
Electrical Impedance Tomography

IPAM - Lecture 3

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Question 1

How stable is the EIT problem?

If Λ_σ^D is slightly perturbed, what can we say about the perturbations of σ ?

Instability of the EIT problem

Fact Without a priori restriction on the class of σ , the inverse of the map $\sigma \mapsto \Lambda_\sigma^D$, taken as a map $\left[H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega) \right] \mapsto L^\infty(\Omega)$ is discontinuous. The same is true for NtD map Λ_σ^N .

Example 1 (Alessandrini-1988)

Consider Ω a unit disk and let $\sigma(\mathbf{x}) = \begin{cases} 1 + \alpha & \text{if } |\mathbf{x}| \leq r < 1 \\ 1 & \text{otherwise.} \end{cases}$

Given arbitrary $V(\theta) = \sum_{n=-\infty}^{\infty} V_n e^{in\theta} \in H^{\frac{1}{2}}(\partial\Omega)$,

$$\left(\Lambda_\sigma^D V \right) (\theta) = \sum_{n=-\infty}^{\infty} |n| \frac{2 + \alpha (1 + r^{2|n|})}{2 + \alpha (1 - r^{2|n|})} V_n e^{in\theta}$$

$$\left(\Lambda_1^D V \right) (\theta) = \sum_{n=-\infty}^{\infty} |n| V_n e^{in\theta}.$$

$$\lim_{r \rightarrow 0} \left(\Lambda_\sigma^D - \Lambda_1^D \right) V = 0, \text{ while } \|\sigma - 1\|_{L^\infty(\Omega)} = |\alpha|.$$

Highly oscillatory σ . A mechanism of instability.

- Take an oscillatory conductivity $\sigma\left(\mathbf{x}, \frac{\mathbf{x}}{\delta}\right) = \sigma^\delta(\mathbf{x})$, for $\delta \ll 1$.

A typical result is that, as $\delta \rightarrow 0$, $\phi^\delta \rightharpoonup \phi^*$, weakly in $H^1(\Omega)$, where ϕ^* is the potential for a homogenized, non oscillatory conductivity $\sigma^*(\mathbf{x})$. Due to the compact trace imbedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$, $\phi^\delta|_{\partial\Omega} \xrightarrow{\delta \rightarrow 0} \phi^*|_{\partial\Omega}$, strongly. $\implies \sigma^\delta$ and σ^* are very different, but we get a perfect $L^2(\partial\Omega)$ norm fit!

- Since Λ_σ^N , as an operator from $H^{\frac{1}{2}}(\partial\Omega)$ to $L^2(\partial\Omega)$, is compact, for any bounded sequence $\{\sigma_k\}$, we have a convergent subsequence $\{\Lambda_{\sigma_k}^N I\}$.
- To get convergence of the sequence of σ , we must impose some extra conditions (need compactness: $\sigma \in Q \subset\subset L^2\Omega$.)
- Stability estimates are known for smooth enough σ , but they are logarithmic (Alessandrini-1988) and they are optimal (Mandache-2001).

Resolution limits: distinguishability at noise level δ

We say that a conductivity perturbation $\delta\sigma(\mathbf{x}) = \sigma(\mathbf{x}) - \sigma_0(\mathbf{x})$ is **indistinguishable** from a given σ_0 , if

$$\sup_{\|I\|=1} \left\| \left(\Lambda_{\sigma}^N - \Lambda_{\sigma_0}^N \right) I \right\| \leq \delta.$$

Can we characterize the set of indistinguishable $\delta\sigma$?

- D. Isaacson - 1986 characterizes the indistinguishable set of perturbations $\delta\sigma(\mathbf{x}) = \begin{cases} \alpha & \text{if } |\mathbf{x}| \leq r < 1 \\ 0 & \text{otherwise,} \end{cases}$ for $\Omega = \text{unit disk}$.

Seagar-1983 studies $\delta\sigma$ supported in a disk that is not concentric with Ω . As expected, it is easier to see things near $\partial\Omega$ than deep in the interior of Ω .

