These supplemental notes for the course CAAM 335 Matrix Analysis are, as the name suggests, meant to be used in combination with the lectures and with the CAAM 335 course notes. They contain write-ups of some of the material discussed in class as well as several additional examples. You get the most out of these notes if you look through the relevant sections before each class (to recognize what example is covered in the notes, so that you do not have to spend time in class to copy examples from the board that are already included in the notes) and if you work through the notes after each class. However, these supplemental notes are not meant to teach you the course material alone. Active participation in the lectures and working through the CAAM 335 course notes are expected. Other texts related CAAM 335 are listed on the course web-page and these texts contain additional examples, different derivations, and additional topics.

Finally, these supplemental notes are updated as the semester progresses. If you find typos, please let me know. Check for the ‘generated at date’ to make sure you have the latest version.
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1 Linear Least Squares

Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, we want to find $x \in \mathbb{R}^n$ such that $Ax \approx b$. If $m = n$ and $A$ is invertible, then we can solve $Ax = b$. Otherwise, we may not have a solution of $Ax = b$ or we may have infinitely many of them.

We are interested in vectors $x$ that minimize the norm of squares of the residual $Ax - b$, i.e., we are interested in vectors $x$ which solve

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2. \tag{1}$$

Although the linear least squares problem (1) is a minimization problem, we will show that it can be solved using tools from linear systems. The solution of (1) can be constructed using the Fundamental Theorem of Linear Algebra. By the Fundamental Theorem of Linear Algebra, the right hand side $b$ can be written as $b = b_R + b_N$, where $b_R \in \mathcal{R}(A)$ and $b_N \in \mathcal{N}(A^T)$. See Figure 1. In particular, $b_R^Tb_N = 0$.

![Figure 1: Decomposition $b = b_R + b_N$ of $b$ into $b_R \in \mathcal{R}(A)$ and $b_N \in \mathcal{N}(A^T)$.](image)

Since $b_N \in \mathcal{N}(A^T) \perp \mathcal{R}(A)$ or $Ax - b_R$, we have

$$\|Ax - b\|_2^2 = \|Ax - b_R - b_N\|_2^2 = \|Ax - b_R\|_2^2 - (Ax - b_R)^Tb_N - b_N^T(Ax - b_R) + \|b_N\|_2^2 = 0 + 0 + \|b_N\|_2^2.$$

Note that the problems

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2, \quad \min_{x \in \mathbb{R}^n} \|Ax - b\|_2, \quad \min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2$$

are equivalent in the sense that if $x$ solves one of them it also solves the others.
Since \( \|Ax - b\|_2^2 = \|Ax - b_R\|_2^2 + \|b_N\|_2^2 \) with \( \|b_N\|_2^2 \) independent of \( x \), minimizing \( \|Ax - b\|_2^2 \) is equivalent to minimizing \( \|Ax - b_R\|_2^2 \). Since \( b_R \in \mathcal{R}(A) \) the equation \( Ax = b_R \) has a solution \( x_* \). The solution is unique if \( \mathcal{N}(A) = \{0\} \). Thus, solving the linear least squares problem (1) is equivalent to solving the linear system \( Ax = b_R \). Unfortunately, setting up this linear system requires \( b_R \), which is essentially as complicated to compute as the solution to linear least squares problem (1) itself. The question is, how can we solve \( Ax = b_R \) without computing \( b_R \)?

First, note that
\[
A^Tb = A^T(b_R + b_N) = A^Tb_R + A^Tb_N = A^Tb_R.
\]

Clearly, any \( x \) that solves \( Ax = b_R \) also solves \( A^TAx_* = A^Tb_R = A^Tb \). On the other hand, if \( x_* \) solves \( A^TAx_* = A^Tb = A^Tb_R \), then \( A^T(Ax_* - b_R) = 0 \), i.e., \( Ax_* - b_R \in \mathcal{N}(A^T) \). Also, \( Ax_* - b_R \in \mathcal{R}(A) \). Since, \( \mathcal{N}(A^T) \perp \mathcal{R}(A) \), this implies \( Ax_* - b_R = 0! \)

Thus, we have shown that \( x_* \) solves \( Ax = b_R \) if and only if \( x_* \) solves
\[
A^TAx = A^Tb.
\]

The equation (2) is called the normal equation. The normal equation always has a solution (since \( Ax = b_R \) has a solution), and it solution is unique if \( \mathcal{N}(A^TA) = \mathcal{N}(A) = \{0\} \).

