Imaging in random media

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Waves in random media: Layered media, the paraxial approximation, radiative transport

Time reversal in random media, super-resolution, statistical stability

We consider the wave equation in a random medium

$$\frac{1}{c^2(\vec{\mathbf{x}})}\frac{\partial^2 u}{\partial t^2} - \triangle u = 0 \ , \quad t > 0 \ , \quad \vec{\mathbf{x}} \in \mathbb{R}^{d+1} \ ,$$

with $d = 1, 2$ and the local wave speed

$$c^{-2}(z, \mathbf{x}) = c_0^{-2}\left[1 + \sigma_0 \mu\left(\frac{z}{l_z}, \frac{\mathbf{x}}{l_x}\right)\right].$$

Here z and $\mathbf{x} \in \mathbb{R}^d$ are, respectively, the coordinates along and transverse to the direction of propagation, and $\vec{\mathbf{x}} = (z, \mathbf{x})$. The random function μ models the fluctuations in the propagation speed.

When the characteristic scale of variation in the transverse direction l_x is large compared to l_z then we have a layered random medium. When $l_x = l_z = l$ then we have essentially isotropic randomness.

The three regimes of random wave propagation

- Layered: Very strong scattering in direction of propagation. Wave localization, long wave codas

- Wave transport: Wave energy "diffuses" by radiative transport. The transport mean free path

- The paraxial or parabolic regime: one-way wave propagation for beams, with scattering into lateral directions and no backscattering

- Layered and paraxial are approximations that have very well developed mathematical theories. Real world phenomena are somewhere in between

Paraxial or parabolic approximation

We consider wave fields propagating mainly in the z direction

$$u(t, \mathbf{x}, z) = \frac{1}{2\pi} \int e^{i\omega(z/c_0 - t)} \psi(z, \mathbf{x}; \omega/c_0) d\omega$$

The complex amplitude $\psi(z, \mathbf{x}; k)$ satisfies the Helmholtz equation

$$2ik\psi_z + \Delta_{\mathbf{x}}\psi + k^2(n^2 - 1)\psi = -\psi_{zz}.$$

Here $k = \omega/c_0$ is the wavenumber and $n(\mathbf{x}, z) = c_0/c(\mathbf{x}, z)$ is the random index of refraction relative to a reference speed c_0. The fluctuations of the refraction index have the form

$$n^2(\mathbf{x}, z) - 1 = \sigma_0 \mu \left(\frac{z}{l}, \frac{\mathbf{x}}{l} \right)$$

They are a stationary random field with mean zero, variance σ_0^2 and correlation length l. The normalized and dimensionless covariance is given by

$$R(z, \mathbf{x}) = E\{\mu(z + z', \mathbf{x} + \mathbf{x}')\mu(z', \mathbf{x}')\}.$$

On the numerical simulation of μ

1. Write (in 1D): $R(x) = (1/2\pi)\int dk e^{ikx}\widehat{R}(k)$, with $\widehat{R}(k)$ the power spectral density, and discretize the integral with mesh size Δk

2. Generate independent identically distributed complex random variables $\widehat{\mu}_n$ with mean zero and variance $\widehat{R}(n\Delta k)\Delta k/2\pi$, and so that $\overline{\widehat{\mu}_n} = \widehat{\mu}_{-n}$

3. The process $\mu_{\Delta k}(x) = \sum_n e^{in\Delta kx}\widehat{\mu}_n$ is an approximate realization of $\mu(x)$

Scales

- L_z, the characteristic distance in the direction of propagation.

- $L_{\mathbf{x}}$, the length scale in the directions transverse to the direction of propagation. This is typically taken to be the width of the propagating beam.

- $k_0 = 2\pi/\lambda_0$, the central wavenumber corresponding to the central wavelength λ_0.

- l, the correlation length of the random medium. It characterizes the dominant spatial scale of the random fluctuations.

- σ_0, the dimensionless standard deviation of the random fluctuations in the medium.

