A quantitative study of imaging in random waveguides

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Basic imaging problem - Point spread function

Pressure field $p(t, x, z)$ satisfies the wave equation

$$\left[ \partial_z^2 + \partial_x^2 - \frac{1}{c^2(x, z)} \partial_t^2 \right] p(t, x, z) = e^{-i\omega_o t} f(Bt) \delta(x - x_o) \delta(z),$$

for $x \in (0, \mathcal{D}(z))$, $z \in \mathbb{R}$ and $t > 0$, with zero initial conditions and

$$p(t, \mathcal{D}(z), z) = \partial_x p(t, 0, z) = 0, \quad z \in \mathbb{R}, \quad t > 0.$$

Study imaging in perturbed waveguides.
Random waveguides

• Perturbations of boundaries and wave speed are unknown and usually of no interest.

• Random waveguides are models of our uncertainty of the perturbations.

• The array measures one realization of the random field $p(t, x, z)$.
  - “Near” the source: $p(t, x, z) \approx p_0(t, x, z)$ the unperturbed field.
  - At long ranges $p(t, x, z)$ is quite different than $p_0(t, x, z)$ $\Rightarrow$ perturbations can’t be neglected!

• Coherent imaging relies on expectation $\mathbb{E}[p] = \text{coherent field}$.

Q1: How do we mitigate the random fluctuations $p - \mathbb{E}[p]$?

Q2: How do we use the coherent field? Typically $\mathbb{E}[p] \neq p_0$. 
Waveguide with perturbed wave speed

\[ \varepsilon = 0\% \]

\[ \varepsilon = 1\% \]

\[ \varepsilon = 3\% \]

*Setup: \( c_o = 1.5\text{km/s}, \) extra wide bandwidth \( 1.5 - 4.5\text{kHz} \) \( (\lambda_o = 0.5\text{m}) \). The waveguide is \( 20\lambda_o \) deep and \( z_A = 494\lambda_o \).
• Brief review of the study of long range cumulative scattering effects in random waveguides.

- We consider separately boundary and wave speed perturbations to compare their effects.

• Describe $\mathbb{E}[p]$ and its variance (i.e., strength of fluctuations). Describe the transport of energy in waveguide.

• Apply the results to imaging
Random model of the perturbations

- **Boundary perturbations:** \( \mathcal{D}(z) = D \left[ 1 + \varepsilon \nu \left( \frac{z}{\ell} \right) \right] \).

- Random process \( \nu \) is stationary and mixing with
  \[ \mathbb{E}[\nu(\zeta)] = 0, \quad \mathcal{R}_{\nu}(\zeta) = \mathbb{E}[\nu(0)\nu(\zeta)] = \text{integrable}. \]

- Take \( \nu \) twice differentiable with almost sure bounded derivatives
  \[ p \left( t, \frac{\mathcal{D}(z)}{D} x, z \right) \sim p(t, x, z), \quad x \in (0, D), \quad z \in \mathbb{R}. \]

- **Wave speed perturbations:** \( \frac{1}{c^2(x, z)} = \frac{1}{c_0^2} \left[ 1 + \varepsilon \mu \left( \frac{x}{\ell}, \frac{z}{\ell} \right) \right] \).

- Random process \( \mu(\xi, \zeta) \) is stationary and mixing with
  \[ \mathbb{E}[\mu(\xi, \zeta)] = 0, \quad \mathcal{R}_{\mu}(\xi, \zeta) = \mathbb{E}[\mu(0, 0)\mu(\xi, \zeta)] = \text{integrable}. \]

- Correlation length \( \ell \sim \) central wavelength and \( \varepsilon \ll 1 \).
The Fourier coefficients \( \hat{p}(\omega, x, z) \) satisfy
\[
[\mathcal{M}_o + \epsilon \mathcal{M}_1 + \epsilon^2 \mathcal{M}_2 + \ldots] \hat{p}(\omega, x, z) = 0, \quad x \in (0, D), \quad z \neq 0,
\]
with boundary conditions
\[
\hat{p}(\omega, D, z) = \partial_x p(\omega, 0, z) = 0,
\]
outgoing conditions at \( |z| \to \infty \) and source conditions at \( z = 0 \).

- **Operator** \( \mathcal{M}_o = k^2 + \partial_x^2 + \partial_z^2 \) for \( k = \frac{\omega}{c_o} \) is as in ideal waveguides.

