CAAM 570

Homework 1

Due at the beginning of class on Feb 1. No late submissions accepted.

1.1.7 n-CUBE

The n-*cube* Q_n ($n \geq 1$) is the graph whose vertex set is the set of all n-tuples of 0s and 1s, where two n-tuples are adjacent if they differ in precisely one coordinate.

a) Draw Q_1, Q_2, Q_3, and Q_4.
b) Determine $v(Q_n)$ and $e(Q_n)$.
c) Show that Q_n is bipartite for all $n \geq 1$.

$\star$**1.1.11** TURÁN GRAPH

A k-partite graph is *complete* if any two vertices in different parts are adjacent. A simple complete k-partite graph on n vertices whose parts are of equal or almost equal sizes (that is, $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$) is called a *Turán graph* and denoted $T_{k,n}$.

a) Show that $T_{k,n}$ has more edges than any other simple complete k-partite graph on n vertices.
b) Determine $e(T_{k,n})$.

1.1.18 GRAPHIC SEQUENCE

A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a simple graph with degree sequence $\mathbf{d}$. Show that:

a) the sequences $(7, 6, 5, 4, 3, 3, 2)$ and $(6, 6, 5, 4, 3, 3, 1)$ are not graphic,
b) if $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is graphic and $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}, \quad 1 \leq k \leq n$$

(Erdős and Gallai (1960) showed that these necessary conditions for a sequence to be graphic are also sufficient.)

1.1.19 Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers. Set $\mathbf{d}' := (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$.

a) Show that $\mathbf{d}$ is graphic if and only if $\mathbf{d}'$ is graphic.
b) Using (a), describe an algorithm which accepts as input a nonincreasing sequence $\mathbf{d}$ of nonnegative integers, and returns either a simple graph with degree sequence $\mathbf{d}$, if such a graph exists, or else a proof that $\mathbf{d}$ is not graphic.

(V. HAVEL AND S.L. HAKIMI)

$\mathbf{d} \neq (0, \ldots, 0)$

Recall that the eigenvalues of a square matrix $\mathbf{A}$ are the roots of its characteristic polynomial $\det(\mathbf{A} - x\mathbf{I})$. An *eigenvalue* of a graph is an eigenvalue of its adjacency matrix. Likewise, the *characteristic polynomial* of a graph is the characteristic polynomial of its adjacency matrix.

1.1.24 Show that:

a) no eigenvalue of a graph G has absolute value greater than Δ,
b) if G is a connected graph and Δ is an eigenvalue of G, then G is regular,
c) if G is a connected graph and $-\Delta$ is an eigenvalue of G, then G is both regular and bipartite.

1.2.8 Show that two simple graphs G and H are isomorphic if and only if there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{A}_H = \mathbf{P}\mathbf{A}_G\mathbf{P}^t$.

1.2.17 Edge-Transitive Graph
A simple graph is *edge-transitive* if, for any two edges uv and xy, there is an automorphism α such that $\alpha(u)\alpha(v) = xy$.

a) Find a graph which is vertex-transitive but not edge-transitive.
b) Show that any graph without isolated vertices which is edge-transitive but not vertex-transitive is bipartite. (E. Dauber)

$\star$**1.5.7** Totally Unimodular Matrix
A matrix is *totally unimodular* if each of its square submatrices has determinant equal to 0, +1, or −1. Let $\mathbf{M}$ be the incidence matrix of a digraph.

a) Show that $\mathbf{M}$ is totally unimodular. (H. Poincaré)
b) Deduce that the matrix equation $\mathbf{Mx} = \mathbf{b}$ has a solution in integers provided that it is consistent and the vector $\mathbf{b}$ is integral.

1.5.8 Balanced Digraph
A digraph D is *balanced* if $|d^+(v) - d^-(v)| \leq 1$, for all $v \in V$. Show that every graph has a balanced orientation.

1.6.3 Give an example of a self-complementary infinite graph.