Uniqueness in $\Omega \in \mathbb{R}^n, \ n \geq 3$

- Complex exponentials have been used by Calderón (1980) and by Fadeev (1996) (but he did not require that they be harmonic functions).

- However, Sylvester and Uhlmann (1987) were the first to realize that complex exponentials can be used for the nonlinear EIT problem, by taking a high frequency (WKB type) limit.

**Theorem (SU-87)** Let $\Omega \in \mathbb{R}^n$, $n \geq 3$, where $\partial \Omega$ is smooth. Suppose $\sigma_1, \sigma_2 \in C^{1,1}(\Omega)$; $\sigma_1, \sigma_2 \geq m > 0$ and let $\Lambda^{D}_{\sigma_1}V = \Lambda^{D}_{\sigma_2}V$, for any $V \in H^{\frac{1}{2}}(\partial \Omega)$. Then, $\sigma_1(x) = \sigma_2(x)$, for $x \in \overline{\Omega}$.

**Proof:** Begin by transforming the EIT problem

$$\nabla \cdot [\sigma(x)\nabla \phi(x)] = 0 \text{ in } \Omega,$$

$$\phi(x) = V(x) \text{ on } \partial \Omega,$$

$$\left(\Lambda^{D}_{\sigma}V\right)(x) = \sigma(x)\frac{\partial \phi(x)}{\partial n(x)} \text{ on } \partial \Omega,$$

to Schrödinger’s equation. Let $u(x) = \sigma^{\frac{1}{2}}(x)\phi(x)$. 
The Schrödinger equation

Let $q(x) = \frac{\Delta \sigma^2(x)}{\sigma^2(x)}$ be the Schrödinger potential. We have:

$$\Delta u(x) - q(x)u(x) = 0, \text{ in } \Omega$$
$$u(x) = f(x) = \sigma^2(x)V(x) \text{ on } \partial \Omega$$
$$(\Gamma_q f)(x) = \frac{\partial u(x)}{\partial n(x)} \text{ on } \partial \Omega,$$ where

$$(\Gamma_q f)(x) = \frac{\sigma^{-1}(x) \partial \sigma(x)}{2} f(x) + \sigma^{-\frac{1}{2}}(x) \left( \Lambda^D_{\sigma} \left( \sigma^{-\frac{1}{2}} f \right) \right)(x), \text{ for } x \in \partial \Omega.$$

- We saw that $\sigma$ and $\frac{\partial \sigma}{\partial n}$ are determined uniquely by $\Lambda^D_{\sigma}$. If $q \in L^\infty(\Omega)$ is known, the elliptic problem for $\sigma$
  $$\Delta \sigma^{\frac{1}{2}}(x) - \sigma^{\frac{1}{2}}(x)q(x) = 0 \text{ in } \Omega \text{ and } \sigma^{\frac{1}{2}}(x) = \text{ given on } \partial \Omega,$$
  is well posed.

- Thus, we need to show that $\Gamma_q$ determines $q$ uniquely.
The Schrödinger equation

**Theorem** Let $q_1, q_2 \in L^\infty (\Omega)$ and suppose that $\Gamma_{q_1} f = \Gamma_{q_2} f$ for arbitrary $f \in H^1_2 (\partial \Omega)$. Then, $q_1(x) = q_2(x)$, for $x \in \Omega$.

**Proof:** Take $u_1$ and $u_2$ solutions of Schrödinger eq. for potentials $q_1$ and $q_2$, respectively. By integration by parts,

$$
\int_\Omega [q_1 - q_2] u_1 u_2 \, dx = \int_{\partial \Omega} \left( u_2 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial u_2}{\partial n} \right) \, ds =
$$

$$
\int_{\partial \Omega} (u_2 \Gamma q_1 u_1 - u_1 \Gamma q_2 u_2) \, ds = \int_{\partial \Omega} (u_2 \Gamma q_2 u_1 - u_1 \Gamma q_2 u_2) \, ds.
$$

