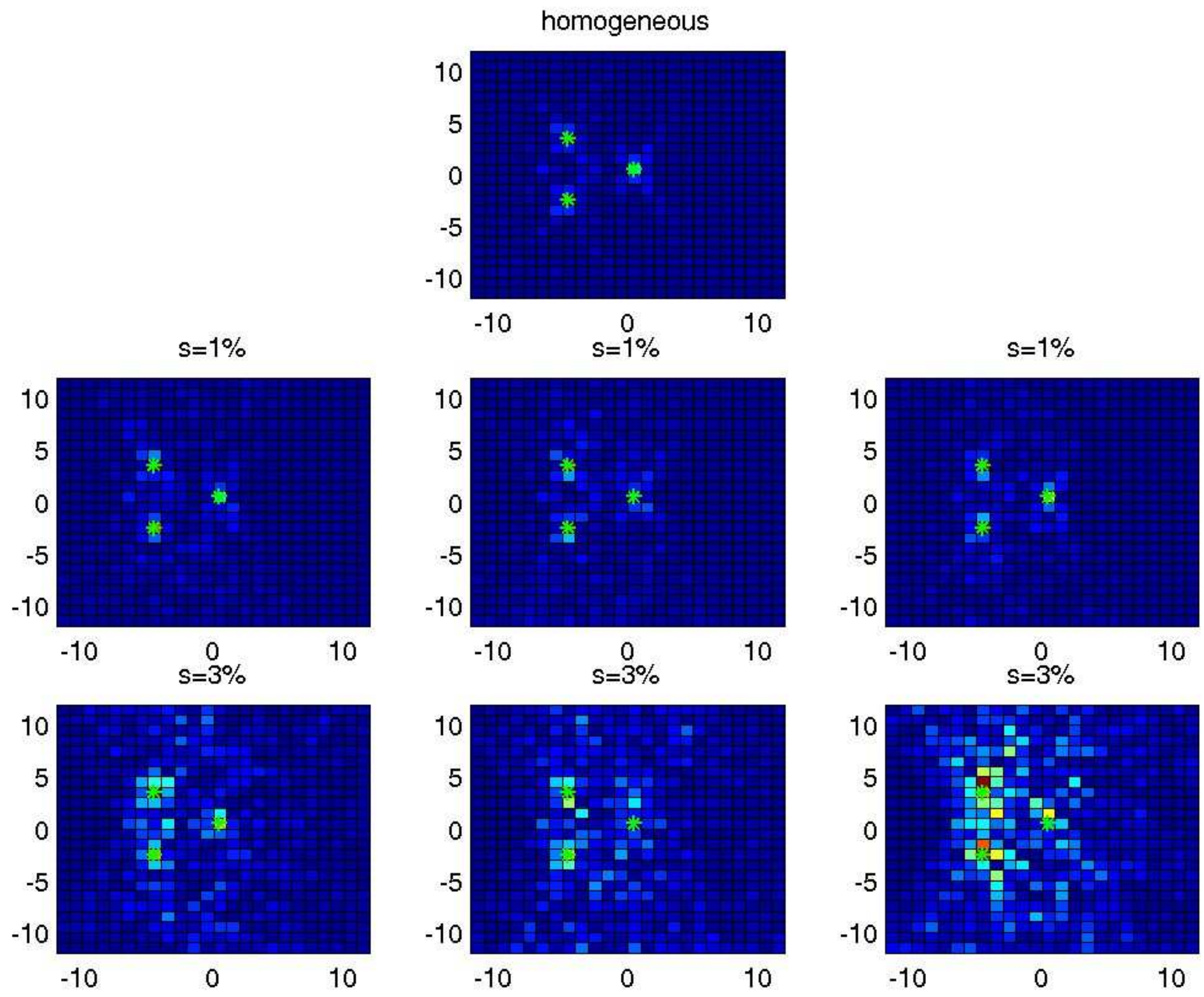
Bottom figures: Random medium with with standard deviation $s = 3\%$.

Why only main (Born) scattering matters in clutter



Down: Homogeneous, 1%, 3%STD.

Kirchhoff migration or travel time imaging (passive array)



Down: Homogeneous, 1%, 2%, 3%STD. Across: different realizations.

Critique of travel time migration imaging

It does not work well in clutter.

It is statistically unstable in clutter.

The reason is that KM tries to cancel the random phase of the signals arriving at the array with a deterministic phase using travel times.

The true Green's function for the random medium is not known and so cannot be used for imaging, which would result is huge resolution enhancement as in physical time reversal.

Time reversal, migration and least squares imaging:

Basic resolution theory

Point spread function in a homogeneous medium

Recall the Fourier transform:

$$\widehat{g}(\omega) = \int_R e^{i\omega t} g(t) dt \; ; \quad g(t) = \frac{1}{2\pi} \int e^{-i\omega t} \widehat{g}(\omega) d\omega$$

The FT of the pulse $e^{-i\omega t} f_B(t)$ (real part) emitted by the source is $\widehat{f}_B(\omega - \omega_0)$ with $\widehat{f}_B = 0$ for $\omega > B/2$. The bandwidth is B.

A point source at $\mathbf{y}$ gives the array data

$$P(\mathbf{x}_r, t) = e^{-i\omega_0 t} \frac{f_B(t - \tau(\mathbf{x}_r, \mathbf{y}))}{4\pi|\mathbf{x}_r - \mathbf{y}|} \; , \; \widehat{P}(\mathbf{x}_r, \omega) = \widehat{f}_B(\omega - \omega_0) \frac{e^{i\omega\tau(\mathbf{x}_r, \mathbf{y})}}{4\pi|\mathbf{x}_r - \mathbf{y}|}$$

where $\tau(\mathbf{x}_r, \mathbf{y})$ is the travel time from $\mathbf{y}$ to $\mathbf{x}_r$.

For a very broadband pulse (impulse response) the signal arrives at $\mathbf{x}_r$ at this travel time. Therefore $|\mathbf{x}_r - \mathbf{y}|^2 = c_0^2 t^2$. If L is the distance of $\mathbf{y}$ to the array and x_r is the distance from $\mathbf{x}_r$ to the nearest point from $\mathbf{y}$ to the array, then $x_r^2 + L^2 = c_0^2 t^2$ or

$$\frac{t^2}{L^2/c_0^2} - \frac{x_r^2}{L^2} = 1$$

which is a hyperbola in (x_r, t).

