Imaging in Random Media

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The singular value decomposition, in detection and imaging

Let $\widehat{\Pi}(\mathbf{x}_r, \mathbf{x}_s, \omega)$ be the array impulse response matrix over the bandwidth $\omega_0 - B/2 < \omega < \omega_0 + B/2$, with N_s sources and N_r receivers which we will assume are collocated and $N_s = N_r = N$.

If $\widehat{f}(\omega) = (\widehat{f}(\mathbf{x}_s, \omega))$ is a vector of illuminations in the frequency domain, then

$$\widehat{P}_f(\omega) = \left(\sum_{\mathbf{x}_s} \widehat{\Pi}(\mathbf{x}_r, \mathbf{x}_s, \omega) \widehat{f}(\mathbf{x}_s, \omega) \right)$$

is the vector of received signals at the array, in the frequency domain. The total power of these signals is $\mathcal{P}_{tot}(f) = \int d\omega ||\widehat{\Pi}(\omega)\widehat{f}(\omega)||^2$

Problem: Find $\mathcal{P} = \max_f \mathcal{P}_{tot}(f)$ with $||f||^2 = \int d\omega ||\widehat{f}(\omega)||^2 = 1$

This problem of optimal illumination for received power, that is, for detection, is solved using the SVD of $\widehat{\Pi}$. We assume that we have a fixed bandwidth of size B.

The SVD of the array response matrix

The array impulse response matrix is symmetric but not hermitian $\widehat{\Pi}^T(\omega) = \widehat{\Pi}(\omega)$. Let $\hat{\mathbf{v}}_j$ and $\hat{\mathbf{u}}_j$ be its right and left singular vectors respectively. Then its singular value decomposition is

$$\widehat{\Pi}(\omega) = \sum_{j=1}^{p} \sigma_j(\omega)\hat{\mathbf{u}}_j(\omega)\hat{\mathbf{v}}_j^*(\omega)$$

Here $p \leq N$ is the rank of $\widehat{\Pi}$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p > 0$.

Suppose that $\omega^* = \text{argmax}\sigma_1(\omega)$ over the bandwidth and let

$$\widehat{f}(\omega) = \frac{1}{2\delta}\hat{\mathbf{v}}_1(\omega^*) \ , \quad \omega \in [\omega^* - \delta, \omega^* + \delta]$$

and zero outside this interval. Then for this illumination f we have that $\mathcal{P}_{tot}(f) \to \mathcal{P} = \sigma_1^2(\omega^*)$ as $\delta \to 0$. The optimal illumination is a narrow band signal proportional to $\hat{\mathbf{v}}_1(\omega^*)$.

Iterative time reversal, frequency domain

Consider the following iterative process (experiment). It is done in the time domain but we describe it frequency by frequency:

1. Start with illumination $\widehat{f}$. The received signal at the array is $\widehat{\Pi}\widehat{f}$
2. Use the time reversed field as illumination. The received field is $\widehat{\Pi}\overline{\widehat{\Pi}\widehat{f}}$
3. Repeat these two steps n times.

The field received at the array has the form

$$\widehat{K}^n(\omega)\overline{\widehat{f}(\omega)} \ , \quad \widehat{K}(\omega) = \widehat{\Pi}(\omega)\overline{\widehat{\Pi(\omega)}}$$

where $\widehat{K}(\omega)$ is the time reversal operator. It is hermitian and positive definite for each frequency, and its eigenvalues are the squares of the singular values. Therefore for large n we have

$$\widehat{K}^n(\omega)\overline{\widehat{f}(\omega)} \approx \sigma_1^{2n}(\omega)\hat{\mathbf{v}}_1(\omega)\hat{\mathbf{v}}_1^*(\omega)\overline{\widehat{f}(\omega)}$$

In the time domain the signal received at the array after a large number n of iterative TR has the approximate form

$$\int d\omega e^{-i\omega t}\sigma_1^{2n}(\omega)\hat{\mathbf{v}}_1(\omega)\hat{\mathbf{v}}_1^*(\omega)\overline{\hat{f}(\omega)}$$

By the Laplace asymptotic method it can be further approximated, up to a constant, by

$$e^{-i\omega^* t}\sigma_1^{2n}(\omega^*)\hat{\mathbf{v}}_1(\omega^*)\hat{\mathbf{v}}_1^*(\omega^*)\overline{\hat{f}(\omega^*)}$$

which is a time harmonic signal at the frequency where $\sigma_1(\omega)$ takes it maximum value.

With ITR we can get $\hat{\mathbf{v}}_1(\omega)$ directly from the physical experiment without doing the SVD. This however requires some special adjustments in order to get it over the full bandwidth. The other singular vectors can also be obtained with ITR.

But why are we interested in the SVD of the response matrix and ITR, which is a physical way of getting the SVD?