We have

$$
\int_{\partial \Omega} u_2(x) (\Gamma q_2 u_1)(x) \, ds = \int_\Omega \nabla \cdot [u_2(x) \nabla w(x)] \, dx,
$$

where

$$
\Delta w(x) - q_2(x) w(x) = 0 \text{ in } \Omega
$$

$$
w(x) = u_1(x), \quad (\Gamma q_2 u_1)(x) = \frac{\partial w(x)}{\partial n(x)} \text{ on } \partial \Omega.
$$
Uniqueness of \( q \) in Schrödinger’s equation

Then,

\[
\int_{\partial \Omega} u_2 \Gamma_{q_2} u_1 ds = \int_\Omega \nabla \cdot (u_2 \nabla w) \, dx = \int_\Omega [u_2q_2w + \nabla u_2 \cdot \nabla w] \, dx
\]

\[
= \int_\Omega [\nabla \cdot (w \nabla u_2) + w (-\Delta u_2 + q_2 u_2)] \, dx
\]

\[
= \int_{\partial \Omega} w \frac{\partial u_2}{\partial n} \, ds = \int_{\partial \Omega} u_1 \Gamma_{q_2} u_2 ds \quad \text{and so}
\]

\[
\int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx = \int_{\partial \Omega} \left( u_1 \Gamma_{q_2} u_2 - u_1 \Gamma_{q_2} u_2 \right) \, ds = 0.
\]

- What functions \( u_1 \) and \( u_2 \) to choose? We would like to use complex exponentials (recall Calderón’s method).

- Sylvester and Uhlmann show that Schrödinger’s equation has solutions which are approximately complex exponentials, for large complex frequencies.
The complex exponential solutions

- Schrödinger’s equation admits solutions of the form

\[ u(x) = e^{x \cdot \rho} [1 + \psi(x, \rho)], \text{ where } \rho \in \mathbb{C}^n \text{ and } \rho \cdot \rho = 0. \]

Functions \( \psi(x, \rho) = 0 \) only when \( q = 0 \) (\( \sigma = \text{constant} \)) but, for sufficiently large \( |\rho| \), \( \| \psi(x, \rho) \|_{L^2(\Omega)} \leq \frac{c}{|\rho|}. \)

- Choose solutions \( u_1 \) and \( u_2 \) of Schrödinger’s equation as

\[ u_j(x) = e^{x \cdot \rho_j} [1 + \psi(x, \rho_j)], \quad \rho_j \cdot \rho_j = 0, \quad j = 1, 2 \]

\[ \rho_1 + \rho_2 = ik, \quad |\rho_1|, |\rho_2| \to \infty, \quad \text{while } k \in \mathbb{R}^n \text{ is arbitrary.} \]

- Note that for \( n \geq 3 \), we can set

\[ \rho_1 = \frac{1}{2} \eta + \frac{i}{2} (\alpha \xi + k), \quad \rho_2 = -\frac{1}{2} \eta + \frac{i}{2} (-\alpha \xi + k) \]

\( k \in \mathbb{R}^n \) is arbitrary, \( \eta, \xi \in \mathbb{R}^n \) satisfy

\[ \eta \cdot \xi = \eta \cdot k = \xi \cdot k = 0, \quad |\xi| = 1, |\eta|^2 = \alpha^2 + |k|^2, \quad \alpha \to \infty. \]
From $\int_{\Omega} (q_1 - q_2)(x)u_1(x)u_2(x)dx = 0$, we have

$$\int_{\Omega} (q_1 - q_2)(x)e^{ik \cdot x}dx = \int_{\Omega} e^{ik \cdot x}(q_1 - q_2)(x)\left[\psi(x, \rho_1) + \psi(x, \rho_2)\right]dx.$$  

• Since $\sigma$ and all its derivatives are uniquely determined at $\partial \Omega$, we have that $q_1(x) = q_2(x)$ for $x \in \partial \Omega$. The integral in the left hand side can then be extended to $\mathbb{R}^n$, by taking $(q_1 - q_2)(x) = 0$ outside $\Omega$.