- Dobson-1990 studied the set of indistinguishable $\delta\sigma$ for the linearized problem, where he looked at the Frechet derivative of the quadratic form:

$$DQ_{\sigma_0}[\delta\sigma](I, J) = \int_{\partial\Omega} J(\mathbf{x}) \left(D\Lambda_{\sigma_0}^N[\delta\sigma]I \right) (\mathbf{x}) d\mathbf{x}.$$

Distinguishability for the linearized problem

Let $\| DQ_{\sigma_0}[\delta\sigma] \| = \sup_{\substack{I, J \in \mathcal{S}_I \\ \|I\|, \|J\| = 1}} | DQ_{\sigma_0}[\delta\sigma](I, J) |$ and define the set of indistinguishable $\delta\sigma$ as

$$\mathcal{S}_\delta = \{ \delta\sigma \in L^\infty(\Omega) \text{ s.t. } \| DQ_{\sigma_0}[\delta\sigma] \| \leq \delta \}.$$

Dobson found two operators A and B that are diagonal in some basis in $L^2(\Omega)$, such that

$$\| B[\delta\sigma] \| \leq \| DQ_{\sigma_0}[\delta\sigma] \| \leq \| A[\delta\sigma] \|.$$

Then, if

$$\begin{aligned} \mathcal{A} &= \{ \delta\sigma \in L^\infty(\Omega) \text{ s.t. } \| A[\delta\sigma] \| \leq \delta \} \\ \mathcal{B} &= \{ \delta\sigma \in L^\infty(\Omega) \text{ s.t. } \| B[\delta\sigma] \| \leq \delta \}, \end{aligned}$$

$$\mathcal{A} \subset \mathcal{S}_\delta \subset \mathcal{B}$$

Distinguishability for the linearized problem

- The bound $\| B[\delta\sigma] \| \leq \| DQ_{\sigma_0}[\delta\sigma] \|$ is given by Calderon.

Take $\eta, \xi \in \mathbb{R}^n$ s.t. $\xi \cdot \eta = 0, |\eta| = |\xi|/2$ and set

$$I(\mathbf{x}) = \alpha \left(\eta + i \frac{\xi}{2} \right) \cdot \mathbf{n}(\mathbf{x}) e^{(\eta + i \frac{\xi}{2}) \cdot \mathbf{x}}$$
$$J(\mathbf{x}) = \alpha \left(-\eta + i \frac{\xi}{2} \right) \cdot \mathbf{n}(\mathbf{x}) e^{(-\eta + i \frac{\xi}{2}) \cdot \mathbf{x}}.$$

Assume that $\Omega \in B_R$, where B_R is the ball of radius R . Since $\| I \|, \| J \| = 1$, α is of the form $\beta \sqrt{2} e^{-\frac{R}{2}|\xi|}$, for $\beta = O(1)$.

Then, $| DQ_{\sigma_0}[\delta\sigma](I, J) | = \beta |\xi|^2 e^{-R|\xi|} |\delta\hat{\sigma}(\xi)| \leq \| DQ_{\sigma_0}[\delta\sigma] \|$.

We have **exponentially increasing error** in $|\delta\hat{\sigma}(\xi)|$ so we are inclined to eliminate large frequencies ξ .

- The upper bound $\| DQ_{\sigma_0}[\delta\sigma] \| \leq \| A[\delta\sigma] \|$ is not so pessimistic. Since we have more distinguishability near $\partial\Omega$ than inside, we need a **space-frequency analysis**. The construction of operator A is therefore done in a wavelet basis.

How to use distinguishability studies in practical reconstructions?

- Proper discretization is key. Intuitively, grids should be fine near the boundary and coarse deep inside the medium.

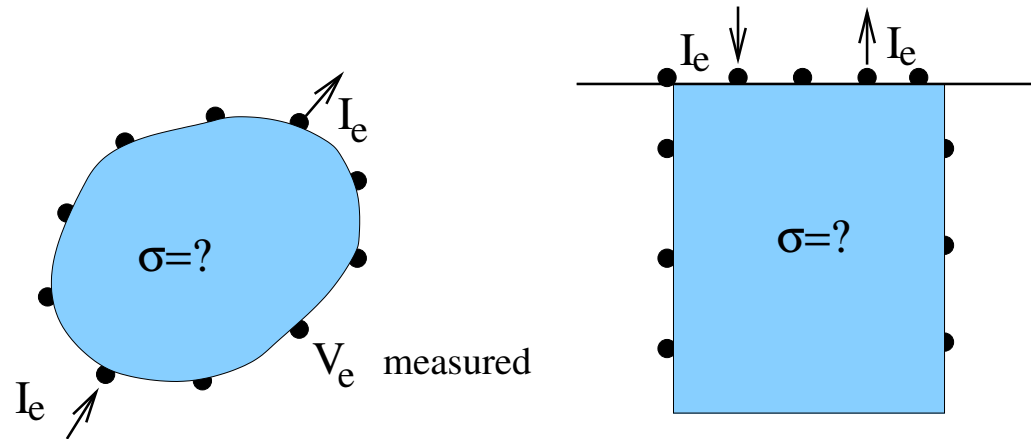
Use of proper grids prevents oscillations of reconstructed σ ...

