Lecture VI: Illuminating and Imaging Edges

George Papanicolaou
Stanford University

CBMS: Imaging in random media

May 14, 2008

Based on a paper that appeared recently in the SIAM Journal on Imaging Sciences by Liliana Borcea (Rice University), Fernando Guevara Vasquez (University of Utah) and G.P.

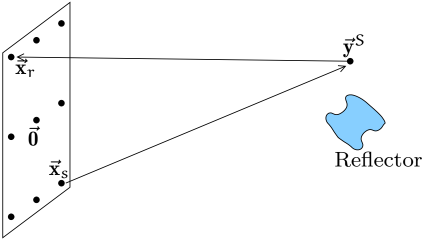
Why image edges?

- In array imaging of extended reflectors, information about the edges and therefore its shape is masked by strong reflections from the parts of the reflector facing the array
- We use the singular value decomposition (SVD) of the array data operator in order to distinguish edge reflections from bulk reflections
- We substantiate mathematically the empirical notion that the number of significant singular values is related to the reflector information content in the array data
- We also generalize the result of BPT-06 regarding optimal properties of edge illumination for point scatterers to edge illumination of scattering by edges

Outline

1. Brief mathematical formulation of array imaging
2. Imaging with selective subspace migration
3. Mathematical analysis in the Fraunhofer diffraction regime and substantiation of the empirical relation between singular values and object information content
4. The extended Fraunhofer regime

Array imaging



Array with N_r receivers $\vec{x}_r$ and N_s sources $\vec{x}_s$.

- Denote by $\widehat{\Pi}(\vec{x}_r, \vec{x}_s, \omega)$ the Fourier transform in time of time signals recorded at receiver $\vec{x}_r$ generated at $\vec{x}_s$ by the pulse

$$g(t) = \exp[i\omega_0 t] \frac{\sin(Bt/2)}{\pi t}.$$

- Form the array response matrix $\widehat{\Pi}(\omega) \in \mathbb{R}^{N_r \times N_s}$

$$(\widehat{\Pi}(\omega))_{r,s} \equiv \widehat{\Pi}(\vec{x}_r, \vec{x}_s, \omega).$$

- With collocated sources and receivers, $N = N_r = N_s$, it is a complex symmetric matrix $\widehat{\Pi}(\omega)^T = \widehat{\Pi}(\omega)$ for each frequency in the bandwidth $\omega_0 - B/2 \leqslant \omega \leqslant \omega_0 + B/2$.

Array imaging

- The array data $\widehat{\Pi}(\omega)$ is, in general, a nonlinear functional of the reflectivity $\rho(\vec{\mathbf{y}})$ of the object to be imaged, $\widehat{\Pi}(\omega) = \widehat{\Pi}(\omega; \rho)$ We want to recover if possible the reflectivity from the array data or, more often, just its support.
- Imaging methods: General (nonlinear) least squares approximation of array data using a forward, wave propagation model for the scattering that produces the array data. It is clear that we need to know the background in which lies the reflecting object to be imaged. We are interested mainly in the reflectivity of the object but have to get the background as well, if we do not already know it.
- Full LSQ is almost never done. Instead, a much simplified (linearized in ρ, Born approximation) version of it is done, which amounts to backpropagation from the array. This is equivalent to action on the data by the adjoint to the reflectivity-to-data operator, which is a data-to-reflectivity operator and hence an imager.

Kirchhoff or travel-time Migration

Most commonly used since the early seventies: backpropagate the data using travel times, $\tau(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = d(\vec{\mathbf{x}}, \vec{\mathbf{y}})/c$ (distance over speed) to a search point $\vec{\mathbf{y}}^S$ in the image domain:

$$\mathcal{J}_{\mathrm{KM}}(\vec{\mathbf{y}}^S; \widehat{f}) = \int_{|\omega-\omega_0|\leqslant B/2} d\omega \sum_{s=1}^{N} \sum_{r=1}^{N} \exp[i\omega(\tau(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S) + \tau(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}^S)] \overline{\widehat{\Pi}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega)} \widehat{f}(\vec{\mathbf{x}}_s, \omega).$$

Here $\widehat{f}(\vec{\mathbf{x}}_s, \omega)$ is the illumination vector at the array.
This imaging functional reconstructs well the reflectivity when the array and the bandwidth are large. Its basic mathematical theory, including its relation to least squares, was carried out in 1980's by Beylkin, Burridge, Bleistein, Symes

Optimal illumination

Choose the illumination vector $\widehat{f}(\vec{\mathbf{x}}_s,\omega)$ so as to minimize some norm of the (normalized) image:

$$\text{minimize} \quad J(\widehat{f}) \quad \text{over} \quad \widehat{f}$$

where

$$J(\widehat{f}) = \int \left| \frac{\mathcal{J}_{\mathrm{KM}}(\vec{\mathbf{y}}^S;\widehat{f})}{\max_{\vec{\mathbf{y}}} |\mathcal{J}_{\mathrm{KM}}(\vec{\mathbf{y}};\widehat{f})|} \right|^2 d\vec{\mathbf{y}}^S \ ,$$

with $\int \sum_s |\widehat{f}(\vec{\mathbf{x}}_s,\omega)|^2 d\omega = 1$
Basic result (BPT-06): For a point scatterer the optimal illumination is concentrated at the edges of the array