In the asymptotic regimes that we consider here L_z and L_x are large compared to l and λ_0, and σ_0 is small.

Scaled, dimensionless wave equation

We obtain the dimensionless form of the equation by introducing dimensionless variables by $\mathbf{x} = L_{\mathbf{x}}\mathbf{x}'$, $z = L_z z'$, $k = k_0 k'$ and rewriting it as

$$2ik\frac{\partial \psi}{\partial z} + \frac{L_z}{k_0 L_{\mathbf{x}}^2}\Delta_{\mathbf{x}}\psi + k^2 k_0 L_z \sigma_0 \mu\left(\frac{zL_z}{l}, \frac{\mathbf{x}L_{\mathbf{x}}}{l}\right)\psi = -\frac{1}{L_z k_0}\frac{\partial^2 \psi}{\partial z^2},$$

after dropping the primes. We identify now the following three, usually small, dimensionless parameters in the problem:

- $\varepsilon = \dfrac{l}{L_z}$, the ratio of the correlation length to the propagation distance,
- $\delta = \dfrac{l}{L_{\mathbf{x}}}$, the ratio of the correlation length to the transverse length scale, which is usually the beam width,
- $\theta = \dfrac{L_z}{k_0 L_{\mathbf{x}}^2} = \dfrac{\lambda_0 L_z}{2\pi L_{\mathbf{x}}^2}$, the reciprocal of the Fresnel number, the ratio of the diffraction focal spot of the beam to its width.

In terms of these parameters we have

$$2ik\psi_z + \theta\Delta_{\mathbf{x}}\psi + \frac{k^2\sigma_0\delta^2}{\theta\varepsilon^2}\mu(\frac{z}{\varepsilon}, \frac{\mathbf{x}}{\delta})\psi = -\frac{\theta\varepsilon^2}{\delta^2}\psi_{zz}.$$

We assume that ε is the smallest parameter in the problem. It then follows formally, but it is quite difficult to prove, that the ψ_{zz} term is a lower order term and can be neglected.

$$2ik\psi_z + \theta\Delta_{\mathbf{x}}\psi + \frac{k^2\sigma\delta}{\theta\sqrt{\varepsilon}}\mu\left(\frac{z}{\varepsilon}, \frac{\mathbf{x}}{\delta}\right)\psi = 0\,, z > 0$$

with ψ at $z = 0$ given and where

$$\sigma = \frac{\sigma_0\delta}{\varepsilon^{3/2}}.$$

This scaled noise strength parameter is assumed to be independent of ε and δ as these parameters tend to zero in the asymptotic analysis.

The white noise limit

We consider the limit $\varepsilon \to 0$ while δ and θ are fixed. This means that ε is the smallest of the three parameters $\varepsilon, \theta, \delta$. Assume that the CLT applies to the random field μ:

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} \int_0^z \mu\left(\frac{s}{\varepsilon}, \mathbf{x}\right) ds = B(z, \mathbf{x}),$$

weakly in law, where B is a Brownian random field parameterized by $\mathbf{x}$. This means that for any test function $h(\mathbf{x})$, in law

$$\frac{1}{\sqrt{\varepsilon}} \int_0^z \mu_h(s/\varepsilon) ds \mapsto B_h(z), z \geq 0,$$

$$\mu_h(z) = \int_{\mathbb{R}^d} \mu(z, \mathbf{x}) h(\mathbf{x}) d\mathbf{x} \ , \quad B_h(z) = \int_{\mathbb{R}^d} B(z, \mathbf{x}) h(\mathbf{x}) d\mathbf{x}.$$

The random field $B(z, \mathbf{x})$ is Gaussian with mean zero and

$$E\{B(z_1, \mathbf{x}_1) B(z_2, \mathbf{x}_2)\} = R_0(|\mathbf{x}_1 - \mathbf{x}_2|) \min\{z_1, z_2\}.$$

Here R_0 is the integrated correlation function $R_0(\mathbf{x}) = \int_{-\infty}^{\infty} R(z, \mathbf{x}) dz$.