We summarize our findings in the following theorem.

**Theorem 1** Let \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). A vector \( x_* \) solves the linear least squares problem (1) if and only if \( x_* \) solves the normal equation (2).

The normal equation always has a solution, and the solution is unique if \( \mathcal{N}(A^TA) = \mathcal{N}(A) = \{0\} \), i.e, if \( A \) has linearly independent columns.

That a solution \( x_* \) of the normal equation \( A^TAx_* - A^Tb = 0 \), is in fact a solution of the linear least squares problem can be seen from
\[
\|Ax - b\|_2^2 \leq \|Ax_* + A(x - x_*) - b\|_2^2 = \|Ax_* - b\|_2^2 + 2(x - x_*)^T(A^TAx_* - A^Tb) + \|A(x - x_*)\|_2^2 \\
\geq \|Ax_* - b\|_2^2 + \|A(x - x_*)\|_2^2
\]

for all \( x \in \mathbb{R}^n \).

We can also give an argument based on optimization to show that if the linear least squares problem (1), then \( x_* \) solves the normal equation (2). We present this argument next, before discussing some examples of linear least squares problems. If \( x_* \) solves the linear least squares problem (1), then
\[
\|Ax_* - b\|_2^2 \leq \|A(x_* + tv) - b\|_2^2 \quad \text{for all vectors } v \in \mathbb{R}^n \text{ and all scalars } t \in \mathbb{R}.
\]

Since \( \|A(x_* + tv) - b\|_2^2 = \|Ax_* - b\|_2^2 + 2tv^TA^TAx_* - A^Tb + t^2\|Av\|_2^2 \), (3) is equivalent to
\[
2tv^TA^TAx_* - A^Tb + t^2\|Av\|_2^2 \geq 0 \quad \text{for all vectors } v \in \mathbb{R}^n \text{ and all scalars } t \in \mathbb{R}.
\]
1.1 Examples of Linear Least Squares Problems

We set \( v = A^T(Ax^* - b) \) and by (4)

\[
2t\|A^T(Ax^* - b)\|_2^2 + t^2\|AA^T(Ax^* - b)\|_2^2 \geq 0 \quad \text{all scalars } t \in \mathbb{R}.
\]  

If \( A^T(Ax^* - b) \neq 0 \), the left hand side in (5) is a quadratic function in \( t \) and for all \( t < 0 \) sufficiently close to zero, this function is negative\(^2\), which contradicts (5). Therefore, it must hold that \( A^T(Ax^* - b) = 0 \), i.e., that \( x^* \) solves the normal equation.

1.1 Examples of Linear Least Squares Problems

We have stated the linear least squares problem as

\[
\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2,
\]

where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) are given.

In applications, different notations are used and it is important to map the application specific notation into the generic notation \( \min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 \). It’s a good idea to first identify the unknowns (which correspond to the components of \( x \) in the generic notation).

We study a few examples of linear least squares problems.

**Example 2** Given \( m \) measurements

\[
(t_i, b_i), \quad i = 1, \ldots, m.
\]

we want to find a function \( t \mapsto x_1 \varphi_1(t) + x_2 \varphi_2(t) + \ldots + x_n \varphi_n(t) \) that best fits these data, i.e.,

\[
b_i \approx x_1 \varphi_1(t_i) + x_2 \varphi_2(t_i) + \ldots + x_n \varphi_n(t_i), \quad i = 1, \ldots, m.
\]

See Figure 2.