• $q_1, q_2 \in L^{\infty}(\Omega)$ and due to the decay of $\|\psi(x, \rho)\|_{L^2(\Omega)}$, as $\frac{1}{|\rho|}$, we have in the limit $\alpha \to \infty$, ($|\rho_1|, |\rho_2| \to \infty$),

$$\int_{\mathbb{R}^n} (q_1 - q_2)(x)e^{ik \cdot x}dx = \widehat{q_1}(k) - \widehat{q_2}(k) = 0.$$  

• $k$ is arbitrary and so, $q_1 = q_2$ q.e.d.
Notes on uniqueness in dimension $n \geq 3$

- Extensions to $\sigma \in W^{2,p}(\Omega), \ p > \frac{n}{2}$, Chanillo - 90; $\sigma \in C^{3+\delta}(\Omega)$, Brown - 96. The strongest result is for Lipschitz $\sigma$, Päivärinta, Panchenko, Uhlmann - 02.

- Sylvester and Uhlmann’s method of proof requires $|\rho| \to \infty$, while keeping $k$ arbitrary. This cannot be done in 2-D.

- Nachman - 1995 proved uniqueness in 2-D for $\sigma \in W^{2,p}(\Omega), \ p > 1$, by using techniques of inverse scattering theory and also complex exponential solutions. We do not discuss this proof here. But we give a 3-D reconstruction due to Nachman-88, which contains some of the techniques in his 2-D proof.

- The 3-D uniqueness proof holds for complex $\gamma$. In 2-D, Nachman’s method does not apply to the complex case. Extensions to complex 2-D problem, for small $\omega$, Francini-2000.

- When $\sigma$ or $\gamma$ are anisotropic (matrix valued), uniqueness does not hold. This is because one can do a coordinate transformation which keeps $\partial \Omega$ unchanged and gives the same NtD or DtN maps, even though $\sigma$ or $\gamma$ are different.
3-D reconstruction method: Nachman-1988

- **Step 1:** Reduce problem to Schrödinger equation. The DtN map $\Gamma_q$ is given by $\Lambda_\sigma^D$, $\sigma |_{\partial \Omega}$ and $\frac{\partial \sigma}{\partial n} |_{\partial \Omega}$, as shown before.

- **Step 2:** Given $\Gamma_q$, we wish to reconstruct $q$ in $\Omega$. Since we can find $\sigma$ and its derivatives at $\partial \Omega$, we can assume that $q |_{\partial \Omega} = 0$ and we can extend it to 0 in $\mathbb{R}^n \setminus \Omega$.

The reconstruction of $q$ uses complex exponential solutions of Schrödinger’s equation.

- **Step 3:** Given $q$, find $\sigma$ by solving

$$
\Delta \sigma^{\frac{1}{2}}(x) - \sigma^{\frac{1}{2}}(x)q(x) = 0 \text{ in } \Omega
$$

$$
\sigma^{\frac{1}{2}}(x) = \text{ given on } \partial \Omega.
$$
Reconstruction of $q$ from $\Gamma_q$

- Consider solution $u(x, \rho) = e^{ix \cdot \rho} [1 + \psi(x, \rho)]$, for $\rho \cdot \rho = 0$, of equation $\Delta u - qu = 0$.

- Take a real vector $\xi$ such that $(\xi + \rho) \cdot (\xi + \rho) = |\xi|^2 + 2\xi \cdot \rho = 0$.

Note that for $n \geq 3$, we have enough freedom to find such $\rho \in \mathbb{C}^n$ and $\xi \in \mathbb{R}^n$ and let $|\rho| \to \infty$ while keeping $\xi$ arbitrary.

- Nachman defines the “scattering transform”

$$t(\xi, \rho) = \int_{\mathbb{R}^n} q(x) u(x, \rho) e^{-ix \cdot (\xi + \rho)} dx = \int_{\mathbb{R}^n} q(x) e^{-ix \cdot \xi} [1 + \psi(x, \rho)] dx$$

If $t(\xi, \rho)$ can be determined in terms of $\Gamma_q$, the reconstruction is

$$\hat{q}(\xi) = \lim_{|\rho| \to \infty} t(\xi, \rho).$$
Calculating $t(\xi, \rho)$ in terms of $\Gamma_q$

- Using that $e^{-ix \cdot (\xi + \rho)}$ is harmonic and integrating by parts,

$$t(\xi, \rho) = \int_{\partial \Omega} e^{-ix \cdot (\xi + \rho)} \left((\Gamma_q - \Gamma_0) u |_{\partial \Omega}\right)(x) ds.$$ 

We do not know $u(x, \rho) = e^{ix \cdot \rho} [1 + \psi(x, \rho)]$ at the boundary because we do not know function $\psi$.