The Ito-Schrödinger equation

In the white noise limit $\varepsilon \to 0$ the solution of the random partial differential equation converges in law to the process defined by the stochastic partial differential equation

$$2ikd_z\psi + \theta\Delta_{\mathbf{x}}\psi dz + \frac{k^2\sigma\delta}{\theta}\psi \circ d_z B\left(\frac{\mathbf{x}}{\delta}, z\right) = 0$$

given here in the Stratonovich form. The Itô form is

$$2ikd_z\psi + \theta\Delta_{\mathbf{x}}\psi dz + \frac{ik^3\sigma^2\delta^2}{4\theta^2}R_0(0)\psi dz + \frac{k^2\sigma\delta}{\theta}\psi d_z B\left(\frac{\mathbf{x}}{\delta}, z\right) = 0.$$

There are two small parameters left in the Itô-Schrödinger equation after we have taken the white-noise limit – the reciprocal Fresnel number θ and the non-dimensional correlation length δ.

We can consider the following limits.

High and low frequency; lateral diversity

- The low frequency limit and large lateral diversity limit: $\delta \to 0$ with θ fixed,

- the high frequency or geometric asymptotics limit followed by the large lateral diversity limit: $\theta \ll \delta \ll 1$, that is, $\theta \to 0$ followed by $\delta \to 0$, and

- the combined scaling limit: $\theta \sim \delta \ll 1$ with $\theta \to 0$ and $\delta \to 0$ simultaneously.

We refer to the limit $\theta \to 0$ as the high frequency limit and to the limit $\delta \to 0$ as the limit of large lateral diversity.

We see that if we pass to the limit $\delta \to 0$ with a fixed $\theta > 0$ we arrive at the homogeneous Schrödinger equation

$$2ik\psi_z + \theta\Delta_{\mathbf{x}}\psi = 0.$$

This is because we have an a priori bound $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$ and for any deterministic test function $\eta(z, \mathbf{x})$ we have by the Itô isometry

$$E\left[\frac{k^2\sigma\delta}{\theta}\int_0^z\int \eta(s,\mathbf{x})\psi(s,\mathbf{x})d_zB\left(\frac{\mathbf{x}}{\delta},s\right)d\mathbf{x}\right]^2$$
$$= \left(\frac{k^2\sigma\delta}{\theta}\right)^2 E\int_0^z\int \eta(s,\mathbf{x})\eta(s,\mathbf{x}')\psi(s,\mathbf{x})\psi(s,\mathbf{x}')R_0\left(\frac{\mathbf{x}-\mathbf{x}'}{\delta}\right)d\mathbf{x}d\mathbf{x}'ds \to 0 \text{ as } \delta \to 0.$$

A similar bound holds for the third term and therefore convergence in probability follows.

Phase space

In the high frequency limit $\theta \to 0$ (whether coupled with the limit $\delta \to 0$, or not) solutions of the Itô-Schrödinger equation become oscillatory in time and space. Therefore, rather than studying the limit of the solution itself we consider the limits of its Wigner transform which resolves the wave energy of oscillatory fields in the phase space and (unlike the spatial energy density) satisfies a closed evolution equation.

We define the spatial Fourier transform and its inverse by

$$\widehat{f}(\mathbf{k}) = \int d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \ , \quad f(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} \widehat{f}(\mathbf{k}) \, ,$$

where $d = 1$ or 2 is the number of transverse spatial dimensions.

The Wigner transform relative to the scale θ is

$$W_\theta(z, \mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{y}} \psi(\mathbf{x} - \frac{\theta \mathbf{y}}{2}, z) \overline{\psi(\mathbf{x} + \frac{\theta \mathbf{y}}{2}, z)} d\mathbf{y}$$

The Wigner distribution is real, may be interpreted as phase space wave energy. It is well suited for random media.