- Optimal grids for 1-D (layered) problem have been proposed and studied for inversion by B., Druskin, Knizhnerman -2002, 2003. Convergence of inversion algorithms on such grids will be discussed in November.
- Optimal grids - forward problems: Davydcheva, Druskin, Ingerman, Moskow, Knizhnerman. Extensions to higher dimensions, anisotropic elasticity problems and to electro-magnetics are currently being used in Schlumberger.

Practical Reconstructions of σ

- In practice, we have a finite number N of electrodes at the boundary. Therefore, our knowledge of Λ_{σ}^N or Λ_{σ}^D is **limited**. What is the information content?
- What are the best excitations? **Optimal currents** $I(\mathbf{x})$.

Limited knowledge of Λ_{σ}^N for a system with N electrodes.



- In each experiment e , we have $I_e = (I_{e,1}, I_{e,2}, \dots, I_{e,N})$, where, by conservation of charge, $\sum_{j=1}^N I_{e,j} = 0$. The number of linearly independent I_e is $N - 1$.
- For each I_e , we measure $V_e = (V_{e,1}, V_{e,2}, \dots, V_{e,N})$, where by the choice of ground $\sum_{j=1}^N V_{e,j} = 0$.
- Our measurements give the $(N - 1) \times (N - 1)$ I to V symmetric (**reciprocity**) matrix \implies information content = $N(N - 1)/2$.

Choosing the excitation I_e

- It is not surprising that the resolution of the images improves as N increases and the electrodes fill $\partial\Omega$ (Issacson, Cheney-91, Allers, Santosa-91).
- The results depend dramatically on our choice of I_e .
 - Least-Squares reconstruction methods use that $\Lambda_\sigma^N - \Lambda_{\sigma_0}^N$ is a Hilbert Schmidt operator over $L^2(\partial\Omega)$, for a large class of σ , and seek to minimize an approx to H-S norm:

$$\sum_{e=1}^N \left\| \left(\Lambda_\sigma^N - \Lambda_{\sigma_0}^N \right) I_e \right\|_{L^2(\partial\Omega)}^2$$

- The **optimal currents** I_e are the eigenfunctions corresponding to the leading eigenvalues, because they give a large distinguishability:

$$d(I_e) = \left\| \left(\Lambda_\sigma^N - \Lambda_{\sigma_0}^N \right) I_e \right\|,$$

where $\|I_e\| = 1$ is assumed.

Reconstruction algorithms

There are many reconstruction methods that can be roughly classified as:

- Linearization methods (Born approximation)
- Nonlinear methods
 - Iterative methods.
 - Non iterative, direct methods.

In any practical algorithm, the ill posedness must be addressed \implies regularization.

Least Squares, Newton type reconstruction methods

$$\min_{\sigma} \mathcal{R}(\sigma), \quad \mathcal{R}(\sigma) = \frac{1}{2} \sum_{e=1}^N \left\| \Lambda_{\sigma}^N I_e - V_e \right\|_{L^2(\partial\Omega)}^2 + \text{reg. term.}$$

- An iterative, Newton approach generates at step k , $\sigma_k(\mathbf{x})$, and seeks $\sigma_{k+1}(\mathbf{x}) = \sigma_k(\mathbf{x}) + \delta\sigma(\mathbf{x})$, by minimizing over $\delta\sigma$,

$$\mathcal{R}(\sigma_k + \delta\sigma) \approx \mathcal{R}(\sigma_k) + D\mathcal{R}(\sigma_k)\delta\sigma + \frac{1}{2}D^2\mathcal{R}(\sigma_k)(\delta\sigma, \delta\sigma)$$

- Regularity of the forward map: $F_e(\sigma) = \Lambda_{\sigma}^N I_e$?

