CAAM 570  
Homework 1  
Due at the beginning of class on Feb 1. No late submissions accepted.

1.1.7 n-Cube  
The n-cube $Q_n$ ($n \geq 1$) is the graph whose vertex set is the set of all n-tuples of 0s and 1s, where two n-tuples are adjacent if they differ in precisely one coordinate.

a) Draw $Q_1$, $Q_2$, $Q_3$, and $Q_4$.
b) Determine $\nu(Q_n)$ and $e(Q_n)$.
c) Show that $Q_n$ is bipartite for all $n \geq 1$.

1.1.11 Turán Graph  
A $k$-partite graph is complete if any two vertices in different parts are adjacent. A simple complete $k$-partite graph on $n$ vertices whose parts are of equal or almost equal sizes (that is, $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$) is called a Turán graph and denoted $T_{k,n}$.

a) Show that $T_{k,n}$ has more edges than any other simple complete $k$-partite graph on $n$ vertices.
b) Determine $e(T_{k,n})$.

1.1.18 Graphic Sequence  
A sequence $d = (d_1, d_2, \ldots, d_n)$ is graphic if there is a simple graph with degree sequence $d$. Show that:

a) the sequences $(7, 6, 5, 4, 3, 3, 2)$ and $(6, 6, 5, 4, 3, 3, 1)$ are not graphic,  
b) if $d = (d_1, d_2, \ldots, d_n)$ is graphic and $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}, \quad 1 \leq k \leq n$$

(Erdős and Gallai (1960) showed that these necessary conditions for a sequence to be graphic are also sufficient.)

1.1.19 Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers. Set $d' := (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$.

a) Show that $d$ is graphic if and only if $d'$ is graphic.
b) Using (a), describe an algorithm which accepts as input a nonincreasing sequence $d$ of nonnegative integers, and returns either a simple graph with degree sequence $d$, if such a graph exists, or else a proof that $d'$ is not graphic.

(V. Havel and S.L. Hakimi)
Recall that the eigenvalues of a square matrix $A$ are the roots of its characteristic polynomial $\det(A - \lambda I)$. An eigenvalue of a graph is an eigenvalue of its adjacency matrix. Likewise, the characteristic polynomial of a graph is the characteristic polynomial of its adjacency matrix.

### 1.1.24 Show that:

a) no eigenvalue of a graph $G$ has absolute value greater than $\Delta$,
b) if $G$ is a connected graph and $\Delta$ is an eigenvalue of $G$, then $G$ is regular,
c) if $G$ is a connected graph and $-\Delta$ is an eigenvalue of $G$, then $G$ is both regular and bipartite.

### 1.2.8 Show that two simple graphs $G$ and $H$ are isomorphic if and only if there exists a permutation matrix $P$ such that $A_H = PA_GP^T$.

### 1.2.17 Edge-Transitive Graph

A simple graph is **edge-transitive** if, for any two edges $uv$ and $xy$, there is an automorphism $\alpha$ such that $\alpha(u)\alpha(v) = xy$.

- a) Find a graph which is vertex-transitive but not edge-transitive.
- b) Show that any graph without isolated vertices which is edge-transitive but not vertex-transitive is bipartite.  

(E. Dauber)

### 1.5.7 Totally Unimodular Matrix

A matrix is **totally unimodular** if each of its square submatrices has determinant equal to 0, +1, or −1. Let $M$ be the incidence matrix of a digraph.

- a) Show that $M$ is totally unimodular.  
(b) Deduce that the matrix equation $Mx = b$ has a solution in integers provided that it is consistent and the vector $b$ is integral.  

(É. Poincaré)

### 1.5.8 Balanced Digraph

A digraph $D$ is **balanced** if $|d^+(v) - d^-(v)| \leq 1$, for all $v \in V$. Show that every graph has a balanced orientation.

### 1.6.3 Give an example of a self-complementary infinite graph.