Stochastic transport equation

Using the Itô calculus we find from the Ito-Schrödinger equation that the scaled Wigner distribution satisfies the stochastic transport equation

$$
\begin{aligned}
dW_\theta(z,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}} W_\theta(z,\mathbf{x},\mathbf{p})dz &= \frac{k^2\sigma^2\delta^2}{4\theta^2}\int \left(W_\theta\left(z,\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{\delta}\right) - W_\theta(z,\mathbf{x},\mathbf{p})\right)\frac{\widehat{R}_0(\mathbf{q})d}{(2\pi)^d} \\
&+ \frac{ik\sigma\delta}{2\theta}\int \frac{d\mathbf{q}}{(2\pi)^d} e^{i\mathbf{q}\cdot\mathbf{x}/\delta}\left(W_\theta\left(z,\mathbf{x},\mathbf{p}-\frac{\theta\mathbf{q}}{2\delta}\right) - W_\theta\left(z,\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{2\delta}\right)\right) d\widehat{B}(\mathbf{q},z).
\end{aligned}
$$

We do the high frequency and large diversity limits with the Itô-Wigner equation as a starting point.

We note that the L^2 norm of the Wigner distribution is conserved

$$
\|W_\theta(z)\|_{L^2(\mathbb{R}^{2d})} = \|W_\theta(0)\|_{L^2(\mathbb{R}^{2d})}
$$

High frequency limit

When we take the high frequency limit we find that W_θ converges weakly to W_δ satisfying the Itô-Liouville equation

$$dW_\delta(z,\mathbf{x},\mathbf{p})+\frac{\mathbf{p}}{k}\cdot\nabla_\mathbf{x}W_\delta(z,\mathbf{x},\mathbf{p})dz+\frac{k^2\sigma^2}{8}R_0''(0)\triangle_\mathbf{p}W_\delta dz=-\frac{k\sigma}{2}d\nabla_\mathbf{x}B\left(\frac{\mathbf{x}}{\delta},z\right)\cdot\nabla_\mathbf{p}W_\delta.$$

We remark that $R''(0)<0$ so that this equation is well-posed.

This SPDE is connected to stochastic flows where solutions of SDE's play the role of characteristics (Kunita).

Large diversity limit

The limiting Wigner distribution solves a stochastic PDE, in which the coefficient of the random term fluctuates on the small scale δ. When we subsequently take the limit of large lateral diversity we find that the limiting Wigner distribution actually becomes deterministic. We refer to this as the stabilization of the Wigner distribution. Define W as the deterministic solution of

$$\frac{\partial W}{\partial z}(z,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}} W(z,\mathbf{x},\mathbf{p}) + \frac{k^2\sigma^2}{8} R_0''(0)\triangle_{\mathbf{p}} W = 0.$$

There are two simple and practical items to remember when with waves in random media and how affect TR and imaging calculations.
One is the moment formula:

$$E\{\overline{\widehat{G}(\mathbf{x}_r,\mathbf{y},\omega)}\widehat{G}(\mathbf{x}_r,\mathbf{y}^S,\omega)\} \approx \overline{\widehat{G}_0(\mathbf{x}_r,\mathbf{y},\omega)}\widehat{G}_0(\mathbf{x}_r,\mathbf{y}^S,\omega)e^{\frac{-k^2\xi^2 a_e}{2L^2}}$$