Selected Subspace Migration

Image with the filtered data

$$\mathcal{J}_{SM}(\vec{\mathbf{y}}^S; D, \hat{f}) = \int_{|\omega-\omega_0|\leqslant B/2} d\omega \sum_{s=1}^{N} \sum_{r=1}^{N} \exp[i\omega(\tau(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S) + \tau(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}^S)] \overline{\left(D\left[\hat{\Pi}(\omega); \omega\right]\right)_{r,s}} \hat{f}(\vec{\mathbf{x}}_s, \omega).$$

Here the filtering operator D acting on data $\hat{\Pi}(\omega)$ is

$$D\left[\hat{\Pi}(\omega); \omega\right] = \sum_{j=1}^{N} d_j(\omega)\sigma_j(\omega)\hat{\mathbf{u}}_j(\omega)\hat{\mathbf{v}}_j^*(\omega),$$

where $\hat{\Pi}(\omega) = \sum_{j=1}^{N} \sigma_j(\omega)\hat{\mathbf{u}}_j(\omega)\hat{\mathbf{v}}_j^*(\omega)$ is the SVD of $\hat{\Pi}(\omega)$:
$\left(\hat{\Pi}(\omega)\hat{\mathbf{v}}_j(\omega) = \sigma_j(\omega)\hat{\mathbf{u}}_j(\omega)\right)$

Selected subspace migration

$$\mathcal{J}_{\mathrm{SM}}(\vec{\mathbf{y}}^S; \mathrm{D}, \widehat{f}) = \int_{|\omega-\omega_0|\leqslant B/2} d\omega \sum_{s=1}^{N}\sum_{r=1}^{N} \exp[i\omega(\tau(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S) + \tau(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}^S)]$$

$$\overline{\left(\mathrm{D}\left[\widehat{\Pi}(\omega);\omega\right]\right)_{r,s}}\,\widehat{f}(\vec{\mathbf{x}}_s, \omega)$$

We look at a simple special case

- $\widehat{f}(\vec{\mathbf{x}}_s, \omega) = 1;$ uniform illumination form the array
- Binary weights:

$$d_j(\omega) = \begin{cases} 1 & \text{if } j \in J(\omega) \subset \{1, \ldots, N\} \\ 0 & \text{otherwise.} \end{cases}$$

Note: If D is the identity I: $\mathcal{J}_{\mathrm{SM}}(\vec{\mathbf{y}}^S; \mathrm{I}, \widehat{f}) = \mathcal{J}_{\mathrm{KM}}(\vec{\mathbf{y}}^S; \widehat{f})$.

Subspace selection strategies

- Kirchhoff migration:

$$J^{KM}(\omega) = \{1, \ldots, N\}.$$

- Detection, DORT (Fink-Prada 94): Keep strongest reflection (good for detection, bad for imaging in general)

$$J^{\text{Detect.}}(\omega) = \{1\}.$$

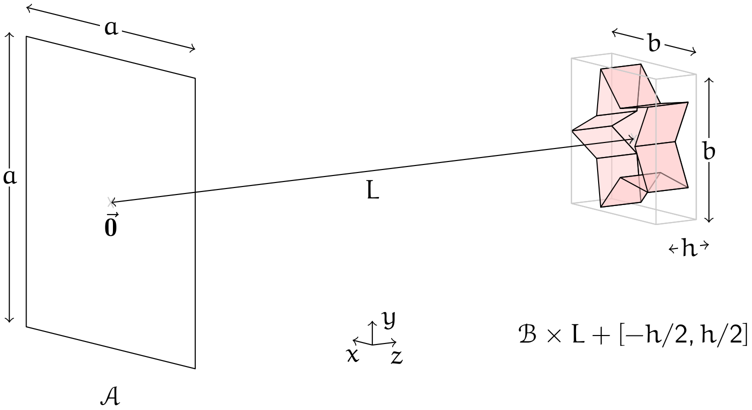
- Edge illumination: Give up some robustness to noise (detection capability) for the ability to focus selectively on the edges (imaging capability)

$$J^{SM}(\omega; [a, b]) = \left\{ j \mid \frac{\sigma_j(\omega)}{\sigma_1(\omega)} \in [a, b] \right\},$$

here $[a, b] \subset (0, 1)$.

Basic problem: How to choose the interval $[a, b]$.

Setup for numerical simulations



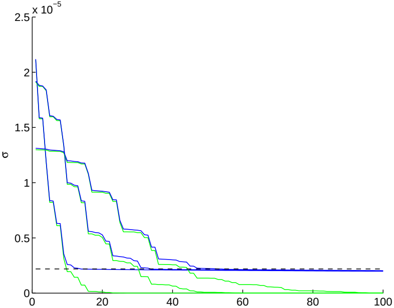
- Frequency 1.5MHz - 4.5MHz, $c_0 = 1.5km/s$ (ultrasound regime)
- Central wavelength: $\lambda_0 = 0.5$mm
- $L = 100\lambda_0 = 5$cm, $a = 25\lambda_0 = 1.25$cm, $b = 20\lambda_0 = 1$cm, $h = \lambda_0/5 = 0.1$mm

Signal-to-noise ratio (SNR)