- **Operator** \( \mathcal{M}_1 \) has coefficients that are linear in \( \nu \) or \( \mu \):
\[
\mathcal{M}_1 = -2\frac{D}{\ell} \nu'\left(\frac{z}{\ell}\right) \partial_x^2 - 2 \nu\left(\frac{z}{\ell}\right) \partial_x^2 - \frac{1}{\ell^2} \nu''\left(\frac{z}{\ell}\right) x \partial_x
\]
or
\[
\mathcal{M}_1 = \epsilon k^2 \mu \left(\frac{x}{\ell}, \frac{z}{\ell}\right).
\]

- **Operator** \( \mathcal{M}_2 \) has coefficients that are quadratic in \( \nu \) or \( \mu \).
Unperturbed problem: separation of variables

- We have for $z > 0$

$$\hat{p}_o(\omega, x, z) = \sum_{j=1}^{N} a_{j,o}(\omega) \phi_j(x)e^{i\beta_jz} + \sum_{j>N} e_{j,o}(\omega) \phi_j(x)e^{-\beta_jz}$$

for $\phi_j(x) = \sqrt{\frac{2}{D}} \cos\left[\frac{\pi(j-1/2)x}{D}\right]$ the eigenfunctions of $\partial_x^2 + k^2$.

- Mode wavenumbers $\beta_j = \sqrt{k^2 - \left[\frac{\pi(j-1/2)}{D}\right]^2}$ for $j \geq 1$.

- Modes $\phi_j(x)e^{i\beta_jz} = \text{superpos. of plane waves in direction of}$

$$K_j = \left(\pm\frac{\pi(j - 1/2)}{D}, \beta_j\right), \quad j = 1, \ldots, N = \lfloor kD/\pi + 1/2 \rfloor.$$
Propagating modes in ideal waveguides

- First mode $\sim$ plane waves along slowness vector
  \[ K_1 = \left( \pm \frac{\pi}{2D}, \beta_1 \right), \quad \frac{\pi}{2D} \approx \frac{k}{2N} \ll \beta_1 \approx k. \]
  The waves travel at speed $\approx c_0$, where approx is for $N \gg 1$.

- The last mode $\sim$ plane waves along slowness vector
  \[ K_N = \left( \pm \frac{\pi(N - 1/2)}{D}, \beta_N \right), \quad \frac{\pi(N - 1/2)}{D} \approx k \gg \beta_N. \]
  The waves strike the boundary many times, at almost normal incidence and travel at very small speed.

- The mode amplitudes are constant
  \[ a_{j,o}(\omega) = \frac{\phi_j(x_o)}{2iB\beta_j} \hat{f} \left( \frac{\omega - \omega_o}{B} \right) \]
Perturbed problem in random waveguides

- Expansion in $L^2(0, D)$ basis $\{\phi_j(x)\}_{j \geq 1}$ gives

$$\hat{p}(\omega, x, z) = \sum_{j=1}^{N} a_j(\omega, z) \phi_j(x) e^{i\beta_j z} + \sum_{j=1}^{N} b_j(\omega, z) \phi_j(x) e^{-i\beta_j z} + \sum_{j>N} e_j(\omega, z) \phi_j(x) e^{-\beta_j z}, \quad z > 0.$$ 

- We have forward going waves, backward going waves and evanescent waves.

- Mode amplitudes are random functions of $z$.

They satisfy a system of stochastic ODE’s that describes mode coupling due to scattering.
Perturbed problem in random waveguides

\[
\partial_z \begin{bmatrix} a(\omega, z) \\ b(\omega, z) \end{bmatrix} = \varepsilon \Upsilon(\omega, \nu(z), z) + \varepsilon^2 \gamma(\omega, \nu(z), z) + \ldots \begin{bmatrix} a(\omega, z) \\ b(\omega, z) \end{bmatrix}
\]

- The leading order matrix is

\[
\Upsilon(\omega, \nu(z), z) = \begin{bmatrix} \Upsilon^{(a)}(\omega, \nu(z), z) & \Upsilon^{(b)}(\omega, \nu(z), z) \\ \Upsilon^{(b)}(\omega, \nu(z), z) & \Upsilon^{(a)}(\omega, \nu(z), z) \end{bmatrix}
\]

Blocks depend linearly on fluctuations \( \nu \) of boundary or \( \mu \) of speed

\[
\Upsilon^{(a)}_{jl}(\omega, \nu(z), z) = iC_{jl}(\omega, \nu(z))e^{i[\beta_l(\omega) - \beta_j(\omega)]z}
\]

\[
\Upsilon^{(b)}_{jl}(\omega, \nu(z), z) = iC_{jl}(\omega, \nu(z))e^{-i[\beta_l(\omega) + \beta_j(\omega)]z}
\]

- The second order matrix is quadratic in \( \nu \) or \( \mu \)

*After solving for evanescent amplitudes*
The solution is like in ideal waveguides for ranges \( z \ll O(\varepsilon^{-2}) \).

