
Electrical Impedance Tomography

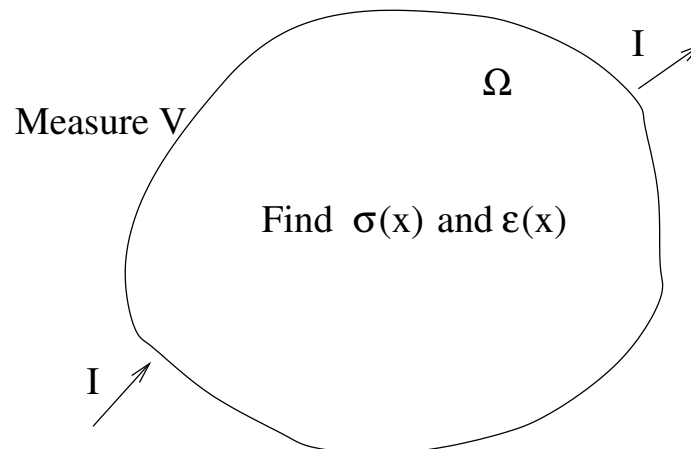
IPAM - Lecture 1

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Electrical Impedance Tomography (EIT)-Definition

- Image the **electrical properties** in the **interior** of a body Ω , given measurements of electric currents I and voltages V at the boundary $\partial\Omega$.
 - $\Omega \subset \mathbb{R}^n$ is bounded, simply connected, with smooth $\partial\Omega$.
 - For $\mathbf{x} \in \Omega$, find **admittivity** $\gamma = \sigma + i\omega\epsilon$ or $1/\gamma =$ the **impedance**.
- (a) Electrical conductivity $\sigma(\mathbf{x})$: measures how easily the material in Ω conducts electricity.
- (b) Electrical permittivity (dielectric constant) $\epsilon(\mathbf{x})$: measures how readily an applied electric field induces dipoles in Ω .



EIT-Applications

- **Medical imaging** Detection of pulmonary emboli, non invasive monitoring of heart function and blood flow, breast cancer detection, ...
- **Geophysics** Imaging underground conducting fluid plumes (near surface) for environmental cleaning. Information about rock porosity, fracture formation, ...
- **Non destructive testing** Identification of defects (voids, cracks) and corrosion in materials.

Different materials have different σ and $\epsilon \implies$ map of $\sigma(\mathbf{x})$ and $\epsilon(\mathbf{x})$ for $\mathbf{x} \in \Omega$ can be used to infer the internal structure in Ω .

Examples of electrical resistivity $\frac{1}{\sigma}$

Medical applications (Barber and Brown - 1984)

Tissue	Resistivity [Ohm-m]
Cerebrospinal fluid	0.65
Blood	1.5
Liver	3.5
Lung (expiration-inspiration)	7.27-23.63
Fat	20.6
Bone	16.6

Geophysics (G. V. Keller - 1988)

Rock or fluid	Resistivity [Ohm-m]
Marine sand	1 - 10
Volcanic rocks, basalt	10 - 200
Limestone dolomite	50 - 5000
Chloride water (oil fields)	0.16

Outline

- **Mathematical models:** continuum and electrode models
- **The forward map:** the DtN and NtD map
- **Uniqueness:** injectivity of the forward map
 - Complex exponentials: Calderón (1980), Sylvester and Uhlmann (1987), Nachman (1988)
 - High contrast tomography
 - Network tomography
- **Stability:** examples and definition of distinguishability
- **Reconstruction methods**

The Mathematical Model

Take time harmonic electric and magnetic fields, at frequency ω :

$$\mathcal{E}(\mathbf{x}, t) = \text{real} \left\{ \mathbf{E}(\mathbf{x}, \omega) e^{i\omega t} \right\}$$

$$\mathcal{H}(\mathbf{x}, t) = \text{real} \left\{ \mathbf{H}(\mathbf{x}, \omega) e^{i\omega t} \right\}$$

Maxwell's equations give:

$$\nabla \times \mathbf{H}(\mathbf{x}, \omega) = [\sigma(\mathbf{x}) + i\omega\epsilon(\mathbf{x})] \mathbf{E}(\mathbf{x}, \omega)$$

$$\nabla \times \mathbf{E}(\mathbf{x}, \omega) = -i\omega\mu(\mathbf{x})\mathbf{H}(\mathbf{x}, \omega)$$

μ = magnetic permeability, usually assumed slowly varying.