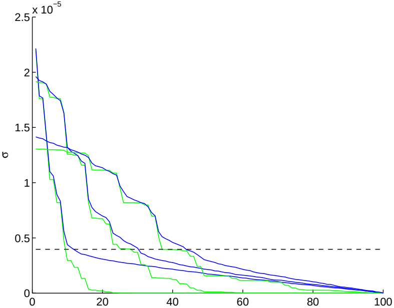
- The array data $\widehat{\Pi}(\omega)$ is not recorded perfectly at the array. There is always noise, which becomes important when we try to image with subspaces corresponding to small singular values.
- Noise here means: uncorrelated (white), zero mean, additive Gaussian noise.
- Clutter effects (random inhomogeneities in the ambient medium) are much harder to model and to analyze (BPT 03,05,06,07)
- Measure noise in decibels (db):

$$\mathrm{SNR\ in\ db} = 10 \log_{10} \left(\frac{\text{signal power}}{\text{noise power}} \right)$$

Singular values of $\widehat{\Pi}(\omega)$ (box target)



0dB SNR, 50×50 array

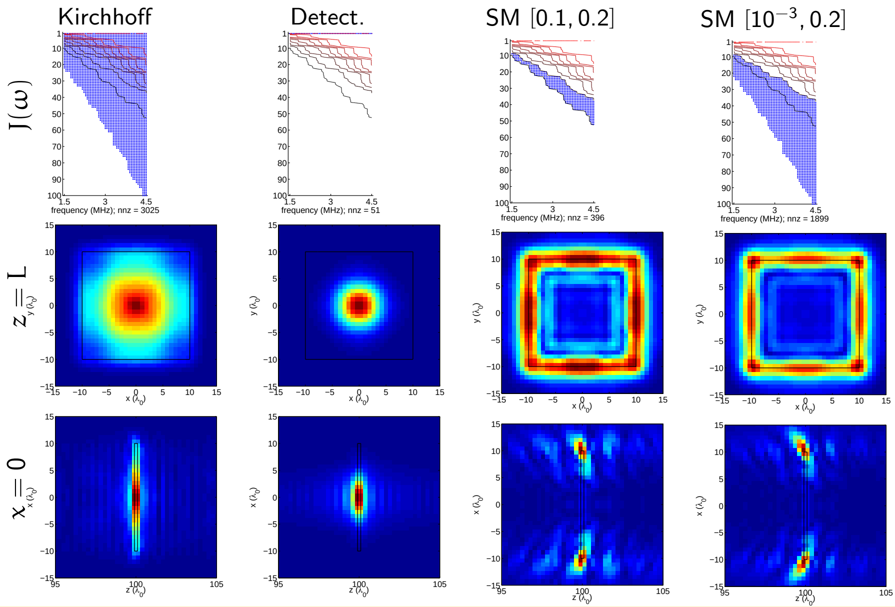


10dB SNR, 10×10 array

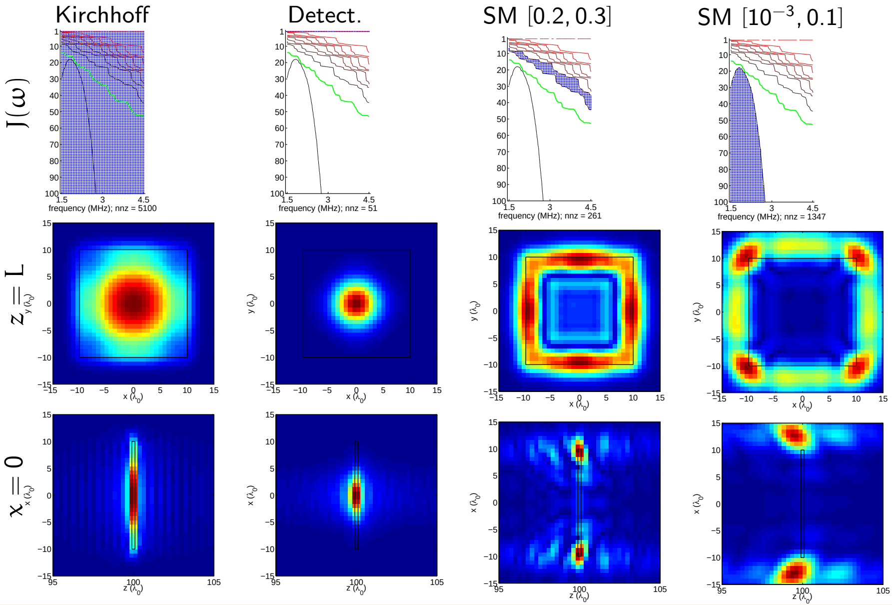
Green: true singular values. Blue: singular values for noisy data.

Dotted: largest singular value of the noise.