Dobson-92: assuming a σ_k such that $\nabla\phi$ is bounded,

$$\|F(\sigma_k + \delta\sigma) - F(\sigma_k) - DF(\sigma_k)\delta\sigma\|_{L^2(\partial\Omega)} \leq C\|\delta\sigma\|_{L^2(\Omega)}^2,$$

where $DF(\sigma)\delta\sigma = \delta\phi|_{\partial\Omega}$ and $\delta\phi$ solves the linearized problem

$$\begin{aligned} \nabla \cdot [\sigma(\mathbf{x})\nabla\delta\phi(\mathbf{x})] &= -\nabla \cdot [\delta\sigma(\mathbf{x})\nabla\phi(\mathbf{x})] \\ \frac{\partial\delta\phi}{\partial n} &= 0 \text{ on } \partial\Omega, \text{ assuming } \delta\sigma|_{\partial\Omega} = 0. \end{aligned}$$

Least Squares, Newton type reconstruction methods

- The derivative of the objective function at σ_k is

$$D\mathcal{R}(\sigma_k)\delta\sigma = \sum_{e=1}^{N_e} \int_{\Omega} \delta\sigma(\mathbf{x}) DF_e^*(\sigma_k) (F_e(\sigma_k) - V_e) d\mathbf{x} + \text{reg. t.}$$

- To calculate the Hessian, we need the 2nd derivative of F_e

$$D^2F[\sigma](\delta\sigma, \delta\sigma)(\mathbf{x}) = \delta^2\phi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \text{ where}$$

$$\nabla \cdot [\sigma(\mathbf{x}) \nabla \delta^2\phi(\mathbf{x})] = -\nabla \cdot [\delta\sigma(\mathbf{x}) \nabla \delta\phi(\mathbf{x})] \quad \text{in } \Omega,$$

$$\frac{\partial \delta^2\phi(\mathbf{x})}{\partial n(\mathbf{x})} = 0 \quad \text{on } \partial\Omega.$$

- We have

$$D^2\mathcal{R}(\sigma_k)\delta\sigma(\mathbf{x}) = \sum_{e=1}^N DF_e^*(\sigma_k) DF_e(\sigma_k) \delta\sigma(\mathbf{x}) + \sum_{e=1}^N \left(D^2F_e[\sigma] \right)^* (F_e(\sigma_k) - V_e, \delta\sigma)(\mathbf{x}) + \text{reg. t.}$$

The adjoint

Assuming σ is known at $\partial\Omega$,

$$DF_e^*(\sigma) (F_e(\sigma) - V_e) (\mathbf{x}) = -\nabla\phi_e(\mathbf{x}) \cdot \nabla\eta_e(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

$$\text{where } \nabla \cdot [\sigma(\mathbf{x})\nabla\eta_e(\mathbf{x})] = 0 \text{ in } \Omega$$

$$\sigma(\mathbf{x})\frac{\partial\eta_e(\mathbf{x})}{\partial n(\mathbf{x})} = (F_e(\sigma) - V_e) (\mathbf{x}) \text{ on } \partial\Omega.$$

For $(D^2F_e(\sigma))^* (\cdot, \cdot)$, which is a bilinear form on $L^2(\partial\Omega) \times L^2(\Omega)$

$$\langle D^2F_e(\sigma)(\delta\sigma, \delta\sigma), F_e(\sigma) - V_e \rangle = \left(\delta\sigma, (D^2F_e(\sigma))^* (F_e(\sigma) - V_e, \delta\sigma) \right),$$

we have

$$(D^2F_e[\sigma])^* (F_e(\sigma) - V_e, \delta\sigma)(\mathbf{x}) = -\nabla\delta\phi_e(\mathbf{x}) \cdot \nabla\eta_e(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

where $\delta\phi_e$ solves the linearized problem.

Variational methods

- Unlike the least squares approach, variational methods seek a “stronger” data fit.
- Reconstruct σ such that the Dirichlet and Neumann potentials are the same in Ω .

Equivalently, the minimizing current density \mathbf{j} in the Thomson variational principle (Neumann problem with data I) is related to the minimizing potential ϕ in the Dirichlet variational problem (with data V), by Ohm’s law:

$$\mathbf{j}(\mathbf{x}) = -\sigma(\mathbf{x})\nabla\phi(\mathbf{x}).$$

Variational Constraints (Berryman and Kohn 1990)

- σ is **Dirichlet feasible** for data $V_e \in H^{\frac{1}{2}}(\partial\Omega)$, $I_e \in H^{-\frac{1}{2}}(\partial\Omega)$ if

$$(V_e, \Lambda_{\sigma}^D V_e) = \min_{\phi|_{\partial\Omega} = V_e} \int_{\Omega} \sigma |\nabla \phi|^2 d\mathbf{x} \geq (V_e, \Lambda_{\sigma^*}^D V_e) = \int_{\partial\Omega} I_e V_e ds = P_e$$