Place a point source ar $\mathbf{x}_r$ and let $g(t, \mathbf{x}_r)$ be the pulse emitted from it. The signal received at a search point $\mathbf{y}^S$ is

$$\sum_{\mathbf{x}_r} \frac{g(t - \tau(\mathbf{x}_r, \mathbf{y}), \mathbf{x}_r)}{4\pi|\mathbf{x}_r - \mathbf{y}^S|}$$

How do we choose $g(t, \mathbf{x}_r)$ so as to beamform a pulse to $\mathbf{y}$?

Use time reversal: $g(t, \mathbf{x}_r) = P(\mathbf{x}_r, -t)$. Then the signal at $\mathbf{y}^S$ is

$$e^{-i\omega_0 t} \sum_{\mathbf{x}_r} \frac{f_B(-t + \tau(\mathbf{x}_r, \mathbf{y}) - \tau(\mathbf{x}_r, \mathbf{y}^S))}{(4\pi)^2|\mathbf{x}_r - \mathbf{y}^S|^2}$$

and in the Fourier domain

$$\overline{\widehat{f}_B(\omega - \omega_0)} \sum_{\mathbf{x}_r} \overline{\widehat{G}_0(\mathbf{x}_r, \mathbf{y}, \omega)} \widehat{G}_0(\mathbf{x}_r, \mathbf{y}^S, \omega)$$

How well does this focus around $\mathbf{y}$?

The travel time migration functional is

$$I^{KM}(\mathbf{y}^S) = \int d\omega \sum_{\mathbf{x}_r} \widehat{P}(\mathbf{x}_r, \omega) e^{-i\omega\tau(\mathbf{x}_r, \mathbf{y}^S)} = \sum_{\mathbf{x}_r} P(\mathbf{x}_r, \tau(\mathbf{x}_r, \mathbf{y}^S))$$

and up to multiplicative factors this is the conjugate of beamforming.

Physical time reversal is not an imaging functional but a physical process:

$$\Gamma^{TR}(\mathbf{y}^S) = \int d\omega \sum_{\mathbf{x}_r} \overline{\widehat{P}(\mathbf{x}_r, \omega)} \widehat{G}(\mathbf{x}_r, \mathbf{y}^S, \omega)$$

There is no difference between them in a homogeneous medium.

Basic fact:
I^{KM} loses resolution in random media
Γ^{TR} gains resolution in random media!

Detailed analysis of

$$I = \sum_{\mathbf{x}_r} e^{ik(|\mathbf{x}_r - \mathbf{y}| - |\mathbf{x}_r - \mathbf{y}^S|)} \ , \quad k = \frac{\omega}{c_0} \ , \quad \lambda = \frac{2\pi}{k}$$

under the following conditions. If the origin of coordinates is at the center of a linear array, $\mathbf{y} = (L, 0)$, $\mathbf{y}^S = (L + \eta, \xi)$, and $\mathbf{x}_r = (0, rh/2)$ for $r = 0, \pm 1, \pm 2, ..., \pm N$, let $a = Nh$ and assume that $a \ll L$. Assume also that $\lambda \ll a$ and that the spacing $h/2$ between sensors smaller than a half wavelength, $h < \lambda$ so that $\mathbf{x}_r = (0, rh/2) = (0, x)$ with $-a/2 \le x \le a/2$.

We then have:

$$|\mathbf{x}_r - \mathbf{y}| = (L^2 + x^2)^{1/2} = L(1 + (\frac{x}{L})^2)^{1/2} \approx L + \frac{x^2}{2L}$$

and similarly

$$|\mathbf{x}_r - \mathbf{y}^S| = ((L + \eta)^2 + (x - \xi)^2)^{1/2} \approx L + \eta + \frac{(x - \xi)^2}{2(L + \eta)}$$

so that

$$|\mathbf{x}_r - \mathbf{y}| - |\mathbf{x}_r - \mathbf{y}^S| \approx -\eta - \frac{\xi^2}{2(L + \eta)} + \frac{x\xi}{L + \eta} + \frac{\eta \xi^2}{2L(L + \eta)}$$

The sum above can be approximated by an integral

$$I \approx \frac{2}{h} e^{-ik(\eta + \frac{\xi^2}{2(L+\eta)})} \int_{-a/2}^{a/2} e^{ik(\frac{x\xi}{L+\eta} + \frac{\eta\xi^2}{2L(L+\eta)})} dx$$

Continue analysis of time harmonic psf

Change variables in the integral, $x = ax'$. After dropping primes and taking absolute values the integral is

$$\frac{2a}{h}\left|\int_{-1/2}^{1/2} e^{i\pi\left(\frac{ka\xi}{\pi(L+\eta)}x+\frac{k}{2\pi}\frac{a^2\eta}{2L(L+\eta)}x^2\right)}\ dx\right|$$

But $\lambda = 2\pi/k$ so the exponent is

$$\frac{2\xi a}{\lambda(L+\eta)}x + \frac{\eta a^2}{\lambda L(L+\eta)}x^2$$

For $\eta \ll L$ this simplifies to

$$2\frac{\xi}{\lambda L/a}x + \frac{\eta}{\lambda(L/a)^2}x^2$$

We see that the cross range coordinate ξ scales with $\lambda L/a$ and the range resolution with η with $\lambda(L/a)^2$. These are the classical time-harmonic resolution limits in array (aperture) imaging, in the regime $\lambda \ll a \ll L$.

It is simply c_0/B where B is the bandwidth.

This can be seen from the formula

$$e^{-i\omega_0 t}\sum_{\mathbf{x}_r}\frac{f_B(-t+\tau(\mathbf{x}_r,\mathbf{y})-\tau(\mathbf{x}_r,\mathbf{y}^S))}{(4\pi)^2|\mathbf{x}_r-\mathbf{y}^S|^2}$$

after noting the width of f_B is proportional to that of its Fourier transform, which is B.

In units of length it is c_0/B.