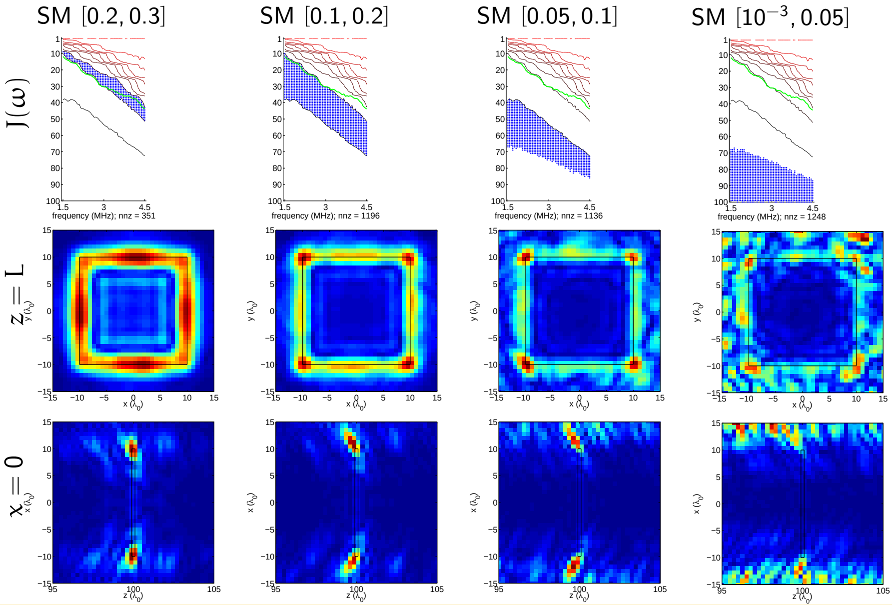
Imaging a box target (50 × 50 array)



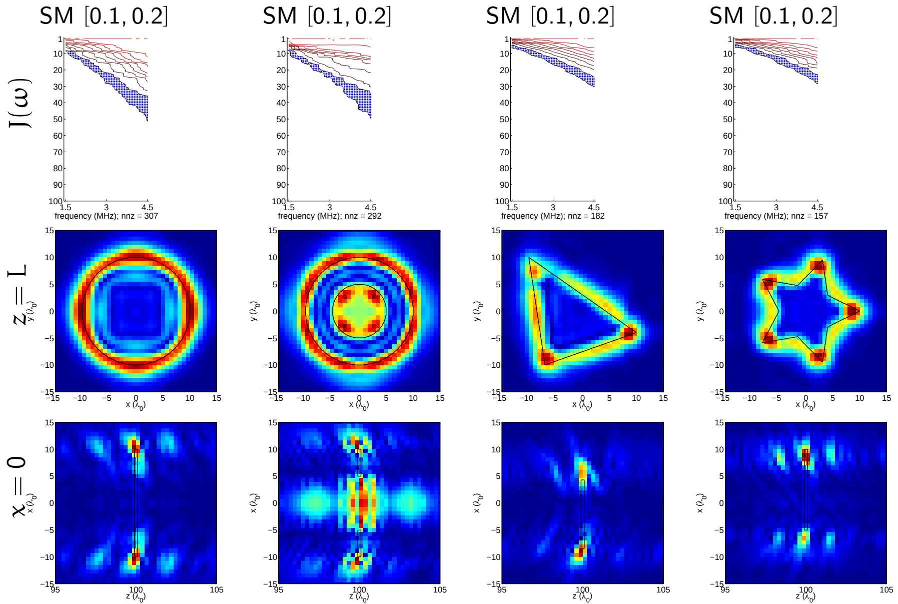
Imaging a box target with 0dB SNR (50 × 50 array)



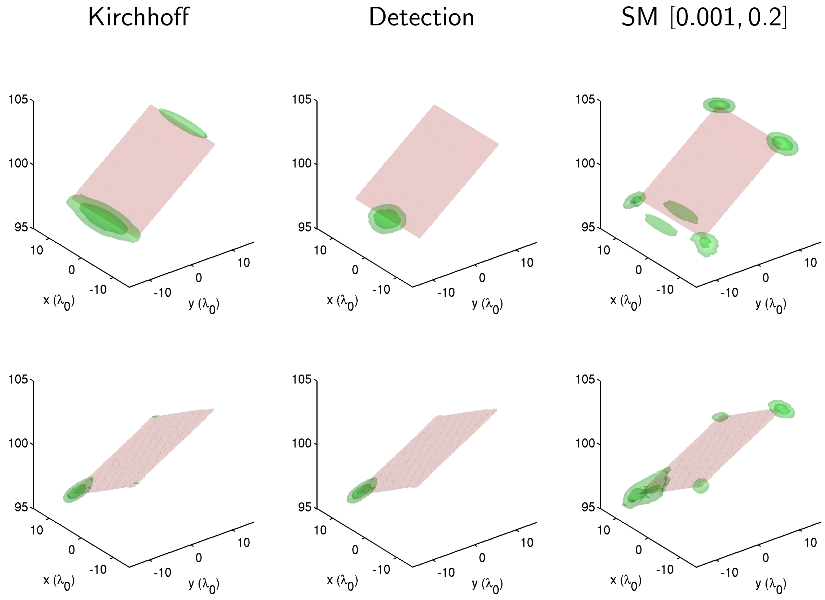
Imaging a box target with 10dB SNR (10×10 array)



Imaging other targets (50×50 array)



