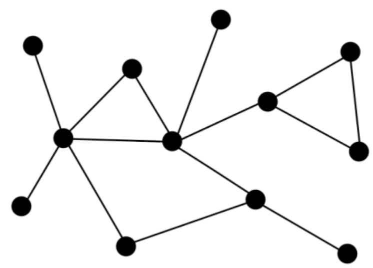
CAAM 570
Final exam

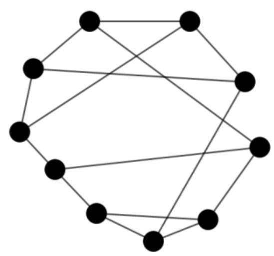
- **You may use up to 5 hours for completing this exam.**
- **If you want, after you're done working, you may neatly rewrite or typeset parts of your exam that look illegible due to bad handwriting; this won't count toward your 5 hour limit, as long as you don't change what you've written before.**
- **The exam must be completed and turned in by 5pm on April 26, 2018. You can leave it in my mailbox in Duncan Hall or in my office.**
- **You may use your class notes, homeworks, and the textbook by Bondy&Murty.**
- **Do not use any other sources, including the internet.**
- **Write the honors pledge affirming that you have followed these directions.**
- **Don't forget to (legibly) write your name on whatever you turn in.**
- **Don't panic, just do the best you can; there will be a curve, etc.**

(Q1 – 20 pts: 5, 5, 10)

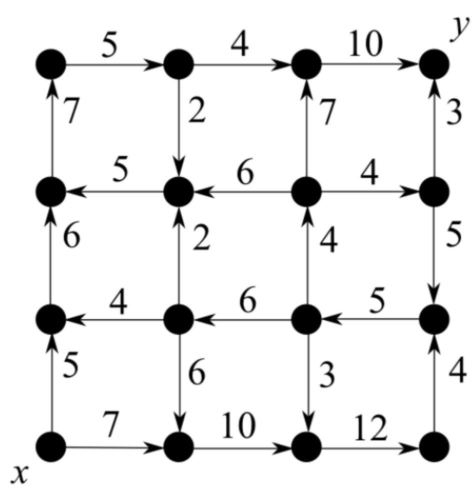
a) Draw the block tree of the following graph.



b) Find the vertex and edge connectivity of the following graph.



c) Find a maximum flow between x and y in the following network. Prove your answer is correct using the Max Flow Min Cut theorem.



(Q2 – 15 pts)

Show that every graph without cut edges has a uniform cycle covering.

(Q3 – 15 pts)

Let G be a connected plane graph and G^* be its plane dual. Show that G is bipartite if and only if G^* is Eulerian.

(Q4 – 30 pts: 15, 15)

a) Give an $O(n)$ time algorithm for finding the longest (simple) path in a tree on n vertices. Prove the correctness of your algorithm.

b) Give a polynomial time algorithm for finding the longest (simple) path in a graph whose blocks have size bounded by a constant. Prove the correctness of your algorithm.

(Q5 – 20 pts: 10, 5, 5)

Consider the following three problems:

K-CYCLE

Instance: A simple graph $G = (V, E)$ and a set of vertices $\{v_1, \dots, v_k\}$.

Question: Is there a cycle in G containing the vertices $\{v_1, \dots, v_k\}$ (and possibly other vertices)?

ORDERED K-CYCLE

Instance: A simple graph $G = (V, E)$ and a sequence of distinct vertices $v_1, \dots, v_k$.

Question: Is there a cycle in G containing the vertices $\{v_1, \dots, v_k\}$ in the order of the sequence $v_1, \dots, v_k$ (and possibly other vertices between them)?

DISJOINT PATHS

Instance: A simple graph $G = (V, E)$ and a set of pairs of vertices $\{(s_1, t_1), \dots, (s_k, t_k)\}$.

Question: Is there a set of pairwise vertex-disjoint paths $\{P_1, \dots, P_k\}$ in G such that for $1 \le i \le k$, the endpoints of P_i are the vertices s_i and t_i?

a) Prove that the K-CYCLE problem is NP-Complete.

b) Are there any major flaw(s) in the following argument? If so, where?

Claim: DISJOINT PATHS $\preceq$ K-CYCLE.

Proof: Let $I = \langle G, \{(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)\}\rangle$ be an instance of DISJOINT PATHS.
Transform this instance into an instance $I' = \langle G', \{s_1, t_1, s_2, t_2, \dots, s_k, t_k\}\rangle$ of K-CYCLE, where $G' = (V(G), E(G) \cup \{\{t_1, s_2\}, \{t_2, s_3\}, \dots, \{t_{k-1}, s_k\}, \{t_k, s_1\}\})$. We will show I is a yes-instance of DISJOINT PATHS if and only if I' is a yes-instance of K-CYCLE.

Suppose I is a yes-instance of DISJOINT PATHS. Then there are vertex-disjoint paths $P_1, \dots, P_k$ in G such that for $1 \le i \le k$, the endpoints of P_i are the vertices s_i and t_i. These same paths also exist in G'. Then $P_1, \{t_1, s_2\}, P_2, \{t_2, s_3\}, \dots, P_{k-1}, \{t_{k-1}, s_k\}, P_k, \{t_k, s_1\}$ is a cycle in G' passing through $\{s_1, t_1, s_2, t_2, \dots, s_k, t_k\}$. Thus, I' is a yes-instance of K-CYCLE.

Now suppose I' is a yes-instance of K-CYCLE. Then there is a cycle in G' passing through the vertices $\{s_1, t_1, s_2, t_2, \dots, s_k, t_k\}$. Removing the edges $\{t_1, s_2\}, \{t_2, s_3\}, \dots, \{t_{k-1}, s_k\}, \{t_k, s_1\}$ from this cycle yields paths between s_i and t_i for each pair of vertices $(s_i, t_i), 1 \le i \le k$ in G. Thus I is a yes-instance of DISJOINT PATHS.

c) Prove that ORDERED K-CYCLE $\preceq$ DISJOINT PATHS.