We formulate this as a least squares problem: Find \( x_1, \ldots, x_n \) such that

\[
\sum_{i=1}^{m} (x_1 \varphi_1(t_i) + x_2 \varphi_2(t_i) + \ldots + x_n \varphi_n(t_i) - b_i)^2
\]

is minimized.

If

\[
A = \begin{pmatrix}
\varphi_1(t_1) & \varphi_2(t_1) & \ldots & \varphi_n(t_1) \\
\varphi_1(t_2) & \varphi_2(t_2) & \ldots & \varphi_n(t_2) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1(t_m) & \varphi_2(t_m) & \ldots & \varphi_n(t_m)
\end{pmatrix} \in \mathbb{R}^{m \times n}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m,
\]

\(^2\) \( 2t\|A^T(Ax^* - b)\|_2^2 + t^2\|AA^T(Ax^* - b)\|_2^2 < 0 \) for all \( t \) with \( -2\|A^T(Ax^* - b)\|_2^2 / \|AA^T(Ax^* - b)\|_2^2 < t < 0 \). Note that \( 0 \neq A^T(Ax^* - b) \in \mathcal{R}(A^T) = \mathcal{N}(A)^\perp \) and, thus, \( 0 \neq A^T(Ax^* - b) \) implies \( AA^T(Ax^* - b) \neq 0 \).
1.1 Examples of Linear Least Squares Problems

\[
b(t) = x_1 \varphi_1(t) + x_2 \varphi_2(t) + \ldots + x_n \varphi_n(t)
\]

Figure 2: Fitting a curve to data

then the \(i\)th residual

\[ r_i = x_1 \varphi_1(t_i) + \ldots + x_n \varphi_n(t_i) - b_i \]

is the \(i\)th component of \(Ax - b\) and the vector of residuals has the form

\[ r = Ax - b. \]

Hence curve fitting leads to \(\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2\).

Note that to formulate the curve fitting problem as a linear least squares problem we need that \(x_1 \varphi_1(t) + x_2 \varphi_2(t) + \ldots + x_n \varphi_n(t)\) depends linearly on \(x_1, \ldots, x_n\). The function (of \(t\)), \(t \mapsto x_1 \varphi_1(t) + x_2 \varphi_2(t) + \ldots + x_n \varphi_n(t)\) can be nonlinear. For example, if we want to fit a polynomial \(x_1 + x_2 t + \ldots + x_n t^{n-1}\) to data, we obtain a linear least squares problem \(\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2\) with

\[
A = \begin{pmatrix}
1 & t_1 & \ldots & t_1^{n-1} \\
1 & t_2 & \ldots & t_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & t_m & \ldots & t_m^{n-1}
\end{pmatrix} \in \mathbb{R}^{m \times n}.
\]

Example 3 Consider the truss in Figure 3. The vertical displacement \(x_6\) of node 3 is proportional to the force \(F\). If we know the topology of the truss and the material properties of the bars, we can compute \(x_6\) given \(F\), by solving the equations \(A^T K Ax = f\) for the truss.

If we do not know topology of the truss or the material properties of the bars, we can determine \(x_6\) experimentally.
Since the vertical displacement $x_6$ of node 3 is proportional to the force $F$ we set

$$x_6 = a + bF.$$ 

and determine $a, b$ as the solution of a least squares problem.

Measurements

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<td>$x_6$[m]</td>
<td>0.0003</td>
<td>0.0008</td>
<td>0.0013</td>
<td>0.0018</td>
<td>0.0023</td>
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The least squares problem is

$$\min \left\| \begin{pmatrix} 1 & 98.0665 \\ 1 & 294.1995 \\ 1 & 490.3325 \\ 1 & 686.4655 \\ 1 & 882.5985 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} 0.0003 \\ 0.0008 \\ 0.0013 \\ 0.0018 \\ 0.0023 \end{pmatrix} \right\|_2^2$$