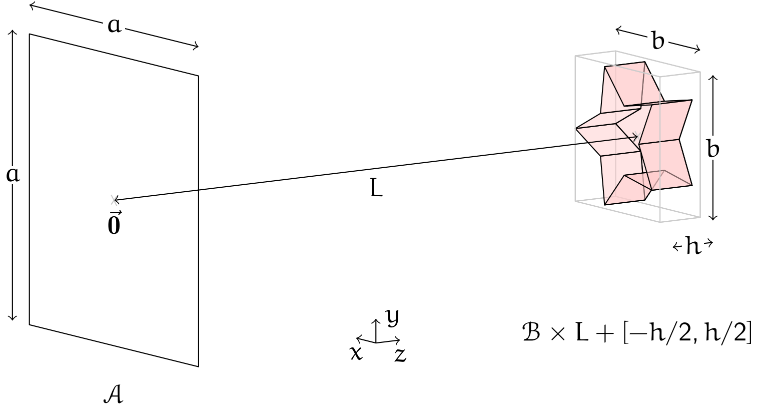
Imaging rotated targets



Some basic facts about image resolution

- Array means that the sensors are so closely spaced that they can be treated as a continuum, as an aperture
- For a linear array of size a, a point target at a distance L is imaged with cross-range resolution $\lambda_0 L/a$, the resolution spot-size (Rayleigh resolution limit). Here λ_0 is the central wave length. The wave number is $k_0 = 2\pi/\lambda_0$
- When imaging with broadband signals the range resolution is c/B, which is the width of the probing pulse in units of length

The Fraunhofer diffraction regime



- Spot-size $\lambda_0 L/a \ll a$ ($\Leftrightarrow$ Fresnel number $\theta_a = k_0 a^2/L \gg 1$), and $a^2/L^2 \ll 1/\theta_a \ll 1$.
- Small reflector $b \ll a$, $\boxed{\theta_b = k_0 b^2/L \ll 1}$, and $b \geqslant \lambda_0 L/a$.
- Thin reflector $h \ll a \ll L$, $\theta_h = k_0 h^2/L \ll 1$ and $h \ll L^2/(k_0^2 a^2)$.

Array response matrix model in the Fraunhofer regime

- For a prism target with reflectivity $\rho(\boldsymbol{\xi}, L+\eta) = \chi_{\mathcal{B}}(\boldsymbol{\xi})\rho_L(\eta)$ and in the Fraunhofer regime approximate $\widehat{\Pi}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega)$ by

$$\widehat{\Pi}_F(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega) = k^2 \int_{\mathcal{B}\times L+[-h/2,h/2]} d\vec{\mathbf{y}}\rho(\vec{\mathbf{y}})\widehat{\mathcal{G}}_0(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}, \omega)\widehat{\mathcal{G}}_0(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}, \omega).$$

- Here with $\vec{\mathbf{x}} = (\mathbf{x}, 0) \in \mathcal{A}$ and $\vec{\mathbf{y}} = (\boldsymbol{\xi}, L+\eta)$ inside the target:

$$\widehat{\mathcal{G}}_0(\vec{\mathbf{x}}, \vec{\mathbf{y}}, \omega) = \frac{1}{4\pi L} \exp\left[ik\left(L + \eta + \frac{|\mathbf{x}|^2}{2L} - \frac{\mathbf{x}\cdot\boldsymbol{\xi}}{L}\right)\right]$$
$$\approx \frac{\exp\left[ik\,|\vec{\mathbf{x}}-\vec{\mathbf{y}}|\right]}{4\pi\,|\vec{\mathbf{x}}-\vec{\mathbf{y}}|} = \widehat{G}_0(\vec{\mathbf{x}}, \vec{\mathbf{y}}, \omega).$$

- We study the continuum version of $\widehat{\Pi}_F(\omega)$

$$(\widehat{\Pi}_F(\omega)f)(\vec{\mathbf{x}}_r) = \int_{\mathcal{A}} d\vec{\mathbf{x}}_s \widehat{\Pi}_F(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega)f(\vec{\mathbf{x}}_s).$$

Singular Value Decomposition of $\widehat{\Pi}_F(\omega)$

- Proposition. If $\mathcal{A}$ is invariant under reflections about the origin ($-\mathbf{x} \in \mathcal{A} \Leftrightarrow \mathbf{x} \in \mathcal{A}$),

$$\widehat{\Pi}_F(\omega) = \frac{\widehat{\rho}_L(-2k)}{4}\mathcal{U}(\omega)\mathcal{R}\mathcal{A}Q_{\frac{k}{L}\mathcal{B}}\mathcal{A}\mathcal{U}(\omega).$$

$$\Rightarrow \text{SVD of } \widehat{\Pi}_F(\omega) \text{ is } \begin{cases} \sigma_n[\widehat{\Pi}_F(\omega)] = \frac{1}{4}\left|\widehat{\rho}_L(-2k)\right| \sigma_n[\mathcal{A}Q_{\frac{k}{L}\mathcal{B}}\mathcal{A}] \\ v_n[\widehat{\Pi}_F(\omega)] = \mathcal{U}^*(\omega)v_n[\mathcal{A}Q_{\frac{k}{L}\mathcal{B}}\mathcal{A}], \quad \text{and} \\ u_n[\widehat{\Pi}_F(\omega)] = \arg(\widehat{\rho}_L(-2k))\mathcal{U}(\omega)\mathcal{R}v_n[\mathcal{A}Q_{\frac{k}{L}\mathcal{B}}\mathcal{A}]. \end{cases}$$