EIT regime: $\omega\mu[\sigma][x]^2 \ll 1$ Cheney, Isaacson, Newell SIAM 1999

$$\nabla \times \mathbf{H}(\mathbf{x}, \omega) = [\sigma(\mathbf{x}) + i\omega\epsilon(\mathbf{x})] \mathbf{E}(\mathbf{x}, \omega)$$

$$\nabla \times \mathbf{E}(\mathbf{x}, \omega) \approx 0$$

For example, typical parameters in medical imaging: $[x] \leq 1\text{m}$, $[\sigma] \leq 1 (\text{Ohm-m})^{-1}$, $\omega = 28.8\text{kHz}$, $\mu \approx \mu_0$.

Mathematical Formulation

- The **electric potential** $\phi(\mathbf{x}, \omega)$ satisfies $\mathbf{E}(\mathbf{x}, \omega) = -\nabla\phi(\mathbf{x}, \omega)$.
- The **electric current density** is $\mathbf{j}(\mathbf{x}, \omega) = \nabla \times \mathbf{H}(\mathbf{x}, \omega)$.
- **Ohm's law:** $\mathbf{j}(\mathbf{x}, \omega) = -\gamma(\mathbf{x}, \omega)\nabla\phi(\mathbf{x}, \omega)$, where
$$\gamma(\mathbf{x}, \omega) = \sigma(\mathbf{x}) + i\omega\epsilon(\mathbf{x}).$$

- The PDE satisfied by ϕ is

$$\begin{aligned}\nabla \cdot [\gamma(\mathbf{x}, \omega)\nabla\phi(\mathbf{x}, \omega)] &= 0, \text{ for } \mathbf{x} \in \Omega, \\ \phi(\mathbf{x}, \omega) &= V(\mathbf{x}, \omega) \text{ for } \mathbf{x} \in \partial\Omega\end{aligned}$$

We may also have Neumann boundary conditions

$$\gamma(\mathbf{x}, \omega)\nabla\phi(\mathbf{x}, \omega)\cdot\mathbf{n}(\mathbf{x}) = I(\mathbf{x}, \omega) \text{ for } \mathbf{x} \in \partial\Omega \text{ and } \int_{\partial\Omega} I(\mathbf{x}, \omega)ds = 0.$$

A unique solution of the Neumann problem is obtained by fixing $\int_{\partial\Omega} \phi(\mathbf{x}, \omega)ds$.

Mathematical model

Both Dirichlet and Neumann problems for $V \in H^{\frac{1}{2}}(\partial\Omega)$ and $I \in H^{-\frac{1}{2}}(\partial\Omega)$, respectively, have a unique solution (weak sense) $\phi(\mathbf{x}, \omega) \in H^1(\Omega)$, provided that

$$\begin{aligned} \text{real}(\gamma(\mathbf{x}, \omega)) &= \sigma(\mathbf{x}) \geq m > 0 \\ \gamma(\mathbf{x}, \omega) &\in L^\infty(\Omega) \end{aligned}$$

- **The continuum model:** we suppose that we know, pointwise at $\partial\Omega$, the voltage V and electric current I .
- **Modeling the electrodes:** we know voltages and current sent through wires attached to electrodes distributed along $\partial\Omega$
 - **Gap model:** voltage and current are constant on the surface of the electrodes. No flux outside the electrodes.
Model is simple but not accurate (Cheney, Isaacson 1991).

The complete electrode model

For $\mathbf{x} \in \partial\Omega \setminus \bigcup_p E_p$, no boundary flux: $\frac{\partial\phi(\mathbf{x},\omega)}{\partial\mathbf{n}(\mathbf{x})} = 0$.

- **Dirichlet problem:** given voltages V_p ,

$$V_p(\omega) = \phi(\mathbf{x},\omega) + z_p(\omega)\gamma(\mathbf{x},\omega)\frac{\partial\phi(\mathbf{x},\omega)}{\partial\mathbf{n}(\mathbf{x})}, \text{ for } \mathbf{x} \in E_p$$

- **Neumann problem:** given currents I_p , s.t. $\sum_p I_p = 0$,

$$I_p(\omega) = \int_{E_p} \gamma(\mathbf{x},\omega)\frac{\partial\phi(\mathbf{x},\omega)}{\partial\mathbf{n}(\mathbf{x})}ds(\mathbf{x})$$

$$V_p(\omega) = \phi(\mathbf{x},\omega) + z_p(\omega)\gamma(\mathbf{x},\omega)\frac{\partial\phi(\mathbf{x},\omega)}{\partial\mathbf{n}(\mathbf{x})}, \text{ for } \mathbf{x} \in E_p$$

Here, V_p are not known a priori.