Solution

$$a = 4.6082 \times 10^{-19},$$
$$b = 2.5684 \times 10^{-06}.$$
Example 4 We want to fit a circle through \((x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\). The equation for the circle around \((c_1, c_2)\) with radius \(r\) is

\[
(x - c_1)^2 + (y - c_2)^2 = r^2.
\]

We rewrite the equation for the circle to get

\[
2xc_1 + 2yc_2 + (r^2 - c_1^2 - c_2^2) = x^2 + y^2.
\]

If we set \(c_3 = r^2 - c_1^2 - c_2^2\), then we can compute the center \((c_1, c_2)\) and the radius \(r = \sqrt{c_3 + c_1^2 + c_2^2}\) of the circle that best fits the data points by solving the least squares problem

\[
\min_{c \in \mathbb{R}^3} \left\| \begin{pmatrix}
2x_1 & 2y_1 & 1 \\
2x_2 & 2y_2 & 1 \\
\vdots & \vdots & \vdots \\
2x_m & 2y_m & 1
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix}
- \begin{pmatrix}
x_1^2 + y_1^2 \\
x_2^2 + y_2^2 \\
\vdots \\
x_m^2 + y_m^2
\end{pmatrix}
\right\|^2.
\]

The following MATLAB code illustrates how to solve a particular example. The corresponding result is shown in Figure 4.

```matlab
% Generate points on a circle and perturb them randomly
t = (0:0.1:0.9)';
x = sin(2*pi*t);
y = cos(2*pi*t);
x = x.*(ones(size(x))+0.2*rand(size(x)));
y = y.*(ones(size(x))+0.2*rand(size(x)));

% set up and solve least squares problem
A = [ 2*x 2*y ones(size(x))];
b = [x.^2+y.^2];
c = A\b;

% plot data and fitted circle
plot(x,y,'*'); hold on
t = (0:0.01:1)';
r = sqrt(c(3)+c(1).^2+c(2).^2);
plot(c(1)+r*sin(2*pi*t),c(2)+r*cos(2*pi*t));
hold off
axis equal
```
Figure 4: The circle that best fits the measurements.
2 Projections

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The vector $x$ solves the linear least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2$$

if and only if $x$ solves the normal equation

$$A^T Ax = A^T b.$$  

If $A^T A$ is invertible (i.e., if $\mathcal{N}(A^T A) = \mathcal{N}(A) = \{0\}$), then the unique solution of the least squares problem is given by

$$x_* = (A^T A)^{-1} A^T b.$$  

We have

$$Ax_* = A(A^T A)^{-1} A^T b \in \mathcal{R}(A).$$  

The matrix

$$P = A(A^T A)^{-1} A^T \in \mathbb{R}^{m \times m}$$

satisfies

$$P^2 = \left( A(A^T A)^{-1} A^T \right) \left( A(A^T A)^{-1} A^T \right) = A(A^T A)^{-1} A^T (A^T A)^{-1} A^T = A(A^T A)^{-1} A^T = P$$

and

$$P^T = \left( A(A^T A)^{-1} A^T \right)^T = A((A^T A)^{-1}) A^T = P.$$  

(Recall that for any invertible matrix $B$, $(B^T)^{-1} = (B^{-1})^T$.)

**Definition 5** A square matrix $P$ that satisfies $P^2 = P$ is called a projection.

A projection $P$ that satisfied $P = P^T$ is called an orthogonal projection.