- The operators $(\mathcal{A}f)(\mathbf{x}) = \chi_{\mathcal{A}}(\mathbf{x})f(\mathbf{x})$ and $Q_{\frac{k}{L}\mathcal{B}} = \mathcal{F}^{-1}\left(\frac{k}{L}\mathcal{B}\right)\mathcal{F}$ are orthogonal projectors in $L^2(\mathbb{R}^2)$.
- The operators $\mathcal{U}(\omega)$ and $\mathcal{R}$, commute and are unitary:

$$(\mathcal{U}(\omega)f)(\mathbf{x}) = (4\pi L)\widehat{\mathcal{G}}_0(\vec{\mathbf{x}}, \vec{\mathbf{y}}^\star, \omega)f(\mathbf{x}) = \exp[ik(L + |\mathbf{x}|^2/(2L))]f(\mathbf{x})$$

$$(\mathcal{R}f)(\mathbf{x}) = f(-\mathbf{x}) \quad \text{(reflection about the origin)}$$

The operator $\mathcal{A}Q_{\frac{k}{L}\mathcal{B}}\mathcal{A}$ is a space and wavenumber limiting operator (Slepian, Landau, Pollak).

Physical interpretation of the factorization on $\widehat{\Pi}_F(\omega)$

- Physical meaning of $\mathcal{U}(\omega)\mathcal{R}\mathcal{A}Q_{\frac{k}{L}\mathcal{B}}\mathcal{A}\mathcal{U}(\omega) = \mathcal{A}\mathcal{U}(\omega)\mathcal{R}Q_{\frac{k}{L}\mathcal{B}}\mathcal{U}(\omega)\mathcal{A}$

- From right to left:

  Restriction (of the illumination) to the array $\mathcal{A}$;
  Propagation to the reflector $\mathcal{U}(\omega)$;
  Reflection-diffraction $\mathcal{R}Q_{\frac{k}{L}\mathcal{B}}$;
  Propagation back to the array $\mathcal{U}(\omega)$;
  Restriction (of the measurements) to the array $\mathcal{A}$

Space and wavenumber limiting operators

- The operator

$$\mathcal{A}Q_{\frac{k}{L}\mathcal{B}}\mathcal{A} = (2\pi)^{-2}((\tfrac{k}{L}\mathcal{B})\mathcal{F}\mathcal{A})^*((\tfrac{k}{L}\mathcal{B})\mathcal{F}\mathcal{A})$$

  is self-adjoint, positive, and Hilbert-Schmidt (trace class) with spectral radius $\leqslant 1$.
- Singular functions
  - are "generalized prolate spheroidal wave functions" (Slepian)
  - can be extended into an orthogonal family in $L^2(\mathbb{R}^2)$.
- Singular values $\sigma_j[\mathcal{A}Q_{\frac{k}{L}\mathcal{B}}\mathcal{A}]$ measure energy concentration of right (or left) singular function of $(\frac{k}{L}\mathcal{B})\mathcal{F}\mathcal{A}$ inside $\mathcal{A}$ (or $\frac{k}{L}\mathcal{B}$).

Singular values are constant in n until they plunge suddenly to zero at a critical index. In the plunge region:

$$\left.\begin{matrix} v_j[(\frac{k}{L}\mathcal{B})\mathcal{F}\mathcal{A}] \\ u_j[(\frac{k}{L}\mathcal{B})\mathcal{F}\mathcal{A}] \end{matrix}\right\} \text{ is localized near the edges of } \left\{\begin{matrix} \mathcal{A} \\ \frac{k}{L}\mathcal{B} \end{matrix}\right. .$$

Localization observed experimentally by Komilikis, Prada and Fink, 1996.

Subspace migration in the Fraunhofer regime

Proposition. Subspace migration images realized with a single singular function $v_j[\widehat{\Pi}_F(\omega)]$ and a single frequency ω are,

$$\mathcal{I}_{SM}(\vec{\mathbf{y}}^S;\omega) \sim \overline{\widehat{\rho}_L(-2k)}\exp[2ik\eta^S]\sigma_j^2[\mathcal{A}Q_{\frac{k}{L}\mathcal{B}}\mathcal{A}]\left|u_j\left[\left(\tfrac{k}{L}\mathcal{B}\right)\mathcal{F}\mathcal{A}\right]\left(\tfrac{k}{L}\boldsymbol{\xi}^S\right)\right|^2,$$

where the search point is $\vec{\mathbf{y}}^S = (\boldsymbol{\xi}^S, L+\eta^S)$ and the symbol $\sim$ means equality up to a positive multiplicative factor, independent of ω, j and $\vec{\mathbf{y}}^S$.

Localization of $u_j\left[\left(\frac{k}{L}\mathcal{B}\right)\mathcal{F}\mathcal{A}\right]\left(\frac{k}{L}\boldsymbol{\xi}^S\right)$ near the edges of $\frac{k}{L}\mathcal{B}$.	$\Rightarrow$	Focusing on the edges of target $\mathcal{B}$.

Imaging with eigenfunctions of the scattering operator: Nachman et al.