This model has been proposed and analyzed by Somersalo, Cheney, Isaacson 1992. It is more complicated but very accurate.

The DtN and NtD maps. Continuum model.

- The **Dirichlet to Neumann map** $\Lambda_\gamma^D : H^{\frac{1}{2}}(\partial\Omega) \mapsto H^{-\frac{1}{2}}(\partial\Omega)$ is

$$\left(\Lambda_\gamma^D V\right)(\xi, \omega) = \gamma(\xi, \omega) \nabla \phi(\xi, \omega) \cdot \mathbf{n}(\xi), \quad \text{for } \xi \in \partial\Omega,$$

where V is arbitrary in $H^{\frac{1}{2}}(\partial\Omega)$ and

$$\begin{aligned} \nabla \cdot [\gamma(\mathbf{x}, \omega) \nabla \phi(\mathbf{x}, \omega)] &= 0, \quad \text{for } \mathbf{x} \in \Omega, \\ \phi(\xi, \omega) &= V(\xi, \omega) \quad \text{for } \xi \in \partial\Omega. \end{aligned}$$

- Let $\phi(\mathbf{x}, \omega)$ and $\psi(\mathbf{x}, \omega)$ be the potentials for Dirichlet data V and W , respectively and take the inner product in $L^2(\partial\Omega)$

$$\begin{aligned} \langle W, \Lambda_\gamma^D V \rangle &= \int_{\partial\Omega} \overline{W(\xi, \omega)} (\Lambda_\gamma^D V)(\xi, \omega) ds = \\ &= \int_{\Omega} \gamma(\mathbf{x}, \omega) \nabla \overline{\psi(\mathbf{x}, \omega)} \cdot \nabla \phi(\mathbf{x}, \omega) \neq \langle \Lambda_\gamma^D W, V \rangle \\ &= \int_{\Omega} \overline{\gamma(\mathbf{x}, \omega) \nabla \psi(\mathbf{x}, \omega)} \cdot \nabla \phi(\mathbf{x}, \omega). \end{aligned}$$

- For $\omega = 0$, ($\gamma = \sigma$), Λ_σ^D is self-adjoint and positive semidefinite.

Dirichlet variational principle

- $\gamma = \sigma$, the power dissipated into heat is:

$$\langle V, \Lambda_{\sigma}^D V \rangle = \min_{\phi|_{\partial\Omega}=V} \int_{\Omega} \sigma(\mathbf{x}) |\nabla\phi(\mathbf{x})|^2 dx.$$

- $\omega \neq 0$, $\gamma = \sigma + i\omega\epsilon$: The DtN map is not self-adjoint. Variational principles do exist in this case but they are more complicated Cherkaev, Gibiansky-94, Fannjiang, Papanicolaou-94

Reformulate the problem in terms of a different, self-adjoint map, with min-min variational formulation. Due to coupling between the real and imaginary parts of ϕ and \mathbf{j} , we have a vectorial formulation.

- Why are variational principles useful: analysis (ex: high contrast), numerical reconstructions, distinguishability studies.

The Neumann to Dirichlet (NtD) map

The **Neumann to Dirichlet** map $\Lambda_\gamma^N : H^{-\frac{1}{2}}(\partial\Omega) \mapsto H^{\frac{1}{2}}(\partial\Omega)$ is the generalized inverse of the DtN map,

$$(\Lambda_\gamma^N I)(\xi, \omega) = \phi(\xi, \omega), \quad \text{for } \xi \in \partial\Omega,$$

and it is defined on the restricted set

$$\mathcal{S}_I = \left\{ I(\xi, \omega) \in H^{-\frac{1}{2}}(\partial\Omega) \text{ s.t. } \int_{\partial\Omega} I(\xi, \omega) ds = 0 \right\}.$$

The potential ϕ satisfies

$$\begin{aligned} \nabla \cdot [\gamma(\mathbf{x}, \omega) \nabla \phi(\mathbf{x}, \omega)] &= 0, \quad \text{for } \mathbf{x} \in \Omega, \\ \gamma(\xi, \omega) \frac{\partial \phi(\xi, \omega)}{\partial n(\xi)} &= I(\xi, \omega) \text{ for } \xi \in \partial\Omega, \\ \int_{\partial\Omega} \phi(\xi, \omega) ds &= \int_{\partial\Omega} V(\xi, \omega) ds = 0. \end{aligned}$$

Note that Λ_γ^N is smoothing and it is better to use in practice, where data is contaminated with noise.