To see why the linear least squares problem gives rise to projections, we provide the following interpretation. The set $\{Ax : x \in \mathbb{R}^n\}$ is the range space $\mathcal{R}(A)$. Hence

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2 = \min_{y \in \mathcal{R}(A)} \|y - b\|_2.$$  

Thus the least squares problem asks us to find a vector $y_* = Ax_*$ in the range space $\mathcal{R}(A)$ that is closest to $b$. If $\mathcal{N}(A^T A) = \mathcal{N}(A) = \{0\}$, this vector is

$$y_* = Ax_* = A(A^T A)^{-1} A^T b,$$  

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Figure 5: The projection $Pb = A(A^T A)^{-1} A^T b$ of $b$ onto $\mathcal{R}(A)$.

i.e., it is the projection $Pb = A(A^T A)^{-1} A^T b$ of $b$ onto $\mathcal{R}(A)$. If we apply $P = A(A^T A)^{-1} A^T$ again, then we find the vector $\tilde{y}_* = A\tilde{x}_*$ in the range space $\mathcal{R}(A)$ that is closest to $Pb = A(A^T A)^{-1} A^T b$. This is the vector $\tilde{y}_* = A(A^T A)^{-1} A^T (A(A^T A)^{-1} A^T b)$. However, since $Pb = A(A^T A)^{-1} A^T b$ is already in $\mathcal{R}(A)$, it is closest to itself, $\tilde{y}_* = A(A^T A)^{-1} A^T b$. This shows that $A(A^T A)^{-1} A^T (A(A^T A)^{-1} A^T b) = A(A^T A)^{-1} A^T b$ for all $b$.

If $P \in \mathbb{R}^{n \times n}$ is a projection, then

$$(I - P)^2 = (I - P)(I - P) = I - 2P + P^2 = I - P,$$

i.e., $I - P \in \mathbb{R}^{n \times n}$ is also a projection. If $P$ is an orthogonal projection, then

$$(I - P)^T = I - P = I - P,$$

i.e., $I - P$ is also an orthogonal projection. Furthermore, if $P \in \mathbb{R}^{m \times m}$ is a projection, then

$$y = Py + (I - P)y \quad \text{for all} \ y \in \mathbb{R}^m,$$

i.e., every $y \in \mathbb{R}^m$ can be written as a sum of a vector in $\mathcal{R}(P)$ and a vector in $\mathcal{R}(I - P)$. Moreover, if $P \in \mathbb{R}^{n \times n}$ is an orthogonal projection, then

$$(Py)^T ((I - P)z) = y^T P^T (I - P)z = y^T P(I - P)z = y^T (P - P)z = 0 \quad \text{for all} \ y, z \in \mathbb{R}^m,$$

i.e., $\mathcal{R}(P) \perp \mathcal{R}(I - P)$. It is because of this orthogonality of the range spaces $\mathcal{R}(P)$ and $\mathcal{R}(I - P)$ that we call projections with $P = P^T$ orthogonal.

Given $A \in \mathbb{R}^{m \times n}$ with $\mathcal{N}(A^T A) = \mathcal{N}(A) = \{0\}$,

$$P = A(A^T A)^{-1} A^T \in \mathbb{R}^{m \times m}$$
is the orthogonal projection onto $\mathcal{R}(A)$, and

$$I - P = I - A(A^T A)^{-1} A^T \in \mathbb{R}^{m \times m}$$

is the orthogonal projection onto $\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$.

Not all projections are orthogonal. For example

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfies $P^2 = P$, i.e., is a projection, but obviously $P^T \neq P$.

**Example 6** We want to compute the orthogonal projection onto the subspace

$$\mathcal{V} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}.$$ 

Note that $\mathcal{V} = \mathcal{R}(A)$, where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 0 \end{pmatrix}.$$ 

Thus we have to compute the projection $P = A(A^T A)^{-1} A^T$ onto $\mathcal{R}(A)$. We compute

$$A^T A = \begin{pmatrix} 5 & 8 \\ 8 & 15 \end{pmatrix}, \quad (A^T A)^{-1} = \begin{pmatrix} 13 & -8 \\ -8 & 5 \end{pmatrix},$$

and

$$P = A(A^T A)^{-1} A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

\[ \diamond \]