Asymptotic results

- Eigenvalue distribution function, for $0 < \delta < 1$,

$$N(\delta;\theta_{ab}) = \#\left\{ j \mid \sigma_j[\mathcal{A} Q_{\frac{k}{L}\mathcal{B}}\mathcal{A}] > \delta \right\}, \text{ where } \theta_{ab} = kab/L.$$

- First order asymptotics (Landau 1975)

$$N(\delta;\theta_{ab}) = (\lambda L)^{-2}\,|\mathcal{A}|\,|\mathcal{B}|\,(1+o(1))$$

$\rightsquigarrow$ well known fact (Tanter, Thomas, Fink, ...) since

$$N(\delta;\theta_{ab}) \approx (\lambda L)^{-2}\,|\mathcal{A}|\,|\mathcal{B}| = \frac{|\mathcal{B}|}{(\lambda L)^2/\,|\mathcal{A}|} = \frac{\text{target area}}{\text{spot area}}.$$

- Second order asymptotics (Widom conjecture 1982)

$$\begin{aligned} N(\delta;\lambda) = {} & (\lambda L)^{-2}\,|\mathcal{A}|\,|\mathcal{B}| \\ & + (\lambda L)^{-1}\frac{\ln\theta_{ab}}{4\pi^2}\ln\frac{1-\delta}{\delta}\int_{\partial\mathcal{A}}\int_{\partial\mathcal{B}} d\mathbf{x}\,d\boldsymbol{\xi}\,|\mathbf{n}_{\mathcal{A}}(\mathbf{x})\cdot\mathbf{n}_{\mathcal{B}}(\boldsymbol{\xi})| \\ & + o(\theta_{ab}\ln\theta_{ab}) \end{aligned}$$

- Cases fully analyzed: 1D (Landau and Widom), half space, square (Widom), bounds for 2nd order term (Gioev).

Main results for imaging in the Fraunhofer regime

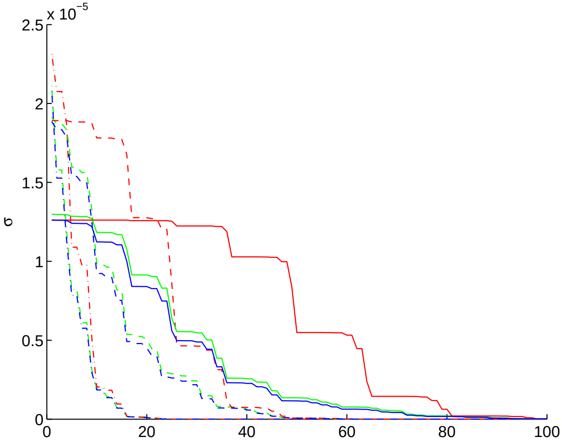
- The index N^* at which the plunge occurs in the distribution of singular vales is related asymptotically to the information content of the reflector in the array data:

$$N^* \sim \left(\frac{b}{\frac{\lambda L}{a}} \right)^2$$

  This had been expected in the imaging community.

- Illuminating with the right singular vector with index N^* means illuminating from the edges of the array. The image with this illumination, at a fixed frequency, is proportional to the square of the absolute value of the left singular vector with the same index. This implies that the resulting image is concentrated near the edges of the reflector.
  This was first obtained analytically without PSWF (BPT 06), using an optimal illumination approach

Fraunhofer regime vs Extended Fraunhofer regime



- Red = Fraunhofer regime
- Green = Extended Fraunhofer regime
- Blue = Computed

Ultrasonic non-destructive testing is in the extended Fraunhofer regime

The Extended Fraunhofer regime (EFR)

- For a prism target with reflectivity $\rho(\boldsymbol{\xi}, L+\eta) = \chi_{\mathcal{B}}(\boldsymbol{\xi})\rho_L(\eta)$ and in the EFR we approximate $\widehat{\Pi}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega)$ by

$$\widehat{\Pi}_{\mathrm{Efr}}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega) = k^2 \int_{\mathcal{B}\times L+[-h/2,h/2]} d\vec{\mathbf{y}}\rho(\vec{\mathbf{y}})\widehat{\mathcal{G}}_0^{\mathrm{Efr}}(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}, \omega)\widehat{\mathcal{G}}_0^{\mathrm{Efr}}(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}, \omega).$$

- Here with $\vec{\mathbf{x}} = (\mathbf{x}, 0) \in \mathcal{A}$ and $\vec{\mathbf{y}} = (\boldsymbol{\xi}, L+\eta)$ inside the target:

$$\begin{aligned}\widehat{\mathcal{G}}_0^{\mathrm{Efr}}(\vec{\mathbf{x}}, \vec{\mathbf{y}}, \omega) &= \frac{1}{4\pi L}\exp\left[ik\left(L+\eta+\frac{|\mathbf{x}|^2}{2L}-\frac{\mathbf{x}\cdot\boldsymbol{\xi}}{L}+\frac{|\boldsymbol{\xi}|^2}{2L}\right)\right] \\ &\approx \frac{\exp\left[ik\,|\vec{\mathbf{x}}-\vec{\mathbf{y}}|\right]}{4\pi\,|\vec{\mathbf{x}}-\vec{\mathbf{y}}|} = \widehat{G}_0(\vec{\mathbf{x}}, \vec{\mathbf{y}}, \omega).\end{aligned}$$

- In the Fraunhofer regime we neglected the term $|\boldsymbol{\xi}|^2/(2L) = \mathcal{O}(\theta_b)$.