Dual variational formulation

- Assuming $\gamma(\mathbf{x}) = \sigma(\mathbf{x})$ i.e. ($\omega = 0$), the NtD map is self-adjoint and positive definite, with variational principle

$$\begin{aligned}
 \langle I, \Lambda_{\sigma}^N I \rangle &= \min_{\substack{\nabla \cdot \mathbf{j} = 0 \\ -\mathbf{j} \cdot \mathbf{n} |_{\partial\Omega} = I}} \int_{\Omega} \frac{1}{\sigma(\mathbf{x})} |\mathbf{j}(\mathbf{x})|^2 dx \\
 &= \sup_{V \in H^{\frac{1}{2}}(\partial\Omega)} \left\{ 2 \int_{\partial\Omega} I(\mathbf{x}) V(\mathbf{x}) ds - \min_{\phi |_{\partial\Omega} = V} \int_{\Omega} \sigma(\mathbf{x}) |\nabla \phi(\mathbf{x})|^2 dx \right\} \\
 &= \sup_{V \in H^{\frac{1}{2}}(\partial\Omega)} \left\{ 2 \langle I, V \rangle - \langle V, \Lambda_{\sigma}^D V \rangle \right\}
 \end{aligned}$$

- In the complex case, we have a primary and dual (min-min) variational formulation, as well (Cherkaev, Gibianski-94, Fan-njiang, Papanicolaou-94).

The DtN and NtD maps. Complete electrode model

- The Dirichlet to Neumann map is an $\mathbb{R}^{L \times L}$ matrix

$$\Lambda_\sigma^D \mathbf{V} = \mathbf{I}, \quad \mathbf{V} = (V_1, \dots, V_L)^T, \quad \mathbf{I} = (I_1, \dots, I_L)^T,$$

which is, as in the continuum setting, self-adjoint and positive semidefinite, with null space spanned by vector $(1, \dots, 1)^T$.

$$\mathbf{V}^T \Lambda_\sigma \mathbf{V} = \min_{u \in H^1(\Omega)} \int_\Omega \sigma |\nabla u|^2 d\mathbf{x} + \sum_{j=1}^L \frac{1}{z_j} \int_{E_j} [u - V_j]^2 ds$$

- The dual variational formulation for the NtD map is

$$\mathbf{I}^T \Lambda_\sigma^N \mathbf{I} = \max_{\substack{\mathbf{V} \in \mathbb{R}^L \\ u \in H^1(\Omega)}} \left\{ 2\mathbf{V}^T \mathbf{I} - \int_\Omega \sigma |\nabla u|^2 d\mathbf{x} - \sum_{j=1}^L \frac{1}{z_j} \int_{E_j} [u - V_j]^2 ds \right\}$$

where $\mathbf{V} = (V_1, \dots, V_L)^T$ is the vector of Lagrange multipliers, corresponding to constraints $\int_{E_j} \sigma \partial u / \partial n ds = I_j$.

Formulation of the Inverse problem

The inverse problem of EIT is formulated as: Given the DtN (or NtD) map, find γ in Ω .

Essential questions:

1. **Injectivity of forward map:** If $\Lambda_{\gamma_1}^D = \Lambda_{\gamma_2}^D \implies \gamma_1 = \gamma_2$?
2. **Stability:** Can we invert the forward map and if so, is the inverse continuous?
3. **Reconstructions**
4. **Resolution:** We have noisy, incomplete data so what can we expect from the reconstructions?