- Nachman constructs $u(x, \rho)$ for $x \in \partial \Omega$ by looking at the exterior problem (recall that the support of $q$ is $\Omega$)

$$\Delta u(x, \rho) = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

$$\frac{\partial u(x, \rho)}{n(x)} = (\Gamma_q u |_{\partial \Omega})(x) \text{ on } \partial \Omega,$$

$$\lim_{R \to \infty} \int_{|y|=R} \left[G_\rho(x,y) \frac{\partial w(y, \rho)}{n(y)} - w(y, \rho) \frac{\partial G_\rho(x,y)}{n(y)} \right] ds(y) = 0, \text{ for all } x,$$

where $w(y, \rho) = u(x, \rho) - e^{ix \cdot \rho}$ and $G_\rho(x,y) = \frac{e^{ix \cdot \rho}}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix \cdot \zeta}}{|\zeta|^2 + 2\rho \cdot \zeta} d\zeta$.

- This is a theoretical method that cannot be used in practice.
Reconstruction methods based on complex exponentials

- Taking the high frequency limit creates huge amplification of noise, by the complex exponentials.

- 2-D uniqueness (Nachman) uses similar scattering techniques, but it takes a low frequency limit.

- Reconstruction algorithm: Mueller, Siltanen, Isaacson (has been tried with experimental data!)

- Algorithm that should work for less smooth $\sigma$, consider a first order system formulation (instead of second order, Schrödinger form). Reconstruction method: Knudsen, Tâmâsan-2003.

- It is not known how to use the complex exponentials for 3-D reconstructions and for complex, 2-D reconstructions.
Can we always recover $\sigma$ from $\Lambda^D_\sigma$?

- Consider the related problem of network tomography. Uniqueness holds only for very special networks: rectangular (Curtis, Morrow-1990; Grünbaum, Zubelli-1990), circular (Ingerman, Curtis, Morrow; Colin de Verdière).

- If there are functions $\sigma$ such that $\Lambda^D_\sigma$ are equivalent to network DtN maps, we can get nonuniqueness.

- Are there media with strong flow channeling, as if we had electric current through an R-net? In Geophysics, one can have such high contrast, porous media.

- B., Papanicolaou, Berryman-1999: proof of equivalence of the DtN map of a class of such $\sigma$, to the map of an R-net.

Other related results: J.B. Keller (periodic arrangement of spheres/cylinders), Berlyand, Kolpakov, Novikov (random arrangement of cylinders). These are all homogenization results (do not consider the DtN map).
High Contrast Model

Shape of high contrast inclusions is not known in inversion. We need a general model.

Model (Kozlov, ’89): $\sigma(x) = \sigma_0 e^{-\frac{S(x)}{\epsilon}}$
- $\sigma_0 =$ constant (or smooth)
- $S(x) =$ smooth function with isolated, non-degenerate critical points.
- $\epsilon \ll 1$
Variational principles

The DtN and NtD maps are denoted by $\Lambda^D_\epsilon$ and $\Lambda^N_\epsilon$.

- The Dirichlet variational principle:
  \[
  (V, \Lambda^D_\epsilon V) = \min_{\phi|\partial\Omega=V} \int_\Omega \sigma_0 e^{-S(x)/\epsilon} \nabla \phi(x) \cdot \nabla \phi(x) \, dx
  \]

- The Thompson variational principle:
  \[
  (I, \Lambda^N_\epsilon I) = \min_{\nabla \cdot j = 0} \int_\Omega \frac{1}{\sigma_0} e^{S(x)/\epsilon} j(x) \cdot j(x) \, dx
  \]
  \[
  j \cdot n \mid_{\partial\Omega} = I
  \]

Result (B., Papanicolaou, Berryman) In the limit $\epsilon \rightarrow 0$,

\[
(V, \Lambda^D_\epsilon V) = < V, \Lambda^D_{d,\epsilon} V > (1 + o(1))
\]

\[
(I, \Lambda^N_\epsilon I) = < I, \Lambda^N_{d,\epsilon} I > (1 + o(1)),
\]

where $\Lambda^D_{d,\epsilon}$ and $\Lambda^D_{d,\epsilon}$ are the DtN and NtD maps of the equivalent R-net, with boundary excitation $I$ and $V$, respectively.
Heuristic Explanation of the R-net approximation