What is the resolution of the Kirchhoff migration (or time reversal) functional in a homogeneous medium when the array encloses the source point? Lets consider time reversal

$$\Gamma^{TR}(\mathbf{y}^S,\omega) = \sum_{\mathbf{x}_r \in \partial D} \widehat{P}(\mathbf{x}_r,\omega)\overline{\widehat{G}_0(\mathbf{x}_r,\mathbf{y}^S,\omega)} \approx \int_{\partial D} dS(x)\widehat{P}(\mathbf{x},\omega)\overline{\widehat{G}_0(\mathbf{x},\mathbf{y}^S,\omega)}$$

Assume that D is a convex region and that

$$\frac{|\mathbf{y}-\mathbf{y}^S|}{|\mathbf{x}-\mathbf{y}^S|} \ll 1$$

This condition says that the search point and the source are away from the array, which is the boundary of D. We will see how to generalize this in lecture III.

We will show that the resolution is $\lambda/2$, which is a well known result.

We can get an approximate expression for the difference of the distance to the array from $\mathbf{y}$ and from $\mathbf{y}^S$ under the assumption that they are relatively far from it. We first write

$$\Gamma^{TR}(\mathbf{y}^S, \omega) = \int_{\partial D} dS(x) \frac{e^{ik(|\mathbf{x}-\mathbf{y}|-|\mathbf{x}-\mathbf{y}^S|)}}{(4\pi)^2 |\mathbf{x}-\mathbf{y}||\mathbf{x}-\mathbf{y}^S|}$$

Then we note that

$$|\mathbf{x}-\mathbf{y}| - |\mathbf{x}-\mathbf{y}^S| \approx (\mathbf{y}-\mathbf{y}^S) \cdot (\frac{\mathbf{y}^S - \mathbf{x}}{|\mathbf{y}^S - \mathbf{x}|} + o(1))$$

which means that we have to evaluate the integral

$$\Gamma^{TR}(\mathbf{y}^S, \omega) \approx \int_{\partial D} dS(x) \frac{e^{ik(\mathbf{y}-\mathbf{y}^S)\cdot \frac{\mathbf{y}^S-\mathbf{x}}{|\mathbf{y}^S-\mathbf{x}|}}}{(4\pi)^2 |\mathbf{x}-\mathbf{y}^S|^2}$$

where we have simplified the denominator as well.

Since we have assumed that array ∂D is convex, we can parametrize it with polar coordinates relative to a fixed point x^* on it:

$$|\mathbf{x} - \mathbf{y}^S| = g(\theta, \phi)$$

with θ and ϕ the polar and azimuthal angles, respectively. In these coordinates the surface element has the form $dS(x) = |\mathbf{x} - \mathbf{y}^S|^2 \sin\theta d\theta d\phi$. Therefore

$$\Gamma^{TR}(\mathbf{y}^S, \omega) \approx \int_0^\pi \int_0^{2\pi} \frac{e^{ik|\mathbf{y}^S - \mathbf{y}|\cos\theta}}{(4\pi)^2} \sin\theta d\theta d\phi$$

The integration gives

$$\Gamma^{TR}(\mathbf{y}^S, \omega) \approx \frac{\sin(k|\mathbf{y}^S - \mathbf{y}|)}{4\pi|\mathbf{y}^S - \mathbf{y}|}.$$

We get a resolution estimate from the first zero of the sinc function, the Rayleigh resolution. We have $k|\mathbf{y}^S - \mathbf{y}| = \pi$ or $|\mathbf{y}^S - \mathbf{y}| = \lambda/2$, which is a well known result.

So far we have considered only time harmonic point spread functions. We will now consider a planar array $A \subset R^2$ and signals with bandwidth B, and we will analyze the weighted Kirchhoff migration functional in the limit $A \to R^2$, $B \to \infty$. The imaging functional is

$$I^{WKM}(\mathbf{y}^S; A, B) = \int_{R^3} \rho(\mathbf{z}) \int_A \int_{|\omega-\omega_0| \le B/2} M(\mathbf{x}, \mathbf{y}^S, \omega)$$

$$\times \frac{e^{i(\omega/c_0)(|\mathbf{x}-\mathbf{z}|-|\mathbf{x}-\mathbf{y}^S|)}}{(4\pi)^2 |\mathbf{x}-\mathbf{z}||\mathbf{x}-\mathbf{y}^S|} d\mathbf{z} d\mathbf{x} d\omega$$

The form of the multiplier M is given below. It compensates for the fact that the array is large. It significance will become clearer in Lecture III.

We assume, as in the full aperture, time harmonic case, that

$$\frac{|\mathbf{y}-\mathbf{y}^S|}{|\mathbf{x}-\mathbf{y}^S|} \ll 1$$

We use this condition it to simplify the imaging functional

$$I^{WKM}(\mathbf{y}^S; A, B) \approx \int_{R^3} \rho(\mathbf{z}) \int_A \int_{|\omega-\omega_0|\leq B/2} M(\mathbf{x}, \mathbf{y}^S, \omega) \frac{e^{i(\omega/c_0)(\mathbf{z}-\mathbf{y}^S)\cdot\frac{\mathbf{x}-\mathbf{y}^S}{|\mathbf{x}-\mathbf{y}^S|}}}{(4\pi)^2|\mathbf{x}-\mathbf{y}^S|^2}$$

We introduce the change of variables from R^3 to itself and its Jacobian

$$(\mathbf{x}, \omega) \in R^3 \to \zeta = \frac{\omega}{c_0} \frac{\mathbf{x}-\mathbf{y}^S}{|\mathbf{x}-\mathbf{y}^S|} \, , \quad d\mathbf{x}d\omega = \frac{\partial(\mathbf{x}, \omega)}{\partial \zeta} = J(\mathbf{x}, \mathbf{y}^S, \omega) d\zeta$$

which is one-to-one and onto as $A \to R^2$ and $B \to \infty$. Let $\mathcal{Z}(A, B)$ be the image of the $(\mathbf{x}, \omega)$ region of integration in ζ space.