**Example 7** We want to compute the orthogonal projection onto the subspace

$$\mathcal{V} = \{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \}.$$ 

Note that $\mathcal{V} = \mathcal{N}(B)$, where

$$B = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \in \mathbb{R}^{1 \times 3}.$$
Thus we have to compute the projection $Q$ onto $\mathcal{N}(B) = \mathcal{R}(B^T)^\perp$. The projection $P$ onto $\mathcal{R}(B^T)$ is $P = B^T(BB^T)^{-1}B$, and the projection $Q$ onto $\mathcal{N}(B) = \mathcal{R}(B^T)^\perp$ is $Q = I - P$. We compute

$$P = B^T(BB^T)^{-1}B = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (1 \ 1 \ 1) = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$Q = I - P = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$
3 Gram-Schmidt Orthogonalization and $QR$–Decomposition

3.1 Gram-Schmidt Orthogonalization

Given linearly independent (the linearly independence assumption will be removed later) vectors $v_1, \ldots, v_k \in \mathbb{R}^n$, the Gram-Schmidt process constructs orthonormal vectors $q_1, \ldots, q_k$, i.e., vectors that satisfy

$$ q_i^T q_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}, $$

that span the same subspace as the vectors $v_1, \ldots, v_k$. More precisely, for $i = 1, \ldots, k$, the Gram-Schmidt process successively computes an orthonormal basis $\{q_1, \ldots, q_i\}$ from $\{v_1, \ldots, v_i\}$ such that both bases span the same subspace. The idea is to use orthogonal projection to remove components along the existing basis vectors, leaving an orthogonal set. The idea is illustrated in Figure 6.

![Figure 6: The projectors $P_{R(q)} = qq^T/(q^T q)$ and $P_{N(q^T)} = I - P_{R(q)}$ decompose the vector $v$ into the component $P_{R(q)} v$ along $q$, and the orthogonal component $P_{N(q^T)} v$.](image)

The steps of the Gram-Schmidt process are described next.

$i = 1$. If $i = 1$, then we need to find a vector $q_1$ with $q_1^T q_1 = 1$ such that

$$ \text{span}\{q_1\} = \text{span}\{v_1\}. $$

The vector $q_1$ is obtained by normalizing $v_1$:

$$ q_1 = v_1/\|v_1\|_2. $$

$i = 2$. Given $q_1$ we want to compute $q_2$ such that $q_1^T q_2 = 0$, $q_2^T q_2 = 1$, and

$$ \text{span}\{q_1, q_2\} = \text{span}\{v_1, v_2\}. $$

First note that since $\text{span}\{q_1\} = \text{span}\{v_1\}$ we have $\text{span}\{v_1, v_2\} = \text{span}\{q_1, v_2\}$. Moreover, by the Fundamental Theorem of Linear Algebra we can write $v_2$ as the sum of vector in
3.1 Gram-Schmidt Orthogonalization

\[ \text{span}\{q_1\} = \mathcal{R}(Q_1), \text{ where } Q_1 \text{ is the matrix } Q_1 = (q_1) \in \mathbb{R}^{n \times 1}, \text{ and a vector in the orthogonal complement of } \mathcal{R}(Q_1). \text{ We can use projections to express these vectors. The projection onto } \mathcal{R}(Q_1) \text{ is given by } P_{\mathcal{R}(Q_1)} = Q_1(Q_1^TQ_1)^{-1}Q_1^T. \text{ Since } q_1^Tq_1 = 1, Q_1^TQ_1 = 1 \text{ and the projection is given by }
\]

\[ P_{\mathcal{R}(Q_1)} = Q_1Q_1^T. \]

Hence,

\[ v_2 = Q_1Q_1^Tv_2 + (I - Q_1Q_1^T)v_2. \]

Since \( Q_1Q_1^Tv_2 \in \mathcal{R}(Q_1) = \text{span}\{q_1\} \) we have

\[ \text{span}\{q_1, v_2\} = \text{span}\{q_1, (I - Q_1Q_1^T)v_2\}. \]

The vector

\[ \tilde{q}_2 = (I - Q_1Q_1^T)v_2 = v_2 - (q_1^Tv_2)q_1 \]

is orthogonal to \( q_1 \). We just need to normalize it to obtain

\[ q_2 = \tilde{q}_2 / \|\tilde{q}_2\|_2. \]