Point scatterer models

Let M point scatterers be at $\mathbf{y}_j, j = 1, 2, ..., M$. Point scatterers means that the reflectivity is

$$\rho(\mathbf{z}) = \sum_{j=1}^{M} \rho_j \mathbf{1}_{|\mathbf{z}-\mathbf{y}_j| \leq \delta_j},$$

with the radii δ_j small compared to wavelengths. In this case the impulse response matrix in the Born approximation is

$$\widehat{\Pi}(\mathbf{x}_r, \mathbf{x}_s, \omega) = \sum_{j=1}^{M} \xi_j(\omega) \widehat{G}(\mathbf{x}_s, \mathbf{y}_j, \omega) \widehat{G}(\mathbf{x}_r, \mathbf{y}_j, \omega)$$

The scattering amplitudes $\xi_j(\omega)$ depend on the reflectivities and radii (or shape, in general), and on the frequency. Define the array vector Green's function

$$\widehat{g}(\mathbf{y}, \omega) = (\widehat{G}(\mathbf{x}_r, \mathbf{y}, \omega))$$

Then the array impulse response matrix has the form

$$\widehat{\Pi}(\omega) = \sum_{j=1}^{M} \xi_j(\omega) \widehat{g}(\mathbf{y}_j, \omega) \widehat{g}^T(\mathbf{y}_j, \omega)$$

The array vectors $\{\widehat{g}(\mathbf{y}_j,\omega)\}$ are not of course orthogonal in general. But

$$\widehat{g}^*(\mathbf{y}_j,\omega)\widehat{g}(\mathbf{y}_l,\omega) = \sum_{\mathbf{x}_r} \overline{\widehat{G}(\mathbf{x}_r,\mathbf{y}_j,\omega)}\widehat{G}(\mathbf{x}_r,\mathbf{y}_l,\omega)$$

is exactly the basic quantity that arises in imaging and time reversal, and whose behavior we have analyzed in Lectures II-III. We know that if the distance $|\mathbf{y}_j - \mathbf{y}_l|$ is large compared to the resolution limit of the array at this frequency, then these array vectors are approximately orthogonal

$$\widehat{g}^*(\mathbf{y}_j,\omega)\widehat{g}(\mathbf{y}_l,\omega) \approx ||\widehat{g}(\mathbf{y}_j,\omega)||^2\delta_{jl}$$

In any case we may assume that the $\{\widehat{g}(\mathbf{y}_j,\omega)\}$ are linearly independent.

In the well separated case the array impulse response matrix is in SVD form.

Well separated scatterers, continued

In the well separated case we have

$$\widehat{\Pi}\overline{\widehat{g}(\mathbf{y}_l)} = \sum_{j=1}^{M} \xi_j \widehat{g}(\mathbf{y}_j)\widehat{g}^T(\mathbf{y}_j)\overline{\widehat{g}(\mathbf{y}_l)} \approx \xi_l ||\widehat{g}(\mathbf{y}_l)||^2 \widehat{g}(\mathbf{y}_l)$$

Therefore assuming that the ξ_l are positive and that $\xi_l||\widehat{g}(\mathbf{y}_l)||^2$ are arranged in decreasing order we have

$$\widehat{\mathbf{v}}_l = \frac{\overline{\widehat{g}(\mathbf{y}_l)}}{||\widehat{g}(\mathbf{y}_l)||}, \quad \widehat{\mathbf{u}}_l = \overline{\widehat{\mathbf{v}}_l}, \quad \sigma_l = \xi_l||\widehat{g}(\mathbf{y}_l)||^2$$

We conclude that the rank of the SVD can be associated uniquely with the number of small scatterers, even of they are not well separated, up to some special configurations.

We now look at time reversal and imaging with the SVD

With illumination f the time reversal field at $\mathbf{y}^S$ is

$$\Gamma_f^{TR}(\mathbf{y}^S) = \int d\omega \widehat{g}^T(\mathbf{y}^S,\omega)\overline{\widehat{\Pi}(\omega)\widehat{f}(\omega)}$$

When $\widehat{f} = \widehat{\mathbf{v}}_l$ then

$$\Gamma_l^{TR}(\mathbf{y}^S) = \int d\omega \sigma_l(\omega)\widehat{g}^T(\mathbf{y}^S,\omega)\overline{\widehat{\mathbf{u}}_l(\omega)}$$

and in the well separated case

$$\Gamma_l^{TR}(\mathbf{y}^S) = \int d\omega \xi_l(\omega)||\widehat{g}(\mathbf{y}_l,\omega)||\widehat{g}^T(\mathbf{y}^S,\omega)\overline{\widehat{g}(\mathbf{y}_l,\omega)}$$

What is interesting here is that by using the SVD we can selectively do time reversal to the l-th scatterer. And by using iterative time reversal we can do this completely in hardware, without doing a numerical SVD. There are advantages to this when SNR issues are important.