With this change of variables we have

$$I^{WKM}(\mathbf{y}^S; A, B) \approx \int_{R^3} \rho(\mathbf{z}) \int_{\mathcal{Z}(A,B)} \frac{e^{i(\mathbf{z}-\mathbf{y}^S)\cdot\zeta}}{(2\pi)^3}$$

provided we choose the multiplier M so that

$$\frac{M(\mathbf{x}, \mathbf{y}^S, \omega) J(\mathbf{x}, \mathbf{y}^S, \omega)}{(4\pi)^2 |\mathbf{x} - \mathbf{y}^S|^2} = \frac{1}{(2\pi)^3}$$

Now as $A \to R^2$ and $B \to \infty$ the inner integral becomes a 3D delta function and therefore

$$I^{WKM}(\mathbf{y}^S; A, B) \approx \rho(\mathbf{y}^S)$$

so that we have an asymptotic recovery of the reflectivity.

This calculation in media with smooth background velocity was carried out by Baylikn in the 80's. It is presented in the book of Bleistein, Cohen and Stockwell (Springer 2001).

By an elementary calculation we find that the multiplier M is proportional to

$$M(\mathbf{x},\mathbf{y}^S,\omega) \sim \mathbf{n}(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{y}^S)$$

where $\mathbf{n}(\mathbf{x})$ is the unit "outward" normal to array at $\mathbf{x}$. By outward we mean that it is pointing in the direction exterior to the region of the reflectors with reflectivity ρ.

This is a rather simple form for the multiplier, which suggests that there should be a more direct and perhaps more general and less computational way to get this asymptotic consistency of the migration imaging functional. In Lecture III we will see that this is indeed the case. There is a simpler and more general way to analyze the large array, large bandwidth behavior of backpropagation (migration) imaging functionals.

Note also that the multiplier M must be consistent with the least squares multiplier $A^H A$ (Lecture I), which we have dropped in travel time migration and in back propagation.

Resolution theory, use of the Kirchhoff-Helmholtz identities

Green's identity

Let D be a closed and bounded region with smooth boundary ∂D and let $c(\mathbf{x})$ be a given speed of propagation that is uniform outside a subregion interior to D. The time harmonic, outgoing Green's function and its conjugate satisfy

$$\Delta_{\mathbf{x}}\widehat{G}(\omega,\mathbf{x},\mathbf{y}_2)+\frac{\omega^2}{c^2(\mathbf{x})}\widehat{G}(\omega,\mathbf{y},\mathbf{y}_2)=-\delta(\mathbf{x}-\mathbf{y}_2)\,,$$

$$\Delta_{\mathbf{x}}\overline{\widehat{G}}(\omega,\mathbf{x},\mathbf{y}_1)+\frac{\omega^2}{c^2(\mathbf{x})}\overline{\widehat{G}}(\omega,\mathbf{x},\mathbf{y}_1)=-\delta(\mathbf{x}-\mathbf{y}_1)\,.$$

We multiply the first equation by $\overline{\widehat{G}}(\omega,\mathbf{x},\mathbf{y}_1)$ and subtract the second equation multiplied by $\widehat{G}(\omega,\mathbf{x},\mathbf{y}_2)$:

$$\begin{aligned}
&\nabla_{\mathbf{y}}\cdot[\overline{\widehat{G}}(\omega,\mathbf{x},\mathbf{y}_1)\nabla_{\mathbf{x}}\widehat{G}(\omega,\mathbf{x},\mathbf{y}_2)-\widehat{G}(\omega,\mathbf{x},\mathbf{y}_2)\nabla_{\mathbf{x}}\overline{\widehat{G}}(\omega,\mathbf{x},\mathbf{y}_1)]\\
&=\widehat{G}(\omega,\mathbf{x},\mathbf{y}_2)\delta(\mathbf{x}-\mathbf{y}_1)-\overline{\widehat{G}}(\omega,\mathbf{x},\mathbf{y}_1)\delta(\mathbf{x}-\mathbf{y}_2)\\
&=\widehat{G}(\omega,\mathbf{x}_1,\mathbf{y}_2)\delta(\mathbf{x}-\mathbf{y}_1)-\overline{\widehat{G}}(\omega,\mathbf{y}_1,\mathbf{y}_2)\delta(\mathbf{x}-\mathbf{y}_2)\,,
\end{aligned}$$

where we have used the reciprocity property $\widehat{G}(\omega,\mathbf{y}_2,\mathbf{y}_1)=\widehat{G}(\omega,\mathbf{y}_1,\mathbf{y}_2)$. Integrate over ∂D and use the divergence theorem:

$$\begin{aligned}
&\int_{\partial D}\mathbf{n}(\mathbf{x})\cdot[\overline{\widehat{G}}(\omega,\mathbf{x},\mathbf{y}_1)\nabla_{\mathbf{x}}\widehat{G}(\omega,\mathbf{x},\mathbf{y}_2)-\widehat{G}(\omega,\mathbf{x},\mathbf{y}_2)\nabla_{\mathbf{x}}\overline{\widehat{G}}(\omega,\mathbf{x},\mathbf{y}_1)]dS(\mathbf{x})\\
&=\widehat{G}(\omega,\mathbf{y}_1,\mathbf{y}_2)-\overline{\widehat{G}}(\omega,\mathbf{y}_1,\mathbf{y}_2)\,,
\end{aligned}$$

where $\mathbf{n}(\mathbf{x})$ is the unit outward normal to ∂D.

The Sommerfeld radiation condition

For the time harmonic Green's function $\widehat{G}(\omega, \mathbf{x}, \mathbf{y})$ this condition is

$$|\mathbf{x}|(\frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla_{\mathbf{x}} - \frac{i\omega}{c_0})\widehat{G}(\omega, \mathbf{x}, \mathbf{y}) \to 0$$

as $|\mathbf{x}| \to \infty$. In a homogeneous medium where

$$\widehat{G}(\omega, \mathbf{x}, \mathbf{y}) = \widehat{G}_0(\omega, \mathbf{x}, \mathbf{y}) = \frac{e^{i(\omega/c_0)|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$$

this simply means that for $|\mathbf{x}-\mathbf{y}| \to \infty$

$$\nabla_{\mathbf{x}}\widehat{G}_0(\omega, \mathbf{x}, \mathbf{y}) \approx \frac{i\omega}{c_0}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\widehat{G}_0(\omega, \mathbf{x}, \mathbf{y})$$