We have

\[ \text{span}\{v_1, v_2\} = \text{span}\{q_1, v_2\} = \text{span}\{q_1, (I - Q_1Q_1^T)v_2\} = \text{span}\{q_1, q_2\}. \]

\[ i > 2. \text{ Given } q_1, \ldots, q_{i-1} \text{ with } q_i^Tq_j = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{otherwise}. \end{cases} \]

and \( \text{span}\{q_1, \ldots, q_{i-1}\} = \text{span}\{v_1, \ldots, v_{i-1}\} \), we want to compute \( q_i \) such that \( q_j^Tq_i = 0, j = 1, \ldots, k-1, q_i^Tq_i = 1 \), and

\[ \text{span}\{q_1, \ldots, q_{i-1}, q_i\} = \text{span}\{v_1, \ldots, v_{i-1}, v_i\}. \]

Since \( \text{span}\{q_1, \ldots, q_{i-1}\} = \text{span}\{v_1, \ldots, v_{i-1}\} \) we have

\[ \text{span}\{v_1, \ldots, v_{i-1}, v_i\} = \text{span}\{q_1, \ldots, q_{i-1}, v_i\}. \]

By the Fundamental Theorem of Linear Algebra we can write \( v_i \) as the sum of vector in \( \text{span}\{q_1, \ldots, q_{i-1}\} = \mathcal{R}(Q_{i-1}), \) where \( Q_{i-1} \) is the matrix \( Q_{i-1} = (q_1, \ldots, q_{i-1}) \in \mathbb{R}^{n \times k-1}, \) and a vector in the orthogonal complement of \( \mathcal{R}(Q_{i-1}). \) We can use projections to express these vectors. The projection onto \( \mathcal{R}(Q_{i-1}) \) is given by \( P_{\mathcal{R}(Q_{i-1})} = Q_{i-1}(Q_{i-1}^TQ_{i-1})^{-1}Q_{i-1}^T. \) Since the columns \( q_1, \ldots, q_{i-1} \) of \( Q_{i-1} \) are orthonormal, \( Q_{i-1}^TQ_{i-1} = I \) and the projection is given by

\[ P_{\mathcal{R}(Q_{i-1})} = Q_{i-1}Q_{i-1}^T. \]
Hence,  
\[ v_i = Q_{i-1}Q_{i-1}^T v_i + (I - Q_{i-1}Q_{i-1}^T) v_i. \]

Since \( Q_1Q_1^T v_2 \in \mathcal{R}(Q_1) = \text{span}\{q_1\} \) we have  
\[ \text{span}\{q_1, \ldots, q_{i-1}, v_2\} = \text{span}\{q_1, \ldots, q_{i-1}, (I - Q_i Q_i^T) v_i\}. \]

The vector  
\[ \tilde{q}_i = (I - Q_i Q_i^T) v_i = v_i - \sum_{j=1}^{i-1} (q_j^T v_i) q_j \]
is orthogonal to \( q_1, \ldots, q_{i-1} \). We just need to normalize it to obtain  
\[ q_i = \tilde{q}_i / \|\tilde{q}_i\|_2. \]

We have  
\[ \text{span}\{v_1, \ldots, v_{i-1}, v_i\} = \text{span}\{q_1, \ldots, q_{i-1}, v_i\} = \text{span}\{q_1, \ldots, q_{i-1}, (I - Q_i Q_i^T) v_i\} = \text{span}\{q_1, \ldots, q_{i-1}, q_i\}. \]

The above steps are summarized in the following algorithm.