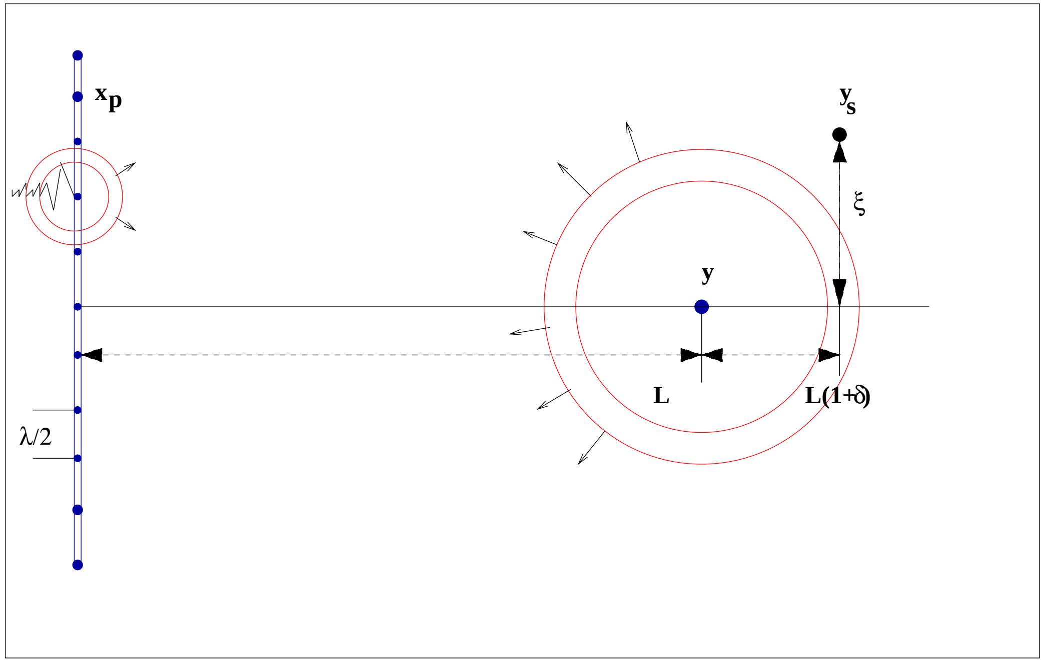
The other is statistical stability: When integrating over a sufficiently wide frequency band we have

$$\int d\omega \overline{\widehat{G}(\mathbf{x}_r,\mathbf{y},\omega)}\widehat{G}(\mathbf{x}_r,\mathbf{y}^S,\omega) \approx \int d\omega E\{\overline{\widehat{G}(\mathbf{x}_r,\mathbf{y},\omega)}\widehat{G}(\mathbf{x}_r,\mathbf{y}^S,\omega)\}$$

Thus, we under favorable conditions we have for example

$$\int d\omega \sum_{\mathbf{x}_r} \overline{\widehat{G}(\mathbf{x}_r,\mathbf{y},\omega)}\widehat{G}(\mathbf{x}_r,\mathbf{y}^S,\omega) \approx \int d\omega \sum_{\mathbf{x}_r} \overline{\widehat{G}_0(\mathbf{x}_r,\mathbf{y},\omega)}\widehat{G}_0(\mathbf{x}_r,\mathbf{y}^S,\omega)e^{\frac{-k^2\xi^2 a_e}{2L^2}}$$

Time Reversal Schematic



Range: L, Carrier wavelength λ, Array size $a = (N-1)\lambda/2$.
Source at y, Search point at y_s, Transducers at x_p.
Remote sensing regime: $\lambda << a << L$.
Random medium: Correlation length $l << L$, fluctuation strength $\sigma << 1$.

Remarks on TR in RM

- Resolution in time reversal: $\frac{\lambda L}{a}$, cross-range. It is the same as the Rayleigh resolution of optical instruments

- Super-resolution in random media because of multiple scattering: $\lambda L / a_e$, cross-range. The effective aperture a_e can be much larger that the physical aperture a. In random media, resolution is better than the diffraction limit

- Statistical stability (self-averaging) of time-reversed and back-propagated field. Broad-band and narrow-band signals. Super-resolution is observed only in regimes where there is statistical stability