The minimizer $j$ in the Thompson variational principle satisfies

$$\nabla \times \left[ \frac{1}{\sigma_0} e^{S(x)/\epsilon} j(x) \right] = 0, \quad \nabla \cdot j(x) = 0 \text{ in } \Omega, \quad -j(x) \cdot n(x) = I(x) \text{ on } \partial \Omega.$$ 

Rewriting the PDE for $j$, $\nabla \times j(x) + \frac{1}{\epsilon} \nabla S(x) \times j(x) = 0$, in $\Omega$.

In the limit $\epsilon \to 0$, the electric current $j$ is parallel to $\nabla S(x)$ and it flows along paths of minimal resistance (steepest descent).
Proof of R-net equivalence

**Step 1:** Prove \((I, \Lambda_{\epsilon}^{N} I) \leq< \mathcal{I}, \Lambda_{d, \epsilon}^{N} \mathcal{I} > (1 + o(1))\)

We use the Thompson variational principle which, in 2-D, with \(x = (x, y)\) and \(j(x) = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) H(x) = \nabla^\perp H(x)\), becomes

\[
(I, \Lambda_{\epsilon}^{N} I) = \min_{H|_{\partial \Omega} = h} \int_{\Omega} \frac{1}{\sigma_{0}} e^{S(x)/\epsilon} \nabla^\perp H(x) \cdot \nabla^\perp H(x) \, dx,
\]

\[-j \cdot n = \frac{\partial H}{\partial s} = I \text{ on } \partial \Omega, \text{ so } H|_{\partial \Omega} = h(s) = \int^{s} I(x(s)) \, ds.\]

Choosing a trial \(H\) Take a vicinity of a ridge of minimal \(\frac{1}{\sigma_{0}} e^{S(x)/\epsilon}\). The local, curvilinear coordinates are \(x = (\xi, \eta)\), where \(|\eta| \leq \delta\) and \(\delta \to 0\) such that \(\frac{\delta^2}{\epsilon} \to \infty\) as \(\epsilon \to 0\).

\[
H = C_{2}
\]

Away from the ridge, \(H\) is a constant.
Proof of the upper bound

- Near the ridge, \( S(\xi, \eta) = S(\xi, 0) + \frac{k(\xi)}{2} \eta^2 + \frac{1}{6} \frac{\partial^3 S(\xi, 0)}{\partial \eta^3} \eta^3 + \ldots \)
  and we choose

\[
H(\xi, \eta) = -\frac{f(\xi)}{2} \text{erf} \frac{\eta}{\sqrt{2\varepsilon \kappa(\xi)}} + \text{constant}.
\]

\( f(\xi) \) is the net electric current flowing through the thin channel around the ridge. \( f(\xi) \) is constant \((C_1 - C_2)\) along the ridge, except at ramifications.

- The trial \( j = \nabla H \) is concentrated through the channels around the ridges of minimal resistance and it is zero elsewhere in \( \Omega \).

\[
j(\xi, \eta) = \frac{f(\xi)}{\sqrt{2\pi\varepsilon}} e^{-\frac{k(\xi)\eta^2}{2\varepsilon}} \left[ \sqrt{k(\xi)} \hat{\xi} - \frac{\eta}{2\sqrt{k(\xi)}} \frac{dk(\xi)}{d\xi} \hat{\eta} \right] + O(\delta)
\]

\[
\approx \frac{f(\xi)\sqrt{k(\xi)}}{\sqrt{2\pi\varepsilon}} e^{-\frac{k(\xi)\eta^2}{2\varepsilon}} \hat{\xi}.
\]

- In our calculation of the upper bound, we add the contribution of each channel of strong electric current \( j \).
Proof of the upper bound

- After using the expansion of $S(\xi, \eta)$ in the vicinity of each ridge of minimal $\frac{1}{\sigma_0} e^{S(x)/\epsilon}$ and after integrating over $\eta \in [-\delta, \delta]$,

$$
(I, \Lambda_\epsilon^N I) \leq \sum_{\text{ridges}} \int \frac{[f(\xi)]^2}{\sigma_0} \sqrt{\frac{k(\xi)}{2\pi\epsilon}} e^{\frac{S(\xi,0)}{\epsilon}} d\xi [1 + o(1)].
$$

The main contribution to the Laplace integral comes from maxima of $S(\xi, 0)$, which are the saddle points.