Uniqueness at the boundary

- Kohn, Vogelius - 1984 showed that if σ is C^∞ near the boundary and $\Lambda_{\sigma_1}^D V = \Lambda_{\sigma_2}^D V$, for arbitrary $V \in H^{\frac{1}{2}}(\partial\Omega)$,

$$\frac{\partial^k \sigma_1(\mathbf{x})}{\partial n^k(\mathbf{x})} = \frac{\partial^k \sigma_2(\mathbf{x})}{\partial n^k(\mathbf{x})}, \text{ for } k \geq 0 \text{ and } \mathbf{x} \in \partial\Omega.$$

Their proof relies on energy estimates and careful selection of boundary data. To prove uniqueness at point $\mathbf{y} \in \partial\Omega$, they choose highly oscillatory V , with vanishing moments and with small support near \mathbf{y} . Such Dirichlet data ensures the fast decay of the potential in Ω and the result can be proved.

- Uniqueness in the interior of the domain Ω , for real analytic functions $\sigma(\mathbf{x})$ follows immediately.

Uniqueness and stability at the boundary

- Sylvester and Uhlmann - 1988 assume that $\partial\Omega$ is C^∞ and $\sigma \in C^\infty(\Omega) \rightsquigarrow \Lambda_\sigma^D$ is a classical pseudodifferential operator of order one (Calderón-1963).

From the definition of the symbol of Λ_σ^D , they find **explicit** formulae for σ and its normal derivatives at $\partial\Omega$.

- Nachman-1988: obtains explicit formulae for σ and $\partial\sigma/\partial n$ at $\partial\Omega$ (reconstruction error vanishes in L_2 norm). The regularity assumptions are relaxed to $\sigma \in C^{1,1}(\Omega)$.

- Stability: Sylvester and Uhlmann-1988: if σ is continuous in Ω , we have stability of boundary reconstructions:

$$\| \sigma_1 - \sigma_2 \|_{L^\infty(\partial\Omega)} \leq c \| \Lambda_{\sigma_1}^D - \Lambda_{\sigma_2}^D \|_{\frac{1}{2}, -\frac{1}{2}}.$$

If σ is Lipschitz continuous, reconstruction of σ is stable in norm $\| \cdot \|_{W^{1,\infty}(\partial\Omega)}$, as well.

Uniqueness in interior: Linearization at constants

- Calderón considered polarization of quadratic form $\langle V, \Lambda_\sigma^D V \rangle$:

$$Q_\sigma(V, W) = \int_\Omega \sigma(\mathbf{x}) \nabla \phi(\mathbf{x}) \cdot \nabla \psi(\mathbf{x}) d\mathbf{x}, \text{ where}$$

$$\begin{aligned} \nabla \cdot [\sigma(\mathbf{x}) \nabla \phi(\mathbf{x})] &= \nabla \cdot [\sigma(\mathbf{x}) \nabla \psi(\mathbf{x})] = 0 \text{ in } \Omega \\ \phi(\mathbf{x}) &= V(\mathbf{x}), \quad \psi(\mathbf{x}) = W(\mathbf{x}) \text{ on } \partial\Omega. \end{aligned}$$

Given $Q_\sigma(V, W)$ for all $V, W \in H^{\frac{1}{2}}(\partial\Omega) \rightsquigarrow \Lambda_\sigma^D$.

- Linearization: let $\sigma(\mathbf{x}) = \sigma_0(\mathbf{x}) + \epsilon h(\mathbf{x})$, for $\epsilon \ll 1$.

The Fréchet derivative of Q_σ at σ_0 (show this as an exercise)

$$dQ_{\sigma=\sigma_0}(h)(V, W) = \int_\Omega h(\mathbf{x}) \nabla \phi_0(\mathbf{x}) \cdot \nabla \psi_0(\mathbf{x}) d\mathbf{x},$$

where ϕ_0 and ψ_0 solve the conductivity problem for $\sigma_0(\mathbf{x})$ and Dirichlet data V and W , respectively.

- Calderón (1980) proved injectivity of the linearized map for $\sigma_0 = 1$ (constant), so ϕ_0 and ψ_0 are harmonic.