The limit of \( a^\varepsilon(\omega, Z) = a(\omega, z = Z/\varepsilon^2) \) satisfying

\[
\partial_Z \begin{bmatrix} a^\varepsilon(\omega, Z) \\ b^\varepsilon(\omega, Z) \end{bmatrix} = \left[ \frac{1}{\varepsilon} \mathcal{R} \left( \omega, \nu \left( \frac{Z}{\varepsilon^2} \right), \frac{z}{\varepsilon^2} \right) + \gamma \left( \omega, \nu \left( \frac{Z}{\varepsilon^2} \right), \frac{Z}{\varepsilon^2} \right) \right] \begin{bmatrix} a^\varepsilon(\omega, Z) \\ b^\varepsilon(\omega, Z) \end{bmatrix}
\]

follows from the diffusion approximation theorem.

Assuming smooth covariance \( \mathcal{R}_\nu \), the forward and backward going mode amplitudes decouple \( \sim \) forward scattering approximation.

The limit in distribution of \( a^\varepsilon(\omega, Z) \) is a Markov diffusion whose moments can be computed explicitly.
The coherent field at the array

- For an array at range $z_A$, we have

$$
\mathbb{E} [\hat{p}(\omega, x, z_A)] \approx \sum_{j=1}^{N} \phi_j(x) \mathbb{E} [a_j(\omega, z_A)] e^{i\beta_j z_A}
$$

where

$$
\mathbb{E}[a_j(\omega, z_A)] \approx a_{j,o}(\omega) \exp \left[ -\frac{z_A}{S_j(\omega)} + i \frac{z_A}{L_j(\omega)} \right].
$$

- The mean amplitudes decay with $z_A$ on the mode dependent scales $S_j(\omega) =$ scattering mean free paths.

- They also display a net phase that increases with $z_A$ on the mode dependent scale $L_j(\omega)$. 
Mode dependent scattering mean free paths

• For boundary perturbations

\[
\frac{1}{S_j(\omega)} = \frac{\pi^4 \ell (j - 1/2)^2}{D^4 \beta_j(\omega)} \sum_{l=1}^{N} \frac{(l - 1/2)^2}{\beta_l(\omega)} \widehat{\mathcal{R}}_\nu [(\beta_j(\omega) - \beta_l(\omega)) \ell]
\]

• For perturbed wave speed

\[
\frac{1}{S_j(\omega)} = \frac{k^4 \ell}{8 \beta_j(\omega)} \sum_{l=1}^{N} \frac{1}{\beta_l(\omega)} \widehat{\mathcal{R}}_{\mu_{jl}} [(\beta_j(\omega) - \beta_l(\omega)) \ell]
\]

with power spectral density \( \widehat{\mathcal{R}}_{jl} \) of the stationary process

\[
\mu_{jl}(\zeta) = \int_0^D dx \, \phi_j(x) \phi_l(x) \mu \left( \frac{x}{\ell}, \zeta \right)
\]

• In both cases we sum non-negative terms but the scales are different for the two types of perturbations.
Mode dependent net phase scales

- For perturbed wave speed

\[
\frac{1}{\mathcal{L}_j(\omega)} = \frac{k^4 \ell}{8 \beta_j(\omega)} \sum_{l=1}^{N+1} \frac{1}{\beta_l(\omega)} \gamma_{jl} \left[ \beta_j(\omega) - \beta_l(\omega) \right] + \kappa_{jl}^{(e)}(\omega)
\]

where

\[
\gamma_{jl}(\beta) = 2 \int_0^\infty du \sin(\beta \ell u) \mathcal{R}_{\mu_{jl}}(u)
\]

and

\[
\kappa_{jl}^{(e)}(\omega) = \frac{k^4 \ell}{2 \beta_j(\omega)} \sum_{l=N+1}^{\infty} \frac{1}{\beta_l(\omega)} \int_0^\infty du e^{-\beta_l(\omega)u} \mathcal{R}_{\mu_{jl}}(u) \cos \left[ \ell \beta_j(\omega) u \right]
\]

is due to the interaction with the evanescent modes.