In order to image we have to back propagate in a known medium, which we take as a homogeneous one and let

$$\widehat{g}_0(\mathbf{y},\omega) = (\widehat{G}_0(\mathbf{x}_r,\mathbf{y},\omega))$$

The Kirchhoff migration functional in then given by

$$I^{KM}(\mathbf{y}^S) = \int d\omega \widehat{g}_0^T(\mathbf{y}^S,\omega)\overline{\widehat{\Pi}(\omega)}\widehat{g}_0(\mathbf{y}^S,\omega)$$

$$= \int d\omega \sum_{j=1}^{p} \sigma_j(\omega)\widehat{g}_0^T(\mathbf{y}^S,\omega)\overline{\widehat{\mathbf{u}}_j(\omega)}\widehat{\mathbf{v}}_j^T(\omega)\widehat{g}_0(\mathbf{y}^S,\omega)$$

In the case of well separated scatterers we have

$$I^{KM}(\mathbf{y}^S) = \int d\omega \sum_{j=1}^{p} \xi_j(\omega)|\widehat{g}_0^*(\mathbf{y}^S,\omega)\widehat{g}_0(\mathbf{y}_j,\omega)|^2$$

We see now how the basic resolution theory of the source point spread function can be carried over to KM.

We see that KM imaging, which is the the unfiltered least squares imaging functional ($A^H A \approx I$), is rather strange because it illuminates a spot and then back propagates to it with the same array vector $\widehat{g}_0(\mathbf{y}^S, \omega)$. If we choose a general illumination vector f, a linear combination of the right singular vectors for example, we have

$$I^{BP}(\mathbf{y}^S; f) = \int d\omega \widehat{g}_0^T(\mathbf{y}^S, \omega)\overline{\widehat{\Pi}(\omega)}\widehat{f}(\omega)$$

If we let $\widehat{f}(\omega) = \sum d_l(\omega)\widehat{\mathbf{v}}_l(\omega)$ then

$$I^{BP}(\mathbf{y}^S; d) = \int d\omega \sum_{j=1}^{p} \sigma_j(\omega)\widehat{g}_0^T(\mathbf{y}^S, \omega)\overline{\widehat{\mathbf{u}}_j(\omega)}\widehat{\mathbf{v}}_j^T(\omega) \sum_{l=1}^{p} d_l(\omega)\overline{\widehat{\mathbf{v}}_l(\omega)}$$

$$I^{BP}(\mathbf{y}^S; d) = \int d\omega \sum_{j=1}^{p} \sigma_j(\omega) d_j(\omega)\widehat{g}_0^T(\mathbf{y}^S, \omega)\overline{\widehat{\mathbf{u}}_j(\omega)}$$

We can now look for a way to choose the weights $\{d_j(\omega)\}$ so as to optimize the image.

Let

$$\mathcal{G}_j(\omega) = \int d\mathbf{y}^S |\hat{g}_0^T(\mathbf{y}^S,\omega)\overline{\hat{\mathbf{u}}_j(\omega)}|^2$$

with the integration over some window, and let

$$\mathcal{I}(d) = \int d\omega \sum_{j=1}^{p} \sigma_j(\omega) d_j(\omega) \mathcal{G}_j(\omega)$$

Then we can try to find weights $\{d_j(\omega)\}$ that minimize this objective function.

There is no reason to adhere to the array least squares criterion, which leads to filtered back propagation (or back projection), as a basis for imaging.

Criteria based on the quality of the image directly have many advantages, especially in random media.

Another way to introduce an optimization process using the SVD is by subspace selection. Let

$$D\left[\widehat{\Pi}(\omega);\omega\right] = \sum_{j=1}^{p} \sigma_j(\omega) d_j(\omega) \hat{\mathbf{u}}_j(\omega) \hat{\mathbf{v}}_j^*(\omega)$$

a subspace selector with weights $\{d_j(\omega)\}$. Now consider KM imaging with it instead of $\widehat{\Pi}$. We have

$$I^{KM}(\mathbf{y}^S;d) = \int d\omega \widehat{g}_0^T(\mathbf{y}^S,\omega) \overline{D\left[\widehat{\Pi}(\omega);\omega\right]} \widehat{g}_0(\mathbf{y}^S,\omega)$$

or

$$I^{KM}(\mathbf{y}^S;d) = \int d\omega \sum_{j=1}^{p} \sigma_j(\omega) d_j(\omega) (\hat{\mathbf{u}}_j^*(\omega) \widehat{g}_0(\mathbf{y}^S,\omega))^2$$

If we now integrate $|I^{KM}(\mathbf{y}^S;d)|$ over an image window we see that we get back an objective similar to the optimal illumination criterion, which we must minimize over $\{d_j(\omega)\}$.

Summary of SVD, TR and imaging methods

- The SVD of the array response matrix allows for selective focusing with TR on small scatterers

- The SVD singular vectors for TR can be computed directly with ITR without having to know the full response matrix in advance

- The SVD can be used for optimal illumination, or optimal subspace selection, for migration imaging that is based on the quality of the image itself