For a smooth background velocity the high frequency (WKB) approximation of the Green's function is

$$\widehat{G}(\omega, \mathbf{x}, \mathbf{y}) \approx a(\mathbf{x}, \mathbf{y})e^{i\omega\tau(\mathbf{x},\mathbf{y})}.$$

Here $a(\mathbf{x}, \mathbf{y})$ and $\tau(\mathbf{x}, \mathbf{y})$ are smooth except at $\mathbf{x} = \mathbf{y}$. The amplitude $a(\mathbf{x}, \mathbf{y})$ satisfies a transport equation and the travel time $\tau(\mathbf{x}, \mathbf{y})$ the eikonal equation. It is symmetric $\tau(\mathbf{x}, \mathbf{y}) = \tau(\mathbf{y}, \mathbf{x})$ and from Fermat's principle

$$\tau(\mathbf{x}, \mathbf{y}) = \inf\left\{T \text{ s.t.} \exists\, (\mathbf{X}_t)_{t\in[0,T]} \in \mathcal{C}^1, \mathbf{X}_0 = \mathbf{x}, \mathbf{X}_T = \mathbf{y}, |\frac{d\mathbf{X}_t}{dt}| = c(\mathbf{X}_t)\right\}.$$

The radiation condition can now be written as

$$\nabla_{\mathbf{x}}\widehat{G}(\omega, \mathbf{x}, \mathbf{y}) \approx i\omega\nabla_{\mathbf{x}}\tau(\mathbf{x}, \mathbf{y})\widehat{G}(\omega, \mathbf{x}, \mathbf{y})$$

Now lets assume that we can use the radiation condition in Green's identity. We get

$$i\omega \int_{\partial D} \mathbf{n}(\mathbf{x}) \cdot (\nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{y}^S) + \nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{y}))\overline{\widehat{G}(\omega,\mathbf{x},\mathbf{y})}\widehat{G}(\omega,\mathbf{x},\mathbf{y}^S)dS(\mathbf{x})$$
$$= \widehat{G}(\omega,\mathbf{y},\mathbf{y}^S) - \overline{\widehat{G}(\omega,\mathbf{y},\mathbf{y}^S)}\,,$$

Assume that $\mathbf{y}$ and $\mathbf{y}^S$ are near each other as they are the source location and the search point of the imaging function. Then we have

$$2i\omega \int_{\partial D} \mathbf{n}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{y}^S)\overline{\widehat{G}(\omega,\mathbf{x},\mathbf{y})}\widehat{G}(\omega,\mathbf{x},\mathbf{y}^S)dS(\mathbf{x})$$
$$= \widehat{G}(\omega,\mathbf{y},\mathbf{y}^S) - \overline{\widehat{G}(\omega,\mathbf{y},\mathbf{y}^S)}\,,$$

Connection with imaging and time reversal

Clearly

$$I^{WBP}(\mathbf{y}^S) = \int_{\partial D} \mathbf{n}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \tau(\mathbf{x}, \mathbf{y}^S) \overline{\widehat{G}(\omega, \mathbf{x}, \mathbf{y})} \widehat{G}(\omega, \mathbf{x}, \mathbf{y}^S) dS(\mathbf{x})$$

is a weighted imaging functional with back propagation to a search point $\mathbf{y}^S$, when there is a point source at $\mathbf{y}$ and the array is the "full aperture" boundary ∂D.

From the KH identity, assuming that it can used, we have that

$$I^{WBP}(\mathbf{y}^S) \approx \frac{1}{2i\omega}(\widehat{G}(\omega, \mathbf{y}, \mathbf{y}^S) - \overline{\widehat{G}(\omega, \mathbf{y}, \mathbf{y}^S)}).$$

This approximate identity is quite general regarding the background medium, which can be rough and even random. But (i)
the array must be in a homogeneous medium and far from the scattering background. The source to be "imaged" can be in the scattering region. We put quotations on imaged because if the medium is rough and random it will not be known and so $I^{WBP}(\mathbf{y}^S)$ is a time reversal field function, not an imaging functional. If the background is variable but known then it is an imaging functional. In addition (ii)
the array must sufficiently far so that the radiation condition applies.

In a smooth background at high frequencies we have

$$\widehat{G}(\omega, \mathbf{x}, \mathbf{y}) \approx a(\mathbf{x}, \mathbf{y}) e^{i\omega\tau(\mathbf{x},\mathbf{y})}$$

and lets assume that the amplitude a is real. Then

$$I^{WBP}(\mathbf{y}^S) \approx \frac{1}{2i\omega}(\widehat{G}(\omega, \mathbf{y}, \mathbf{y}^S) - \overline{\widehat{G}(\omega, \mathbf{y}, \mathbf{y}^S)})$$
$$\approx \frac{1}{\omega} a(\mathbf{y}, \mathbf{y}^S) \sin(\omega\tau(\mathbf{y}, \mathbf{y}^S))$$

From the first zero of the sine function we get a resolution limit, the Rayleigh resolution limit: $\omega\tau(\mathbf{y}, \mathbf{y}^*) = \pi$. If c_0 is a reference background speed then

$$c_0\tau(\mathbf{y}, \mathbf{y}^*) = \frac{\pi c_0}{\omega} = \frac{\lambda_0}{2},$$

which is an appropriate generalization of the "half wavelength" far field (high frequency) resolution limit for full aperture imaging.

In the high frequency, far field regime we can symmetrize a large planar array about the source and make the array imaging functional (the psf) look approximately like a full aperture functional.

This argument leads to the approximation

$$\int_{|\omega-\omega_0|\le B/2} d\omega \widehat{f}_B(\omega-\omega_0) \int_A \mathbf{n}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{y}^S)\overline{\widehat{G}(\omega,\mathbf{x},\mathbf{y})}\widehat{G}(\omega,\mathbf{x},\mathbf{y}^S)dS(\mathbf{x})$$

$$\approx a(\mathbf{y},\mathbf{y}^S)\int_{|\omega-\omega_0|\le B/2} d\omega \frac{\widehat{f}_B(\omega-\omega_0)}{4i\omega} e^{i\omega\tau(\mathbf{y},\mathbf{y}^S)}$$

We have used here the fact that the pulse is a real function and the omega integration extends to negative frequencies with $\overline{\widehat{f}_B(\omega)} = \widehat{f}_B(-\omega)$.