- For boundary perturbations more complicated formula.
The SNR

- In imaging we have only one realization of the wave field.
- Coherent imaging makes sense if \( p \) is close to \( \mathbb{E}[p] \).
- If cumulative scattering plays a role, we should keep only the modes with high SNR in the imaging function

\[
\text{SNR}[a_j(\omega, z_A)] = \frac{\left| \mathbb{E}[a_j(\omega, z_A)] \right|}{\sqrt{\mathbb{E} \left[ |a_j(\omega, z_A)|^2 \right] - \left| \mathbb{E}[a_j(\omega, z_A)] \right|^2}}
\]
The intensity

- The intensity of the amplitudes is

\[
\mathbb{E} \left[ |a_j(\omega, z_A)|^2 \right] \approx \sum_{l=1}^{N} |a_{l,o}(\omega)|^2 \left[ e^{\Gamma^{(c)}(\omega) z_A} \right]_{jl}
\]

with matrix \( \Gamma^{(c)} \) defined in terms of \( \hat{R}_\nu \) or \( \hat{R}_{\mu,jl} \).

It is a negative semi-definite Perron-Frobenius matrix, with null space = \( \text{span}\{1/\sqrt{N}(1,\ldots,N)\} \) and second eigenvalue \(-\Lambda_2 < 0\).

- Thus

\[
\sup_{j,l=1,\ldots,N} \left| \left[ e^{\Gamma^{(c)}(\omega) z_A} \right]_{jl} - \frac{1}{N} \right| \leq O \left( e^{-z_A/L_{\text{equip}}} \right),
\]

where \( L_{\text{equip}}(\omega) = 1/\Lambda_2(\omega) \) is the equipartition distance, and

\[
\mathbb{E} \left[ |a_j(\omega, z_A)|^2 \right] \xrightarrow{z_A \to \infty} \frac{1}{N} \sum_{l=1}^{N} |a_{l,o}(\omega)|^2.
\]
The SNR and Imaging

- If \( z_A > S_j \) we have

$$\text{SNR}[a_j(\omega, z_A)] = \frac{|\mathbb{E}[a_j(\omega, z_A)]|}{\sqrt{\mathbb{E} \left[ |a_j(\omega, z_A)|^2 \right] - |\mathbb{E}[a_j(\omega, z_A)]|^2}} \sim \exp \left[ - \frac{z_A}{S_j(\omega)} \right].$$

- In coherent imaging we should keep the modes with \( S_j > z_A \).

- If these modes have large net scattering phases, we need to take the phases out!

- If none of the modes have high SNR, then the data is incoherent. We may look at imaging using intensities, as long as \( z_A < L_{\text{equip}} \).
Illustration of scattering effects

- Consider $D = 20$ central wavelengths, so that $N = 40$.

- Covariance of boundary fluctuations is of Matérn–7/2 form
  \[ R_\nu(\zeta) = \left( 1 + |\zeta| + \frac{6\zeta^2}{15} + \frac{15}{15} |\zeta|^3 \right) e^{-|\zeta|}, \]
  with power spectral density
  \[ \tilde{R}_\nu(\beta\ell) = \frac{32}{5 \left[ 1 + (\beta\ell)^2 \right]^4}. \]

  The correlation length is $\ell = \lambda_o/\sqrt{5}$, and the amplitude of the fluctuations is scaled by $\varepsilon = 0.013$.

- Covariance of the wave speed fluctuations is
  \[ R_\mu(\xi, \zeta) = e^{-\frac{\xi^2 + \zeta^2}{2}}, \]
  with correlation length $\ell = \lambda_o$ and $\varepsilon = 0.04$. 
• First modes maintain coherence longer $S_1 \gg S_N$

• Filtering makes sense for coherent imaging.

• Incoherent imaging makes no sense here because $S_1 \approx L_{\text{equip}}$. 
• All modes lose coherence at 100 wavelengths from source.

• Incoherent imaging is useful in these waveguides.
Coherent imaging

- Sensors at locations \( \{x_r = (x_r, z_A), r = 1, \ldots, N_R\} \) record \( p(t, x_r) \).

- Coherent imaging time reverses the data and propagates in ideal waveguide to \( x = (x, z) \) sweeping search domain.

\[
\mathcal{I}(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{N_R} \sum_{r=1}^{N_R} \hat{p}(\omega, x_r) \hat{G}_o(\omega, x_r, x) \\
= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{j=1}^{N} \frac{1}{2i\beta_j(\omega)} \hat{p}_j(\omega, z_A) \phi_j(x) e^{i\beta_j(\omega)(z_A-z)}
\]

where

\[
\hat{p}_j(\omega, z_A) = \frac{1}{N_R} \sum_{r=1}^{N_R} \hat{p}(\omega, x_r) \phi_j(x_r).
\]

- Adaptive coherent imaging seeks weights \( w_j \) of \( \hat{p}_j(\omega, z_A) \) that are optimal in trade-off between stability and resolution.
Adaptive coherent imaging