- σ is **Thompson feasible** for measurements V_e, I_e if

$$(I_e, \Lambda_{\sigma}^N I_e) = \min_{\substack{\nabla \cdot \mathbf{j} = 0 \\ -\mathbf{j} \cdot \mathbf{n}|_{\partial\Omega} = I_e}} \int_{\Omega} \frac{1}{\sigma} |\mathbf{j}|^2 d\mathbf{x} \geq (I_e, \Lambda_{\sigma^*}^N I_e) = \int_{\partial\Omega} I_e V_e ds = P_e$$

- σ is **feasible for measurements** V_e, I_e if it is both Dirichlet and Thompson feasible.
- σ is **feasible** if it is Dirichlet and Thompson feasible for all N sets of measurements.

Motivation for Using Variational Constraints

- At the “solution”, the constraints are satisfied with equality.
- Let $\phi_e^{(k)}$ and $\mathbf{j}_e = -\sigma^{(k)}\nabla\psi_e^{(k)}$ be the Dirichlet and Thompson minimizers for a sequence $\{\sigma^{(k)}\}$ and suppose that

$$(V_e, \Lambda_{\sigma^{(k)}}^{\text{DtN}} V_e) \rightarrow P_e, \quad (I_e, \Lambda_{\sigma^{(k)}}^{\text{NtD}} I_e) \rightarrow P_e.$$

Note that

$$(V_e, \Lambda_{\sigma^{(k)}}^{\text{DtN}} V_e) + (I_e, \Lambda_{\sigma^{(k)}}^{\text{NtD}} I_e) - 2P_e = \int_{\Omega} \sigma^{(k)} |\nabla(\phi_e^{(k)} - \psi_e^{(k)})|^2 d\mathbf{x} \rightarrow 0$$

and, by the coercivity of the bilinear form

$$\int_{\Omega} \sigma^{(k)} \nabla u \cdot \nabla v d\mathbf{x}, \quad \text{for } u, v \in H^1(\Omega), \quad \text{s.t.} \quad \int_{\partial\Omega} u ds = \int_{\partial\Omega} v ds = 0,$$

we have $\|\phi_e^{(k)} - \psi_e^{(k)}\|_{H^1(\Omega)} \rightarrow 0$ i.e. $\|V_e - F[\sigma^{(k)}, I_e]\|_{H^{\frac{1}{2}}(\partial\Omega)} \rightarrow 0$.

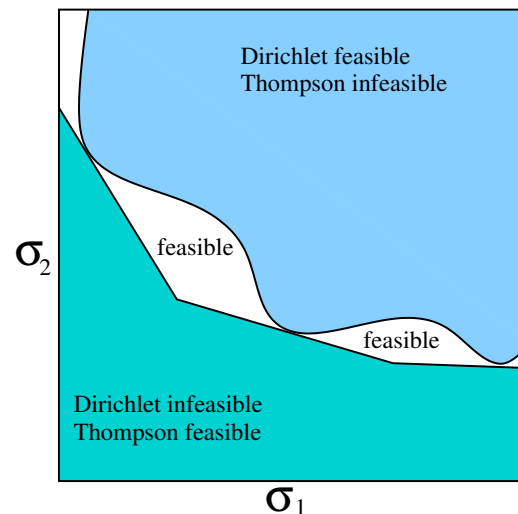
- How to achieve convg. of constraints? (B., Gray, Zhang-02)

Least Squares Formulation

- Typically achieve $\|V_e - F[\sigma^{(k)}, I_e]\|_{L^2(\partial\Omega)} \rightarrow 0$ and if $I_e \in L^2(\partial\Omega)$,