For a suitably chosen probing pulse we see from this approximation that

$$\int_{|\omega-\omega_0|\leq B/2} d\omega \widehat{f}_B(\omega-\omega_0) \int_A \mathbf{n}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{y}^S)\overline{\widehat{G}(\omega,\mathbf{x},\mathbf{y})}\widehat{G}(\omega,\mathbf{x},\mathbf{y}^S)dS(\mathbf{x})$$

$$\approx a(\mathbf{y},\mathbf{y}^S)\delta(\tau(\mathbf{y},\mathbf{y}^S)),$$

as $B \to \infty$ and $A \to R^2$.

This is a generalization of the resolution result mentioned earlier in connection with a change of variables in a homogeneous medium.

Summary of results in resolution theory

- Time-harmonic, paraxial, $\lambda \ll a \ll L$:
  Cross range $(\lambda L)/a$, Range $\lambda(L/a)^2$.

- Broadband: Range c_0/B (arrival time resolution). Cross range resolution is still $(\lambda L)/a$.

- Full aperture: $\lambda/2$

- Large aperture, large bandwidth arrays: exact recovery with suitably weighted migration or back propagation functional.

Noise sources and correlations. Open media and cavities. Velocity estimation and imaging with distributed sensors

Based on a paper with Josselin Garnier that can be obtained from `http://math.stanford.edu/~papanico`

The simplest way randomness can enter is from sources that are (i) randomly distributed and (ii) are stationary random processes in time.

In this case the signals recorded at sensors located at $\{\mathbf{x}_j\}$, usually distributed over some region, are themselves stationary in time random processes. What information can possibly be in these signals? Can we image with them?

First some definitions:
1. Random process $\nu(t), t \in R$ (or $\mu(\mathbf{x}), \in R^d, d > 1$) are stationary if $\nu(t_1), \nu(t_2), \ldots, \nu(t_M)$ has the same joint law as $\nu(t_1+h), \nu(t_2+h), \ldots, \nu(t_M+h)$ for any set of points $\{t_j\}$ and any h. In this case $E\{\nu(t)\}$ is a constant which we take as zero.
2. The correlation $C(\tau) = E\{\nu(t)\nu(t+\tau)\}$ is a function of the lag τ only. Assuming that C is an integrable function, its Fourier transform $\widehat{C}(\omega) = \int e^{i\omega t} C(t) dt$ is always non-negative or more generally a measure (Bohner's theorem).

Let $u(t, \mathbf{x}_1)$ and $u(t, \mathbf{x}_2)$ denote the time-dependent wave fields recorded by two sensors at $\mathbf{x}_1$ and $\mathbf{x}_2$. Their empirical cross correlation function over the time interval $[0, T]$ with time lag τ is given by

$$C_T(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{T} \int_0^T u(t, \mathbf{x}_1) u(t + \tau, \mathbf{x}_2) dt \,.$$

In a homogeneous medium, if the source of the waves is a space-time stationary random field that is also delta correlated in space and time then we will show that

$$\frac{\partial}{\partial \tau} C_T(\tau, \mathbf{x}_1, \mathbf{x}_2) \simeq G(\tau, \mathbf{x}_1, \mathbf{x}_2) - G(-\tau, \mathbf{x}_1, \mathbf{x}_2) \,,$$

where G is the Green's function.

This approximate equality holds for T sufficiently large and provided some limiting absorption is introduced to regularize the integral. The main point here is that the time-symmetrized Green's function can be obtained from the cross correlation if there is enough source diversity. In this case the wave field at any sensor is equipartitioned, in the sense that it is a superposition of uncorrelated plane waves of all directions. We can recover in particular the travel time $\tau(\mathbf{x}_1, \mathbf{x}_2)$ from the singular support of the cross correlation.

Wave equation with noise sources

We consider solutions u of the wave equation in a three dimensional inhomogeneous (possibly random: $c(\mathbf{x})$ random) medium:

$$\frac{1}{c^2(\mathbf{x})}\frac{\partial^2 u}{\partial t^2} - \Delta_{\mathbf{x}} u = n^{\varepsilon}(t,\mathbf{x})\,.$$

The term $n^{\varepsilon}(t,\mathbf{x})$ models a random distribution of noise sources. It is a zero-mean stationary (in time) Gaussian process with autocorrelation function

$$\langle n^{\varepsilon}(t_1,\mathbf{y}_1)n^{\varepsilon}(t_2,\mathbf{y}_2)\rangle = F^{\varepsilon}(t_2-t_1)\Gamma(\mathbf{y}_1,\mathbf{y}_2)\,.$$

Here $\langle\cdot\rangle$ stands for statistical average with respect to the distribution of the noise sources.

We assume that the decoherence time of the noise sources is much smaller than typical travel times between sensors. If we denote with ε the (small) ratio of these two time scales, we can then write the time correlation function F^{ε} in the form

$$F^{\varepsilon}(t_2-t_1) = F(\frac{t_2-t_1}{\varepsilon})\,,$$

where t_1 and t_2 are scaled relative to typical sensor travel times.

Statistics of the noise sources

The Fourier transform $\widehat{F}^{\varepsilon}$ of the time correlation function is a nonnegative, even, real-valued function. It is proportional to the power spectral density of the sources:

$$\widehat{F}^{\varepsilon}(\omega) = \varepsilon \widehat{F}(\varepsilon\omega) \,,$$

where the Fourier transform is defined by

$$\widehat{F}(\omega) = \int F(t) e^{i\omega t} dt \,.$$

The spatial distribution of the noise sources is characterized by the autocovariance function Γ. It is the kernel of a symmetric nonnegative definite operator. For simplicity, we will assume that the process n is delta-correlated in space:

$$\Gamma(\mathbf{y}_1, \mathbf{y}_2) = \theta(\mathbf{y}_1)\delta(\mathbf{y}_1 - \mathbf{y}_2) \,,$$

where θ characterizes the spatial support of the sources.