Calderon's imaging algorithm

- Define two harmonic functions as the **complex exponentials**

$$u(\mathbf{x}) = e^{\mathbf{x} \cdot \rho}, \quad \tilde{u}(\mathbf{x}) = e^{-\mathbf{x} \cdot \bar{\rho}}, \quad \mathbf{x} \in \mathbb{R}^n, \quad \rho = \eta + i\frac{k}{2}, \quad \text{s. t. } \rho \cdot \rho = 0, \quad \text{or}$$

$$|\eta| = \frac{|k|}{2}, \quad \eta \cdot k = 0, \quad \text{for } \eta, k \in \mathbb{R}^n.$$

- $dQ_{\sigma=\sigma_0}(h)(u, \tilde{u}) = -|\rho|^2 \int_{\Omega} h(\mathbf{x}) e^{\mathbf{x} \cdot (\rho - \bar{\rho})} d\mathbf{x} = -\frac{|k|^2}{2} \int_{\Omega} h(\mathbf{x}) e^{ik \cdot \mathbf{x}} d\mathbf{x}.$

Set $h = 0$ outside Ω and extend the integral to \mathbb{R}^n

$$dQ_{\sigma=1}(h)(e^{\mathbf{x} \cdot \rho}, e^{-\mathbf{x} \cdot \bar{\rho}}) = -\frac{|k|^2}{2} \hat{h}(k), \quad \underline{\text{so } \hat{h}(k) \text{ is unique.}}$$

- $Q_{\sigma}(e^{\mathbf{x} \cdot \rho}, e^{-\mathbf{x} \cdot \bar{\rho}}) = Q_1(e^{\mathbf{x} \cdot \rho}, e^{-\mathbf{x} \cdot \bar{\rho}}) + \epsilon dQ_{\sigma=1}(h)(e^{\mathbf{x} \cdot \rho}, e^{-\mathbf{x} \cdot \bar{\rho}}) + O(\epsilon^2),$
and Calderon's reconstruction is

$$\epsilon \hat{h}(k) \approx -\frac{2}{|k|^2} \left[Q_{\sigma}(e^{\mathbf{x} \cdot \rho}, e^{-\mathbf{x} \cdot \bar{\rho}}) - Q_1(e^{\mathbf{x} \cdot \rho}, e^{-\mathbf{x} \cdot \bar{\rho}}) \right].$$

Testing Calderon's imaging algorithm

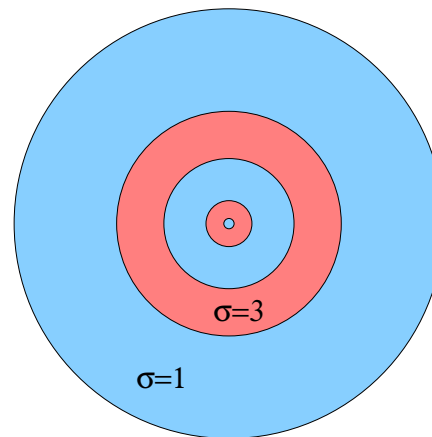
D. Isaacson, E. Isaacson - 1989 test the reconstruction for Ω a unit disk and $\sigma(\mathbf{x}) = 1 + \beta(\mathbf{x})$, where $\beta(\mathbf{x}) = \begin{cases} a & \text{for } |\mathbf{x}| \leq r < 1 \\ 0 & \text{otherwise.} \end{cases}$

The easy geometry allows an explicit formula for the reconstruct.

$$\beta^C(\mathbf{x}) = \frac{2a}{a+2} \sum_{p=0}^{\infty} \left(-\frac{a}{2+a} \right)^p \chi_{r^{p+1}}(\mathbf{x}), \text{ where}$$

χ_{r^p} = characteristic func. of disk of radius r^p , concentric with Ω .

- When $a = O(\epsilon)$, $\beta^C(\mathbf{x}) = a\chi_r(\mathbf{x}) + O(\epsilon^2)$.
- When $a \rightarrow \infty$, $\beta^C(\mathbf{x}) = 2 \left[\chi_r(\mathbf{x}) - \chi_{r^2}(\mathbf{x}) + \chi_{r^3}(\mathbf{x}) - \chi_{r^4}(\mathbf{x}) + \dots \right]$



Uniqueness for more general σ ?

- We have uniqueness for linearization at constants. Can we go further? We cannot obtain local uniqueness by means of the Inverse Function Theorem, due to the compactness of the derivative of the forward map. Other techniques are required.
- Kohn and Vogelius - 1984 used the uniqueness at the boundary to prove interior uniqueness of analytic σ . Extension to piecewise analytic σ , in subdomains with C^∞ boundary is given by Kohn and Vogelius - 1985.
- Druskin - 1984 proved uniqueness of piecewise constant σ .
- Next, we see how the complex exponentials used by Calderon can be used to prove uniqueness in the interior for the non-linear EIT problem.