- Define the weighted imaging function

\[
I(x; w) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{j=1}^{N} \frac{w_j}{2i\beta_j(\omega)} \hat{p}_j(\omega, z_A) \phi_j(x) e^{i\beta_j(\omega)(z_A - z)} ,
\]

for \( w = (w_1, \ldots, w_N)^T \in \mathbb{C}^N \), with Euclidian norm \( \| w \| = 1 \),

- Optimal weights maximize the peak of the amplitude of the image normalized by its \( L^2 \) norm

\[
\mathbf{w}^* = \arg\max_{\mathbf{w} \in \mathbb{W}} M(\mathbf{w}), \quad M(\mathbf{w}) = \frac{|I(x^*; \mathbf{w})|^2}{\|I(\cdot; \mathbf{w})\|^2},
\]

where

\[
\mathbb{W} = \left\{ \mathbf{w} = (w_1, \ldots, w_N)^T \in \mathbb{C}^N, \quad \sum_{j=1}^{N} |w_j|^2 = 1 \right\}.
\]
What do we expect

Requirements for the imaging method to work:

- Only the coherent modes should contribute i.e., weights should null the modes with scattering mean free path $S_j < z_A$.

- The phases of the coherent modes should be compensated if they are large, i.e., $L_j < z_A$.

- Statistical stability requires bandwidth! The decoherence frequency in random waveguides is small $\sim \varepsilon^2 \omega_0$ so the bandwidth doesn’t have to be too large.

Under these conditions we can analyze the optimal weights with the theoretical figure of merit

$$M_{\text{th}}(w) = \frac{\mathbb{E} \left[ I(x_0; w) \right]^2}{\mathbb{E} \left[ \| I(\cdot; w) \|^2 \right]}.$$
Simplifications in analysis

\[ \mathbb{E} [\mathcal{I}(x; w)] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{j=1}^{N} \frac{w_j}{2i\beta_j(\omega)} \mathbb{E} \left[ \hat{p}_j(\omega, z_A) \right] \phi_j(x) e^{i\beta_j(\omega)(z_A-z)}, \]

- Assume a perfect array so that

\[ \mathbb{E} \left[ \hat{p}_j(\omega, z_A) \right] = \frac{1}{N_R} \sum_{r=1}^{N_R} \mathbb{E} \left[ \hat{p}(\omega, x_r) \right] \phi_j(x_r) \approx \int_{D}^D dx \mathbb{E} \left[ \hat{p}(\omega, x) \right] \phi_j(x) = \mathbb{E} [a_j(\omega, z_A)] e^{i\beta_j(\omega)z_A}. \]

- Recall that

\[ \mathbb{E} [a_j] = a_{j,o} e^{-\frac{z_A}{s_j(\omega)} + i\frac{z_A}{l_j(\omega)}}, \quad a_{j,o} = \frac{\phi_j(x_o)}{2iB\beta_j(\omega)} \hat{f} \left( \frac{\omega - \omega_o}{B} \right) \]

- Bandwidth \( B \) satisfies \( \varepsilon^2 \omega_o \ll B \lesssim \varepsilon \omega_o \).
Ideal waveguides

\[
\mathcal{I}(\mathbf{x}; \mathbf{w}) \sim \sum_{j=1}^{N} \frac{w_j}{\beta_j^2(\omega_0)} \phi_j(x) \phi_j(x_o) e^{-i\beta_j(\omega_0)z} f \left[ -\beta_j'(\omega_0)z \right]
\]

- Uniform weights \( w_j = 1/\sqrt{N} \) give cross-range fringes.
- Optimal weights \( w_j = \beta_j/\|\beta_j\| \).
- Optimal for cross-range resolution \( w_j = \beta_j^2/\|\beta^2\| \) is not best for range \( \sim \) compromise.
Perturbed boundary and \( z_A = 100\lambda_0 \) - Simulations

Weights remove incoherent modes.
Perturbed wavespeed and $z_A = 50\lambda_0$- Simulations

Coherent imaging doesn’t work!
Summary

• We presented a quantitative study of imaging in random waveguides.

• The theory allows us to quantify the loss of coherence of the waves.

• Coherent imaging methods makes sense in waveguides with perturbed boundaries up to some ranges. When coherent imaging fails, all methods fail!

• Coherent imaging doesn’t work in waveguides with perturbed wave speed. Here incoherent imaging can help!

• This is in 2-D waveguides and in 3-D waveguides with bounded cross-section. In 3-D waveguides with unbounded cross-section the scattering effects are a bit different.