The stationary solution of the wave equation has the integral representation

$$\begin{aligned} u(t,\mathbf{x}) &= \int\int_{-\infty}^{t} n^{\varepsilon}(s,\mathbf{y})G(t-s,\mathbf{x},\mathbf{y})dsd\mathbf{y} \\ &= \int\int n^{\varepsilon}(t-s,\mathbf{y})G(s,\mathbf{x},\mathbf{y})dsd\mathbf{y}\,, \end{aligned}$$

where $G(t,\mathbf{x},\mathbf{y})$ is the time-dependent Green's function. It is the fundamental solution of the wave equation

$$\frac{1}{c^2(\mathbf{x})}\frac{\partial^2 G}{\partial t^2} - \Delta_{\mathbf{x}} G = \delta(t)\delta(\mathbf{x}-\mathbf{y})\,,$$

starting from $G(0,\mathbf{x},\mathbf{y}) = \partial_t G(0,\mathbf{x},\mathbf{y}) = 0$ (and continued on the negative time axis by $G(t,\mathbf{x},\mathbf{y}) = 0 \ \forall t \leq 0$).

The empirical cross correlation of the signals recorded at $\mathbf{x}_1$ and $\mathbf{x}_2$ for an integration time T is

$$C_T(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{T} \int_0^T u(t, \mathbf{x}_1) u(t + \tau, \mathbf{x}_2) dt \, .$$

It is a statistically stable quantity, in the sense that for a large integration time T, C_T is independent of the realization of the noise sources.

The expectation of C_T (with respect to the distribution of the sources) is independent of T:

$$\langle C_T(\tau, \mathbf{x}_1, \mathbf{x}_2) \rangle = C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) \, ,$$

where $C^{(1)}$ is given by

$$C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \int d\mathbf{y} \int ds ds' G(s, \mathbf{x}_1, \mathbf{y}) G(\tau + s + s', \mathbf{x}_2, \mathbf{y}) F^{\varepsilon}(s') \theta(\mathbf{y}) \, ,$$

Fourier form and self-averaging of the cross correlation

In the frequency domain the cross correlation is given by

$$C^{(1)}(\tau,\mathbf{x}_1,\mathbf{x}_2) = \int d\mathbf{y} \int d\omega \overline{\widehat{G}}(\omega,\mathbf{x}_1,\mathbf{y})\widehat{G}(\omega,\mathbf{x}_2,\mathbf{y})\widehat{F}^{\varepsilon}(\omega)e^{-i\omega\tau}\theta(\mathbf{y})\,.$$

The empirical cross correlation C_T is a self-averaging quantity:

$$C_T(\tau,\mathbf{x}_1,\mathbf{x}_2) \overset{T\to\infty}{\longrightarrow} C^{(1)}(\tau,\mathbf{x}_1,\mathbf{x}_2)\,,$$

in probability with respect to the distribution of the sources.

More precisely, the fluctuations of C_T around its mean value $C^{(1)}$ are of order $T^{-1/2}$ for T large compared to the decoherence time of the sources.

Elementary derivation of the cross correlation identity

We derive the relation between the cross correlation and the Green's function when the medium is homogeneous with background velocity c_0 and the source distribution extends over all space, i.e. $\theta \equiv 1$.

In this case the signal amplitude diverges because the contributions from noise sources far away from the sensors are not damped. For a well-posed formulation we need to introduce some dissipation, so we consider the solution u of the damped wave equation:

$$\frac{1}{c_0^2}\left(\frac{1}{T_a} + \frac{\partial}{\partial t}\right)^2 u - \Delta_{\mathbf{x}} u = n^{\varepsilon}(t, \mathbf{x}) \,.$$

Cross correlation identity in a homogeneous medium

In a three-dimensional medium with dissipation and with noise source distribution extending over all space $\theta \equiv 1$

$$\frac{\partial}{\partial \tau} C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = -\frac{c_0^2 T_a}{4} e^{-\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{c_0 T_a}} \left[F^\varepsilon * G(\tau, \mathbf{x}_1, \mathbf{x}_2) - F^\varepsilon * G(-\tau, \mathbf{x}_1, \mathbf{x}_2) \right],$$

where $*$ denotes convolution in τ and G is the Green's function of the homogeneous medium without dissipation:

$$G(t, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{4\pi |\mathbf{x}_1 - \mathbf{x}_2|} \delta\Big(t - \frac{|\mathbf{x}_1 - \mathbf{x}_2|}{c_0}\Big).$$

Estimating travel times from cross correlations

If the decoherence time of the sources is much shorter than the travel time (i.e., $\varepsilon \ll 1$), then F^ε behaves like a Dirac distribution and we have

$$\frac{\partial}{\partial \tau} C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) \simeq e^{-\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{c_0 T_a}} \left[G(\tau, \mathbf{x}_1, \mathbf{x}_2) - G(-\tau, \mathbf{x}_1, \mathbf{x}_2) \right],$$

up to a multiplicative constant.

It is therefore possible to estimate the travel time $\tau(\mathbf{x}_1, \mathbf{x}_2) = |\mathbf{x}_1 - \mathbf{x}_2|/c_0$ between $\mathbf{x}_1$ and $\mathbf{x}_2$ from the cross correlation, with an accuracy of the order of the decoherence time of the noise sources.

On the extraction of the travel time from cross correlations

The cross correlation is closely related to the symmetrized Green's function from $\mathbf{x}_1$ to $\mathbf{x}_2$ not only for a homogeneous medium but also for inhomogeneous media.

One can give a simple and rigorous proof for an open inhomogeneous medium in the case in which the noise sources are located on the surface of a sphere that encloses both the inhomogeneous region and the sensors, located at $\mathbf{x}_1$ and $\mathbf{x}_2$.

The proof is based on an approximate identity that follows from Green's identity and the Sommerfeld radiation condition. This approximate identity is none other than the Helmholtz-Kirchhoff integral theorem of Lecture III.

Velocity estimation with travel time tomography

If the sensors are at known locations $\{\mathbf{x}_j\}, j = 1, 2, \ldots, N$ and are suitably distributed over a region whose speed of propagation $c(\mathbf{x})$ is unknown, then this speed can be estimated from the travel time $\{\tau(\mathbf{x}_i, \mathbf{x}_j)\}$. There are tomographic algorithms for doing this estimation. The accuracy and robustness of the resulting estimate $\hat{c}(\mathbf{x})$ will depend on (i) the accuracy of the travel time estimates, (ii) the topology of the sensor network, and (iii) the properties of the ambient noise sources, which are also unknown.