On the plane of the source, at a point with transverse coordinates ξ, the time time harmonic field is

$$\psi^B(L,\xi,k) = \int G_\theta(L,x,\xi;k)\overline{G_\theta(L,\eta,x;k)\psi_0(\eta,k)}\chi_A(x)dxd\eta$$

where G_θ is the (random) Green's function. In the time domain it is

$$\Psi^B(L,\xi,t) = \int e^{-i\omega t}\psi^B(L,\xi,\frac{\omega}{c_0})d\omega$$

Because of the form of this field, and for many other reasons, we introduce and use the Wigner distribution of ψ

$$W_\theta(z,x,p) = \int \frac{dy}{(2\pi)^2}e^{ip\cdot y}\psi(z,x-\frac{\theta y}{2},k)\overline{\psi(z,x+\frac{\theta y}{2},k)}$$

and note that ψ^B can be written entirely in terms of W_θ.

The Wigner distribution satisfies a linear stochastic equation, the Ito-Wigner equation, that comes from the Ito-Schrödinger equation using the Ito calculus. In the high frequency limit the Wigner process converges weakly to the solution of the Ito-Liouville equation

$$d_z W + (\frac{p}{k} \cdot \nabla_x W - \frac{k^2 D}{2} \Delta_p W) dz + \frac{k}{2} \nabla_p W \cdot \nabla_x d_z B(\frac{x}{\delta}, z) = 0$$

where $D = -R_0''(0)/4$ and the wave number scales out: $W = W(z, x, p/k; k = 1)$. The expected value $E\{W\}$ solves the PDE

$$W_z + \frac{p}{k} \cdot \nabla_x W - \frac{k^2 D}{2} \Delta_p W = 0$$

with given initial conditions $W(0, x, p; k)$.

The process W depends on δ but $E\{W\}$ does not.

The mean of the time-reversed, back-propagated field

If we take a source field that is a directed beam

$$e^{ip_0 \cdot x/\theta} \psi_0(\frac{x}{\sigma_s}, k),$$

with σ_s the lateral extent of the source, then in the white-noise ($\epsilon \to 0$) and high-frequency ($\theta \to 0$) limits we have

$$E\{\psi^B(L, \xi, k)\} = \psi_0(\cdot, -k) * \mathcal{W}(\cdot)(\xi)$$

where $\mathcal{W}$ is the point spread function

$$\mathcal{W}(\eta) = \left(\frac{k}{2\pi L}\right)^2 \hat{\chi}_A(\frac{\eta k}{L}) e^{-\eta^2/(2\sigma_M^2)}$$

and

$$\sigma_M = \frac{L}{k a_e}\,, \quad a_e = \sqrt{\frac{DL^3}{3}}$$

Here $a_e = a_e(L)$ is the effective aperture of the array.

Interpretation of the point spread function

If there is no scattering medium then $D = 0$ and

$$\mathcal{W}(\eta) = \left(\frac{k}{2\pi L}\right)^2 \widehat{\chi}_A(\frac{\eta k}{L})$$

For a square aperture $A = [-\frac{a}{2}, \frac{a}{2}]^2$

$$\mathcal{W}(\eta) = \mathcal{W}(\eta_1, \eta_2) = \frac{1}{\pi^2 \eta_1 \eta_2} \sin(\frac{\eta_1 k a}{2L}) \sin(\frac{\eta_2 k a}{2L})$$

The first zero of the sine function is at

$$\eta_F = \frac{2\pi L}{ka} = \frac{\lambda L}{a} = \text{Rayleigh resolution}$$

If we define $\sigma_F = L/ka$, the Fresnel spot size, then when $\sigma_F << \sigma_M$, or $a >> a_e$, multipathing does not alter the refocused spot size of diffraction theory.

But if $a_e >> a$ then the point spread function is

$$\mathcal{W} \approx \left(\frac{a}{\sqrt{2\pi} a_e}\right)^2 \frac{e^{-\eta^2/(2\sigma_M^2)}}{2\pi\sigma_M^2}$$

Self-averaging

When is the time-reversed, back-propagated field self-averaging? This is a fundumental issue because it determines when super-resolution is observable.

