CAAM 336.Examination 1

Solutions

1. [30 points]

   (a) i. **Vector space.**

   Operations on the set $C_p[0,1]$ and equality of functions are defined pointwise. Here p stands for the periodic boundary condition $f(0)=f(1)$.

   First, lets show that the operations are well defined: if $f,g\in C_p[0,1]$, then $(f+g),(\alpha f)\in C_p[0,1]$. I.e. the operations on the set do not "through us out of the set".

   1. $(f+g)(0)=f(0)+g(0)=f(1)+g(1)=(f+g)(1),$
   2. $(\alpha f)(0)=\alpha f(0)=\alpha f(1)=(\alpha f)(1),$

   Second, lets verify vector space axioms.
   Functions and scalars below: $f,g,h\in C_p[0,1],\quad \alpha,\beta\in\mathbf{R}$.

   1. $\mathbf{f}+(\mathbf{g}+\mathbf{h})=(\mathbf{f}+\mathbf{g})+\mathbf{h}:$
   $\big(f+(g+h)\big)(x)=f(x)+(g+h)(x)=f(x)+g(x)+h(x)=$
   $(f+g)(x)+h(x)=\big((f+g)+h\big)(x)\quad\forall x.$
   2. $\mathbf{f}+\mathbf{g}=\mathbf{g}+\mathbf{f}:$
   $(f+g)(x)=f(x)+g(x)=g(x)+f(x)=(g+f)(x)\quad\forall x.$
   3. **Zero element:**
   $\mathbf{0}(x)\equiv 0.$
   a) $\mathbf{0}(0)=0=\mathbf{0}(1)\Rightarrow\mathbf{0}\in C_p[0,1].$
   b) $(\mathbf{0}+f)(x)=\mathbf{0}(x)+f(x)=0+f(x)=(f)(x)\quad\forall x.$
   4. **Inverse element:**
   $(-f)(x)=-f(x).$
   a) $(-f)(0)=-f(0)=-f(1)=(-f)(1)\Rightarrow(-f)\in C_p[0,1].$
   b) $\big((-f)+f\big)(x)=(-f)(x)+f(x)=-f(x)+f(x)=0\quad\forall x.$
   5. $\alpha(\mathbf{f}+\mathbf{g})=\alpha\mathbf{f}+\alpha\mathbf{g}:$
   $\big(\alpha(f+g)\big)(x)=\alpha\big((f+g)(x)\big)=\alpha\big(f(x)+g(x)\big)=$
   $\alpha f(x)+\alpha g(x)=\big(\alpha f\big)(x)+\big(\alpha g\big)(x)\quad\forall x.$
   6. $(\alpha+\beta)\mathbf{f}=\alpha\mathbf{f}+\beta\mathbf{f}:$
   $\big((\alpha+\beta)f\big)(x)=(\alpha+\beta)f(x)=\alpha f(x)+\beta f(x)=(\alpha f)(x)+(\beta f)(x)\quad\forall x.$
   7. $\alpha(\beta\mathbf{f})=(\alpha\beta)\mathbf{f}:$
   $\big(\alpha(\beta f)\big)(x)=\alpha(\beta f)(x)=\alpha\beta f(x)=(\alpha\beta)f(x)=\big((\alpha\beta)f\big)(x)\quad\forall x.$
   8. **Multiplicative identity element:**
   $(1f)(x)=1f(x)=f(x)\quad\forall x.$

(a) ii. **Not a vector space.**

Take any $x \in \mathbf{R}^3$ with $\|x\| = 1$. By norm axioms $\|2x\| = 2\|x\| = 2$, therefore $2x$ does not belong to the considered set.

(a) iii. **Vector space.**

As in (a)i., first we show that if $x, y \in L$ then $(x+y), (\alpha x) \in L$. Here L denotes considered subset. This is true because the linear equation is homogeneous.

Second, we verify vector field axioms. Real numbers (and vectors in $\mathbf{R}^n$) satisfy them.

(b) i. **Not an inner product.**

Take any non-zero constant function, for instance $u(x) \equiv 1$. From $u'(x) \equiv 0$ follows $(u, u) = 0$. Thus, the property $(u, u) = 0$ iff $u = 0$ is violated.

(b) ii. **Inner product.**

Linearity in each argument follows directly from the linearity of the integral, commutativity is also obvious.

Assume $(u, u) = \int_0^1 (1 + x^2)u'(x)^2 = 0$. Since the integrand $(1 + x^2)u'(x)^2$ is a non-negative continuous function, it must be identically equal to zero. Then, from $1 + x^2 > 0 \ \forall x$ follows $u'(x) \equiv 0$. So $u(x) \equiv$ const. From boundary condition $u(0) = u(1) = 0$ follows $u \equiv 0$.