The estimation of the surface wave velocity in Southern California from seismic noise correlations over some 150 seismic stations was a breakthrough in 2005 when it was successfully done by Sabra, Gerstoft, Roux, and Kuperman (Surface wave tomography from microseisms in Southern California, *Geophys. Res. Lett.* **32** L14311)

Derivation of the cross correlation identity

The Green's function of the homogeneous medium with dissipation is:

$$G_a(t, \mathbf{x}_1, \mathbf{x}_2) = G(t, \mathbf{x}_1, \mathbf{x}_2) e^{-\frac{t}{T_a}} .$$

The cross correlation function is given by

$$C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \int d\mathbf{y} \int ds ds' G_a(s, \mathbf{x}_1, \mathbf{y}) G_a(\tau + s + s', \mathbf{x}_2, \mathbf{y}) F^\varepsilon(s') .$$

Integrating in s and s' gives

$$C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \int \frac{d\mathbf{y}}{16\pi^2 |\mathbf{x}_1 - \mathbf{y}| \, |\mathbf{x}_2 - \mathbf{y}|} e^{-\frac{|\mathbf{x}_1 - \mathbf{y}| + |\mathbf{x}_2 - \mathbf{y}|}{c_0 T_a}} F^\varepsilon\Big(\tau - \frac{|\mathbf{x}_1 - \mathbf{y}| - |\mathbf{x}_2 - \mathbf{y}|}{c_0}\Big) .$$

We parameterize the locations of the sensors by $\mathbf{x}_1 = (h, 0, 0)$ and $\mathbf{x}_2 = (-h, 0, 0)$, where $h > 0$, and we use the change of variables for $\mathbf{y} = (x, y, z)$:

$$\begin{cases} x = h \sin\theta \cosh\phi , & \phi \in (0, \infty) , \\ y = h \cos\theta \sinh\phi \cos\psi , & \theta \in (-\pi/2, \pi/2) , \\ z = h \cos\theta \sinh\phi \sin\psi , & \psi \in (0, 2\pi) , \end{cases}$$

whose Jacobian is $J = h^3 \cos\theta \sinh\phi (\cosh^2\psi - \sin^2\theta)$. Using the fact that $|\mathbf{x}_1 - \mathbf{y}| = h(\cosh\phi - \sin\theta)$ and $|\mathbf{x}_2 - \mathbf{y}| = h(\cosh\phi + \sin\theta)$, we get

$$C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{h}{8\pi} \int_0^\infty d\phi \sinh\phi \int_{-\pi/2}^{\pi/2} d\theta \cos\theta e^{-\frac{2h\cosh\phi}{c_0 T_a}} F^\varepsilon\Big(\tau + \frac{2h\sin\theta}{c_0}\Big) .$$

After the new change of variables $u = h \cosh \phi$ and $s = (2h/c_0) \sin \theta$, we obtain

$$C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{c_0^2 T_a}{32\pi h} e^{-\frac{2h}{c_0 T_a}} \int_{-2h/c_0}^{2h/c_0} F^\varepsilon(\tau + s) ds \,.$$

By differentiating in τ, we get

$$\frac{\partial}{\partial \tau} C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{c_0^2 T_a}{32\pi h} e^{-\frac{2h}{c_0 T_a}} [F^\varepsilon(\tau + \frac{2h}{c_0}) - F^\varepsilon(\tau - \frac{2h}{c_0})] \,,$$

which is the desired result since $|\mathbf{x}_1 - \mathbf{x}_2| = 2h$.

Use of the Kirchhoff-Helmholtz identity

Let us assume that the medium is homogeneous with background velocity c_e outside the ball $B(\mathbf{0}, r)$ with center $\mathbf{0}$ and radius r. Then, for any $\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{0}, r)$ we have for $L \gg r$ the KH identity:

$$\widehat{G}(\omega, \mathbf{x}_1, \mathbf{x}_2) - \overline{\widehat{G}}(\omega, \mathbf{x}_1, \mathbf{x}_2) = \frac{2i\omega}{c_e} \int_{\partial B(\mathbf{0}, L)} \overline{\widehat{G}}(\omega, \mathbf{x}_1, \mathbf{y}) \widehat{G}(\omega, \mathbf{x}_2, \mathbf{y}) dS(\mathbf{y}) \, .$$

We also assume that the sources are localized with a uniform density on the sphere $\partial B(\mathbf{0}, L)$ with center $\mathbf{0}$ and radius L.

If $L \gg r$, then for any $\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{0}, r)$

$$\frac{\partial}{\partial \tau} C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = -F^{\varepsilon} * G(\tau, \mathbf{x}_1, \mathbf{x}_2) + F^{\varepsilon} * G(-\tau, \mathbf{x}_1, \mathbf{x}_2) \, ,$$

up to a multiplicative factor. Here $*$ stands for convolution in τ.

Directional sources

What if the ambient noise sources are not distributed evenly about the sensors? What if the wave fields recorded at the sensors have a dominant orientation instead of being equdistributed in all directions?

In such cases we cannot expect to be able to recover the full (symmetrized) Green's function between the sensors. At best we can recover the travel time $\tau(\mathbf{x}_1, \mathbf{x}_2)$ if the line (the ray) connecting the two sensors continues into the source region. This is done using the stationary phase method and is discussed further in Garnier's lecture (Thursday afternoon).

What about imaging reflectors with passive sensor networks using ambient noise sources? This can be done using suitable fourth order cross correlations.

Concluding remarks on noise cross correlations

- Cross correlations can be used effectively in closed environments with limited ambient noise source diversity but enhancing by multiple wall reflections (ergodic cavities)
- It is also possible that a scattering medium can enhance ambient noise source diversity and make the estimation of background velocities feasible
- But there is a limitation in how strongly scattering the medium can be: the transport mean free path must be long compared to the distance between sensors (to preserve coherence) but short compared to the distance between noise sources and sensors
- In a scattering medium, the transport mean free path is a rough measure of how far waves have to propagate before they lose their coherence and wave energy diffuses isotropically