In the present setting there are two results:

- If the source is localized, $\sigma_s \sim \theta$, then, in the limit $\delta \to 0$, the time harmonic field ψ^B is self-averaging

$$\lim_{\delta \to 0} E\{(\psi^B - E\{\psi^B\})^2\} = 0$$

- If the source is distributed, $\sigma_s >> \theta$, then only in the time domain, that is for $\Psi^B(L, \xi, t)$, we have self-averaging in mean square sense as $\delta \to 0$.

- What does $\delta \to 0$ mean? Provides cross-range diversity in multipathing.

Field theory for the Ito-Liouville equation

The self-averaging is based on the following theorem for the Ito-Liouville process (with $k = 1$) defined by

$$d_z W + (p \cdot \nabla_x W - \frac{D}{2}\Delta_p W)dz + \frac{1}{2}\nabla_p W \cdot \nabla_x d_z B(\frac{x}{\delta}, z) = 0$$

with $W(0, x, p) = \chi_A(x)$:

For any $z > 0$ the integral

$$J_\delta(z, x) = \int W_\delta(z, x, p)dp$$

exists and

$$\lim_{\delta \to 0} E\{(J_\delta - E\{J_\delta\})^2\} = 0$$

where $E\{J_\delta\}$ is independent of δ.

This is proved by using properties of the SDE's (random characteristics) through which the Ito-Liouville equation can be solved.

Time-reversed, back-propagated pulse

In the time domain and for a distributed source, the self-averaging field, in the white-noise and high-frequency limit, is given by

$$\Psi^B(L,\xi,t) = e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi) \cdot \int_{\{|\omega|<\Omega\}} \frac{d\omega}{2\pi} e^{-i\omega t}\widehat{g}(-\omega)\ \chi_A * \left(\frac{e^{-x^2/2a_e^2}}{2\pi a_e^2}\right)\left(\frac{Lc_0p_0}{\omega_0+\omega}\right)$$