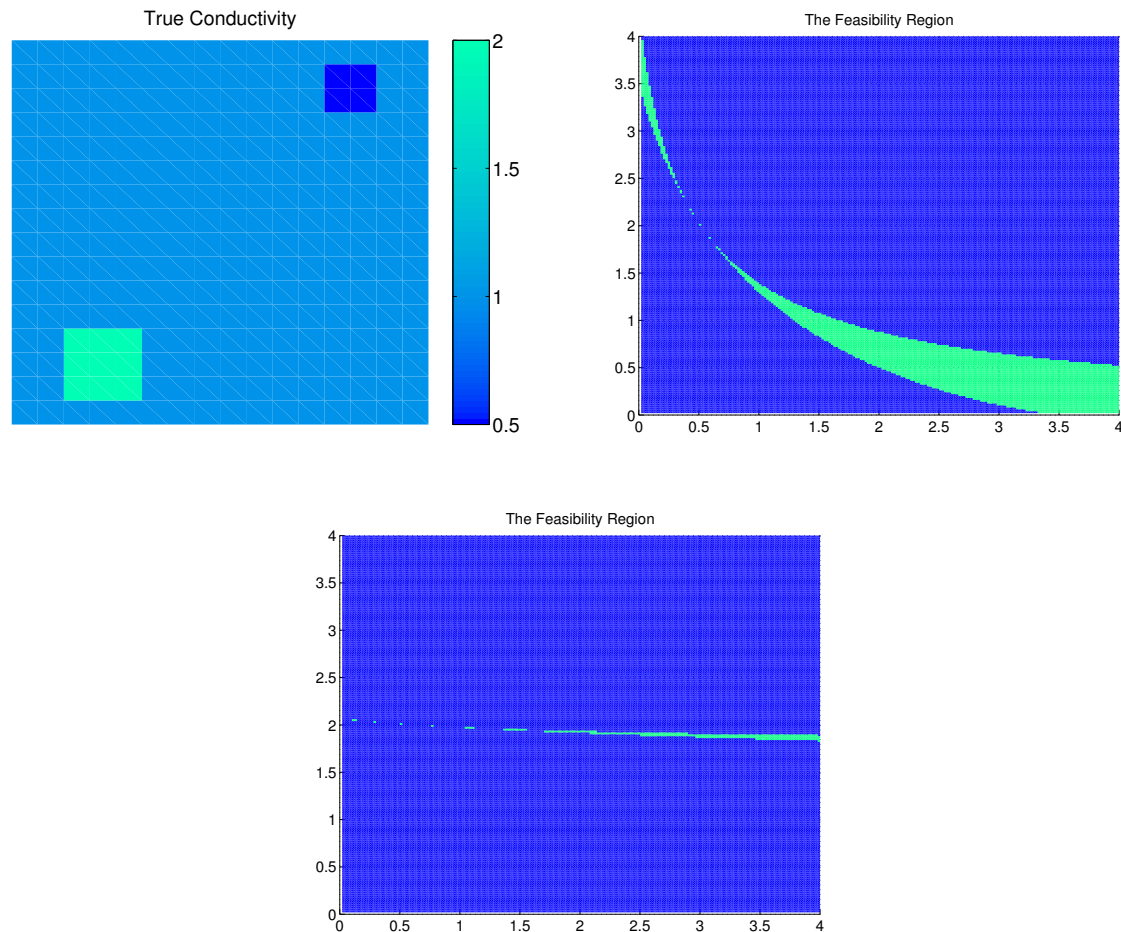
$$(I_e, \Lambda_{\sigma^{(k)}}^N I_e) - P_e = \int_{\partial\Omega} I_e(\mathbf{x}) [F[\sigma^{(k)}, I_e](\mathbf{x}) - V_e(\mathbf{x})] ds(\mathbf{x}) \rightarrow 0.$$
- Duality $(V_e, \Lambda_{\sigma}^D V_e) = \sup_{I \in \mathcal{J}} [2(I, V_e) - (I, \Lambda_{\sigma}^N I)].$

Suppose that $\sigma =$ Thomson infeasible: $(I_e, \Lambda_{\sigma}^N I_e) < P_e, .$
 Then $\sigma =$ Dirichlet feasible $(V_e, \Lambda_{\sigma}^D V_e) > P_e.$



\implies Constrain the iteration to Dirichlet infeas. region.

Example of Feasibility Regions



- Feasibility region depends strongly on excitation I_e . It also depends on σ .

Variational methods

- The variational constraints are another way of achieving the minimum of the variational, equation-error functional considered by Kohn, Vogelius-87 (Wexler-85).
- Another type of regularization:
 - The objective function lacks lower semicontinuity so, unless we regularize, we expect oscillatory sequences of σ .
 - Kohn, Vogelius calculate the relaxation of the problem, where the anticipated oscillations in σ are “built” directly in the functional (homogenization approach).
 - The new, relaxed problem, is lower semicontinuous and σ belongs to a larger, relaxed set, which contains the original admissible set.
 - Problem: Relaxed set includes anisotropic σ , for which uniqueness doesn't hold.