(b) iii. **Inner product.**

Consider any first order polynomial $a_1 x + a_0$. It defines a vector (a_1, a_0) in $\mathbf{R}^2$ and vice versa: any vector defines a polynomial. Sum and scalar multiplication of the polynomials correspond to sum and scalar multiplication of the corresponding vectors in $\mathbf{R}^2$.

Since $a_1 b_1 + a_0 b_0$ is an inner product in $\mathbf{R}^2$, it is an inner product in $\mathbf{P}_1$.

Notation in the problems below: $u' = \dfrac{du}{dx}, \quad u'' = \dfrac{d^2u}{dx^2}$.

(c) i. **Linear operator.**

$$\begin{aligned}
L(u+v) &= -\frac{d}{dx}\Big((1+x^2)(u+v)'\Big) = -\frac{d}{dx}\Big((1+x^2)(u'+v')\Big) = \\
& -\frac{d}{dx}\Big((1+x^2)u' + (1+x^2)v'\Big) = \\
& -\frac{d}{dx}\Big((1+x^2)u'\Big) - \frac{d}{dx}\Big((1+x^2)u'\Big) = Lu + Lv, \\
L(\alpha u) &= -\frac{d}{dx}\Big((1+x^2)(\alpha u)'\Big) = -\frac{d}{dx}\Big((1+x^2)\alpha u'\Big) = \\
& -\alpha\frac{d}{dx}\Big((1+x^2)u'\Big) = \alpha\, Lu.
\end{aligned}$$

(c) ii. **Non-linear operator.**

$$L(2u) = -(2u)\frac{d^2(2u)}{dx^2} = -4u\frac{d^2u}{dx^2} = 4\,Lu \neq 2\,Lu.$$

(d). By definition $u \in \mathcal{N}(L)$ iff $Lu = 0$. So we have a boundary value problem: $u'' + \pi^2 u = 0,\ u(0) = u(1) = 0$.

Solutions of $u'' + \pi^2 u = 0$ are $u(x) = c_1 \sin(\pi x) + c_2 \cos(\pi x)$, where c_1, c_2 are arbitrary constants.

Substituting boundary conditions we get: $c_1 \sin(0) + c_2 \cos(0) = 0$, $c_1 \sin(\pi) + c_2 \cos(\pi) = 0$. Both equations yield $c_2 = 0$.

Therefore, $\mathcal{N}(L) = \{c_1 \sin x\ :\ c_1 \in \mathbf{R}\} = \mathrm{span}\{\sin x\}$.

2. [30 points]

(a) The Gram matrix is:

$$G = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix},$$

and the right-hand side of the normal equations is:

$$b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solving the normal equations yields:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -1 \end{pmatrix}.$$

So, the nearest point is:

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1 \\ 1.5 \end{pmatrix}.$$

(b) The Gram matrix is:

$$G = \begin{pmatrix} (1,1) & (1,x) \\ (x,1) & (x,x) \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix},$$

and the right-hand side of the normal equations is:

$$b = \begin{pmatrix} (1,x^3) \\ (x,x^3) \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/5 \end{pmatrix}.$$

Solving the normal equations yields:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1/5 \\ 9/10 \end{pmatrix}.$$

The solution is: $-1/5 \cdot 1 + 9/10 \cdot x = -1/5 + 9/10x$.

(c) Polynomials 1 and $x - 1/2$ are linearly independent. Dimension of $\mathcal{P}_1$ is two. Therefore, these polynomials form a basis.

Orthogonality is shown below:

$$(1, x - 1/2) = \int_0^1 1 \cdot (x - 1/2)\, dx = \frac{x^2}{2}\Big|_0^1 - \frac{x}{2}\Big|_0^1 = 0.$$

(d) The Gram matrix is:

$$G = \begin{pmatrix} (1,1) & (1, x - 1/2) \\ (x - 1/2, 1) & (x - 1/2, x - 1/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/12 \end{pmatrix},$$

and the right-hand side of the normal equations is:

$$b = \begin{pmatrix} (1, x^3) \\ (x - 1/2, x^3) \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/5 - 1/8 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 3/40 \end{pmatrix}.$$

Solving the normal equations yields:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 9/10 \end{pmatrix}.$$

The solution is: $1/4 \cdot 1 + 9/10 \cdot (x - 1/2) = -1/5 + 9/10\, x$.

Note that we have obtained the same solution as in 2(b).

3. [40 points]

(a) To apply a Fourier series method we make a transformation $u(x) = 1 + v(x)$ to obtain a BVP with homogeneous boundary conditions.