Take a saddle point $x_S = (\xi_S, 0)$ along ridge and let

$$
S(\xi, 0) = S(x_S) - \frac{p(x_S)(\xi - \xi_S)^2}{2} + \ldots
$$

The resistance of this saddle is $R^\epsilon(x_S) = \frac{1}{\sigma(x_S)} \sqrt{\frac{k(x_S)}{p(x_S)}}$.

The upper bound is

$$
(I, (\Lambda)^{-1} I) \leq \sum_{\text{ridges}} \sum_{x_S} R^\epsilon(x_S) [f(x_S)]^2 [1 + o(1)]
$$
The upper bound on the NtD map

The net electric current $f$ along each ridge satisfies Kirchhoff’s node law (conservation of charge).

At $\partial \Omega$, $\mathbf{j} = \nabla H$ flows into the nearest minimum of $\frac{1}{\sigma_0} e^{S(x)/\epsilon}$, along paths of steepest descent.

\[
I_a = H(s_{a'}) - H(s_{b'}) = \int_{s_{a'}}^{s_{b'}} I(s) \, ds.
\]

Inside $\Omega$, $f(x_{S1}) + f(x_{S2}) = \int_{a' e' b'} - \frac{\partial H}{\partial s} \, ds = H(s_{a'}) - H(s_{b'}) = I_a$
The upper bound on the DtN map

We just proved \((I, \Lambda^N \epsilon I) \leq < \mathcal{I}, \Lambda^N \epsilon \mathcal{I}> (1 + o(1))\)

Step 2: Prove \((V, \Lambda^D \epsilon V) \leq < \mathcal{V}, \Lambda^D \epsilon \mathcal{V}> (1 + o(1))\).

Use the Dirichlet variational principle

\[
(V, \Lambda^D \epsilon V) = \min_{\phi | \partial \Omega = V} \int_{\Omega} \sigma_0 e^{-S(x)/\epsilon} \nabla \phi(x) \cdot \nabla \phi(x) \, dx.
\]

Since \(\nabla \phi \cdot \nabla \phi = \nabla^\perp \phi \cdot \nabla^\perp \phi\), the proof is identical to that in the upper bound.

We have here the dual network for the dual flow \(\nabla^\perp \phi\), uniquely defined by the ridges of minimal \(\sigma_0 e^{-S(x)/\epsilon}\). The dual flux at boundary dual node \(a'\) is

\[
\int_{aa'd} - \nabla^\perp \phi(x) \cdot n(x) \, ds = V(a) - V(d).
\]

The boundary voltage for our R-net is therefore \(V_a = \) the measured \(V(a)\).
The lower bounds on the DtN and NtD maps

To get lower bounds, use the duality

\[(V, \Lambda^D_{\epsilon} V) = \sup_{I \in H^{-\frac{1}{2}}(\partial \Omega)} \left[ 2(I, V) - (I, \Lambda^N_{\epsilon} I) \right].\]

Take trial boundary current

\[I(s) = \sum_{j \in N_B} \frac{I_j}{\sqrt{2\pi \delta}} e^{-\frac{(s-s_j)^2}{2\delta}}, \quad \delta \ll 1 \implies \]

\[(I, V) = \sum_{j \in N_B} I_j \int_{\partial \Omega} \frac{V(s)}{\sqrt{2\pi \delta}} e^{-\frac{(s-s_j)^2}{2\delta}} ds = \sum_{j \in N_B} I_j \psi_j(1 + O(\delta)).\]

From \( < V, \Lambda^D_{d,\epsilon} V > = < I, \Lambda^N_{d,\epsilon} I > = \sum_{j \in N_B} I_j V_j, \) and the upper bound on \( (I, \Lambda^N_{\epsilon} I), \) we have

\[(V, \Lambda^D_{\epsilon} V) \geq < V, \Lambda^D_{d,\epsilon} V > (1 + o(1))\]

and the result follows.