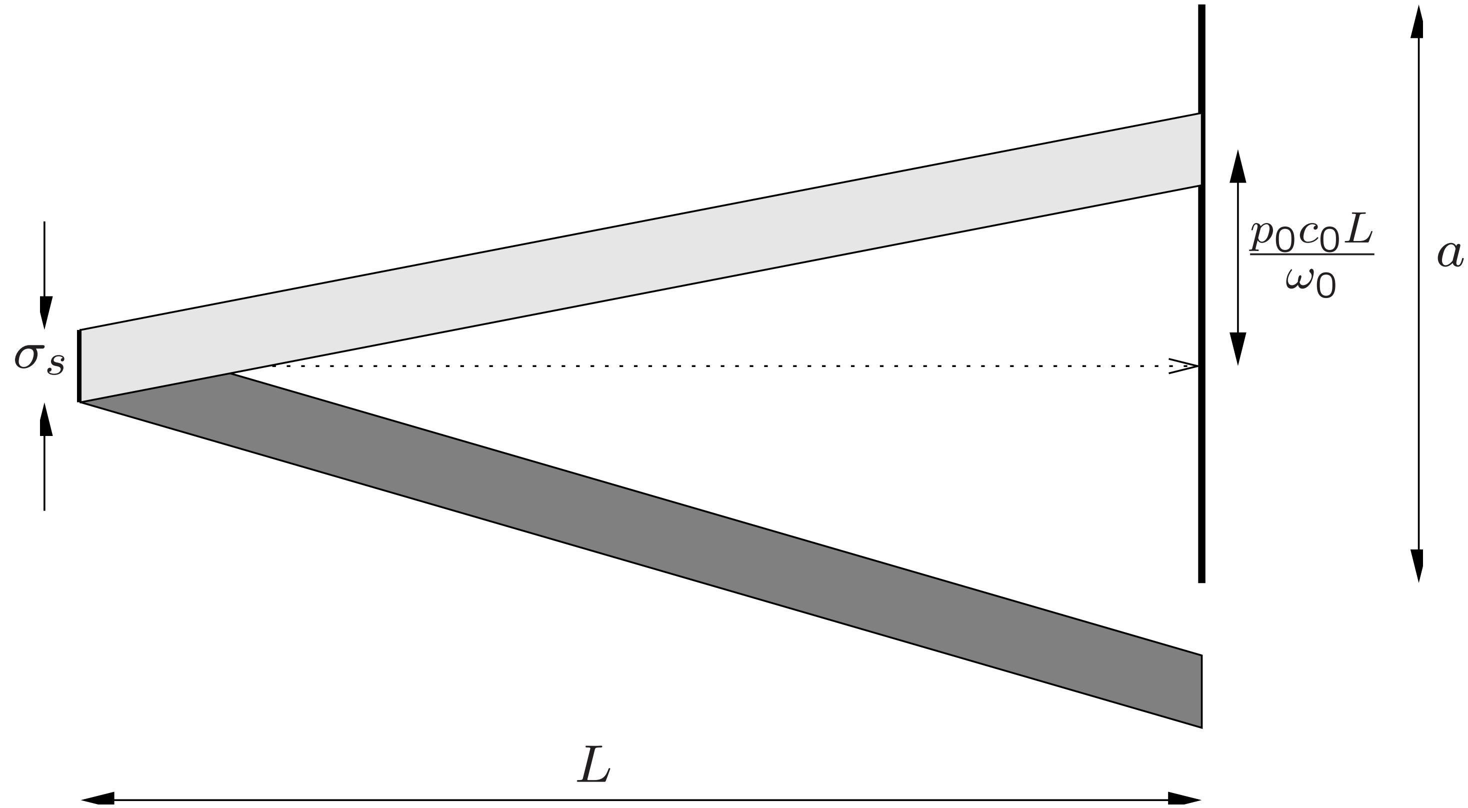
When $a_e << a$, that is, no multipathing, then

$$\Psi^B(L,\xi,t) \sim e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi) \cdot \int_{\{|\omega|<\Omega\}} \frac{d\omega}{2\pi} e^{-i\omega t}\widehat{g}(-\omega)\ \chi_A\left(\frac{Lc_0p_0}{\omega_0+\omega}\right)$$

In this case, if the beam lands entirely withing the TRM then the time-reversed and back-propagated pulse is

$$e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi)g(-t)$$

Time-reversed, back-propagated pulse schematic



A directed field propagates from a distributed source of size σ_s toward the time reversal mirror of size a. The time-reversed, back-propagated field depends on the location of the mirror relative to the direction of the propagating beam.

Time-reversed, back-propagated pulse with multipathing

When multipathing is strong, $a_e >> a$, then the self-averaging time-reversed and back-propagated pulse is given by

$$\Psi^B(L,\xi,t) \sim e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi) \cdot \left(\frac{a}{\sqrt{2\pi}a_e}\right)^2 \int_{\{|\omega|<\Omega\}} \frac{d\omega}{2\pi} e^{-i\omega t}\widehat{g}(-\omega)\ e^{-\frac{1}{2}\left(\frac{Lc_0p_0}{a_e(\omega_0+\omega)}\right)^2}$$

Note that, remarkably, this expression **is almost independent of the time reversal mirror**!

Use this formula to estimate the most important quantity in time reversal with strong multipathing: the effective aperture a_e.

Point the beam in different directions toward the TRM, measure the time reversed pulse and estimate a_e by fitting to the formula.

Summary and conclusions

- Time reversal in a random medium is important because of super-resolution and self-averaging, which are phenomena that are difficult to analyze and understand quantitatively, and require interesting mathematics.

- Applications abound, are very exciting and limited only by the **hardware**, our imagination, and also our analytical understanding: Direct TR applications, Imaging, Communications.