Then $-\dfrac{d^2u}{dx^2} + u = -\dfrac{d^2v}{dx^2} + v + 1$, and we have the following problem:

$$\begin{aligned} -\frac{d^2v}{dx^2} + v = \hat{f}, \quad & x \in (0,1), \\ v(0) = v(1) = 0&, \end{aligned}$$

where $\hat{f}(x) = f(x) - 1$.

The linear operator $L : C_0^2[0,1] \to C[0,1]$ is defined by:

$$Lu = -\frac{d^2u}{dx^2} + u,$$

where $C_0^2[0,1] = \{v \in C^2[0,1] \ : \ v(0) = v(1) = 0\}$.

(b) To prove that the operator is symmetric take $u, v \in C_0^2[0,1]$. Using integration by parts and boundary conditions we obtain:

$$\begin{aligned}
(Lu, v) &= \int_0^1 (-u'' + u)\, v \, dx = -\int_0^1 u'' v \, dx + \int_0^1 uv \, dx = \\
&\quad -u'(1)v(1) + u'(0)v(0) + \int_0^1 u'v' \, dx \int_0^1 uv \, dx = \\
&\quad \int_0^1 u'v' \, dx + \int_0^1 uv \, dx, \\
(u, Lv) &= \int_0^1 u\, (-v'' + v) \, dx = -\int_0^1 uv'' \, dx + \int_0^1 uv \, dx = \\
&\quad -u(1)v'(1) + u(0)v'(0) + \int_0^1 u'v' \, dx \int_0^1 uv \, dx = \\
&\quad \int_0^1 u'v' \, dx + \int_0^1 uv \, dx.
\end{aligned}$$

So $(Lu, v) = (u, Lv)$, i.e. the operator is symmetric.
Substituting $v = u$ above, we obtain:

$$(Lu, u) = \int_0^1 (u')^2 \, dx + \int_0^1 u^2 \, dx \geq 0.$$

So the operator is positive definite. Since equality to zero is possible iff $u \equiv 0$, the operator is strictly positive definite.

(c) To find the eigenvalues λ and eigenfunctions $\phi(x)$ we need to solve the following equation:

$$-\phi'' + \phi = \lambda\phi \;\text{ or }\; \phi'' + (\lambda - 1)\phi = 0, \tag{1}$$

$$\phi(0) = \phi(1) = 0. \tag{2}$$

In case $\lambda \leq 1$ the equation (1) has the following solution:
$\phi(x) = c_1 e^{\sqrt{1-\lambda}\, x} + c_2 e^{-\sqrt{1-\lambda}\, x}$.
Boundary conditions (2) are satisfied only if $c_1 = c_2 = 0$. Since eigenfunctions must be non-zero functions, there are no eigenfunctions of this type.

In case $\lambda > 1$ the equation (1) has the following solution:
$\phi(x) = c_1 \sin(\sqrt{\lambda - 1}\, x) + c_2 \cos(\sqrt{\lambda - 1}\, x)$.
From the left boundary condition $\phi(0) = 0$ follows: $c_2 = 0$.
From the right boundary condition and because we are looking for non-zero solutions ($c_1 \neq 0$) follows: $\sin\sqrt{\lambda - 1} = 0$ and, therefore, $\sqrt{\lambda - 1} = \pi k$.
So the eigenvalues are $\lambda_k = \pi^2 k^2 + 1,\ k = 1, 2, \ldots$
and the corresponding normalized eigenfunctions are $\phi_k(x) = \sqrt{2}\sin(\pi k x)$.

(d) We seek the solution in the form: $v(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$.

Substituting it in the differential equation we get:

$$Lv(x) = \sum_{k=1}^{\infty} c_k(\pi^2 k^2 + 1)\phi_k(x) = \sum_{k=1}^{\infty} (\hat{f}, \phi_k)\phi_k(x).$$

Thus

$$c_k = \frac{(\hat{f}, \phi_k)}{\pi^2 k^2 + 1} = \frac{(f, \phi_k) - (1, \phi_k)}{\pi^2 k^2 + 1}.$$

Note that:

$$(1, \phi_k) = \int_0^1 \sqrt{2} \sin(\pi k x)\, dx = \frac{\sqrt{2}}{\pi k}\Big(1 + (-1)^{k-1}\Big).$$

So the solution is

$$u(x) = 1 + v(x) = 1 + \sum_{k=1}^{\infty} \frac{(f, \phi_k) - (1, \phi_k)}{\pi^2 k^2 + 1} \phi_k(x).$$

(e) We need to find (f, ϕ_k) in the formula above:

$$(f, \phi_k) = \sqrt{2} \int_0^1 \big(\sin(\pi x) + \sin(3\pi x)\big) \sin(\pi k x)\, dx = \begin{cases} \sqrt{2}/2, & k \in \{1, 3\} \\ 0, & \text{otherwise} \end{cases}$$