SVD of $\widehat{\Pi}(\omega)$ in the Extended Fraunhofer regime (EFR)

- Proposition. If $\mathcal{A}$ is invariant under reflections about the origin,

$$\widehat{\Pi}_{\mathrm{Efr}}(\omega) = \frac{\widehat{\rho}_L(-2k)}{4}\mathcal{U}(\omega)\mathcal{R}\mathcal{A}\widetilde{Q}_{\frac{k}{L}\mathcal{B}}\mathcal{A}\mathcal{U}(\omega).$$

- The operator $\widetilde{Q}_{\frac{k}{L}\mathcal{B}}$ is a non-Hermitian one:

$$(\widetilde{Q}_{\frac{k}{L}\mathcal{B}}f)(\mathbf{x}) = (2\pi)^{-2}\int d\mathbf{y} f(\mathbf{y})\widehat{q}(\mathbf{y}-\mathbf{x}),$$

where $q(\boldsymbol{\xi}) = \chi_{\frac{k}{L}\mathcal{B}}(\boldsymbol{\xi})\exp[i\frac{L}{k}|\boldsymbol{\xi}|^2]$.

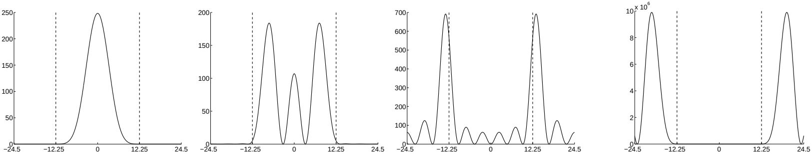
- Compare with $Q_{\frac{k}{L}\mathcal{B}} = \mathcal{F}^{-1}(\frac{k}{L}\mathcal{B})\mathcal{F}$:

$$(Q_{\frac{k}{L}\mathcal{B}}f)(\mathbf{x}) = (2\pi)^{-2}\int d\mathbf{y} f(\mathbf{y})\widehat{\chi}_{\frac{k}{L}\mathcal{B}}(\mathbf{y}-\mathbf{x}).$$

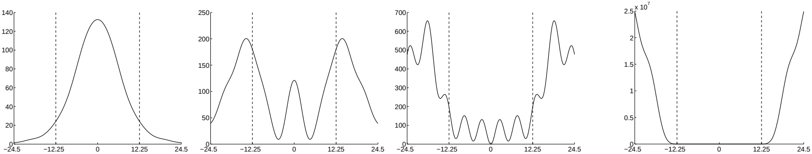
- Eigenvalue distribution result can be formulated in the EFR using pseudospectra (Landau 1975).

Eigenfunctions for the square in the EFR regime

$|\psi_1(2x/a;C)|^2$ $|\psi_3(2x/a;C)|^2$ $|\psi_6(2x/a;C)|^2$ $|\psi_{10}(2x/a;C)|^2$



$\nu_1[C] \approx 1.000$ $\nu_3[C] \approx 0.995$ $\nu_6[C] \approx 0.256$ $\nu_{10}[C] \approx 1.707 \times 10^{-5}$



$|\mu_1| \approx 0.944$ $|\mu_3| \approx 0.568$ $|\mu_6| \approx 0.227$ $|\mu_{10}| \approx 1.188 \times 10^{-5}$

Concluding remarks

- Is it possible to have a general theory with "no" approximations, other than the Born approximation (linearized scattering)?
- Analyze adaptive imaging (waveform design) by optimizing the quality of the image for different pulses $\widehat{f}(\mathbf{x}_s, \omega)$ and subspace filter weights $d_j(\omega) \neq 1$ (BPT IP-07).
- Random media: Extend the theory to CINT (coherent interferometry BPT JASA-07) to edge detection
- Papers to look at are here: `http://math.stanford.edu/~papanico`

Space and frequency limiting

- A space-limited function $f = \mathcal{A}f$ has total energy:

$$\|\mathcal{A}f\|_{L^2}^2 = (2\pi)^{-2} \|\mathcal{F}\mathcal{A}f\|_{L^2}^2 .$$

- The energy of f measured in frequency on the set $\mathcal{B}$ is

$$(2\pi)^{-2} \|\mathcal{B}\mathcal{F}\mathcal{A}f\|_{L^2}^2 .$$

- Finding space-limited functions that are best localized on the set $\mathcal{B}$ in frequency amounts to solving

$$\max_{f=\mathcal{A}f} \left\{ \mathcal{R}(f) = (2\pi)^{-2} \frac{\|\mathcal{B}\mathcal{F}\mathcal{A}f\|_{L^2}^2}{\|\mathcal{A}f\|_{L^2}^2} \right\} .$$

- The functional $\mathcal{R}(f)$ is a Rayleigh quotient of

$$(2\pi)^{-2}\mathcal{A}\mathcal{F}^*\mathcal{B}\mathcal{F}\mathcal{A} = \mathcal{A}Q_{\mathcal{B}}\mathcal{A}$$

  which is maximized by the eigenfunction $\nu_1[\mathcal{A}Q_{\mathcal{B}}\mathcal{A}]$.
- The eigenfunction $\nu_n[\mathcal{A}Q_{\mathcal{B}}\mathcal{A}]$ is the space-limited to $\mathcal{A}$ function that is best localized in frequency on $\mathcal{B}$, while being orthogonal to the first $n-1$ eigenfunctions.
- In general for n in the plunge region, $\nu_n[\mathcal{A}Q_{\mathcal{B}}\mathcal{A}]$ is localized near the edges of $\mathcal{A}$ and its Fourier transform near the edges of $\mathcal{B}$.