---

**Algorithm 8 (Gram-Schmidt)**

0. Given linearly independent vectors \( v_1, \ldots, v_k \in \mathbb{R}^n \).

1. Set \( q_1 = v_1 / \|v_1\|_2. \)

2. For \( i = 1, \ldots, k - 1 \)

   a. Compute \( \tilde{q}_{i+1} = v_{i+1} - \sum_{j=1}^{i} q_j^T v_{i+1} q_j. \)

   b. Set \( q_{i+1} = \tilde{q}_{i+1} / \|\tilde{q}_{i+1}\|_2. \)

3. Return orthonormal vectors \( q_1, \ldots, q_k \in \mathbb{R}^n \) with  
   \[ \text{span}\{v_1, \ldots, v_k\} = \text{span}\{q_1, \ldots, q_k\}. \]
Example 9 We apply the Gram-Schmidt Algorithm 8 to the vectors

\[ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -3 \\ -4 \\ -2 \end{pmatrix}. \]

The Gram-Schmidt Algorithm 8 computes orthonormal vectors \( q_1, q_2, q_3 \) as follows:

\[ q_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \]

\[ \tilde{q}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \quad q_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \]

\[ \tilde{q}_3 = \begin{pmatrix} 1 \\ -3 \\ -4 \\ -2 \end{pmatrix} - \frac{8}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{7}{10} \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 16 \\ -17 \\ -13 \\ 14 \end{pmatrix}, \quad q_3 = \frac{1}{\sqrt{910}} \begin{pmatrix} 16 \\ -17 \\ -13 \\ 14 \end{pmatrix}. \]

\[ \diamond \]

So far we have assumed that the vectors \( v_1, \ldots, v_k \in \mathbb{R}^n \) are linearly independent. The next example explores what happens when they are not.
Example 10 We apply the Gram-Schmidt Algorithm 8 to the vectors

\[ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ -3 \\ -4 \\ -2 \end{pmatrix}. \]

Note that \( v_3 = v_2 - v_1 \), i.e. the vectors are linearly dependent.

The first two steps of the Gram-Schmidt Algorithm 8 give

\[ q_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \tilde{q}_2 = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad q_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}. \]

The next steps leads to

\[ \tilde{q}_3 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} - \frac{8}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad q_3 = \frac{1}{\sqrt{910}} \begin{pmatrix} 16 \\ -17 \\ -13 \\ 14 \end{pmatrix}, \quad q_4 = \frac{1}{\sqrt{910}} \begin{pmatrix} 16 \\ -17 \\ -13 \\ 14 \end{pmatrix}. \]

In the end

\[ \text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{q_1, q_2, q_4\} = \text{span}\{q_j : q_j \neq 0, j \in \{1, \ldots, 4\}\}. \]
The generalization of the Gram-Schmidt Algorithm 8 to (possibly linearly dependent) vectors \( v_1, \ldots, v_i \in \mathbb{R}^n \) is now straightforward and is given in the following algorithm.

**Algorithm 11 (Gram-Schmidt)**

0. Given vectors \( v_1, \ldots, v_i \in \mathbb{R}^n \) with \( v_1 \neq 0 \) (otherwise renumber).

1. Set \( q_1 = v_1 / \| v_1 \|_2 \).

2. For \( i = 1, \ldots, k - 1 \)
   
   a. Compute \( \tilde{q}_{i+1} = v_{i+1} - \sum_{j=1}^{i} q_j^T v_{i+1} q_j \).
   
   b. If \( \tilde{q}_{i+1} \neq 0 \), set \( q_{i+1} = \tilde{q}_{i+1} / \| \tilde{q}_{i+1} \|_2 \).
      If \( \tilde{q}_{i+1} = 0 \), set \( q_{i+1} = 0 \).

3. Return a set of orthonormal vectors \( \{ q_j : q_j \neq 0, j \in \{1, \ldots, k\} \} \) with \( \text{span}\{v_1, \ldots, v_i\} = \text{span}\{q_j : q_j \neq 0, j \in \{1, \ldots, k\}\} \).

### 3.2 \( QR \)-Decomposition via Gram-Schmidt

Assume that \( \tilde{q}_{i+1} \neq 0 \). The identity in step 2a of the Gram-Schmidt Algorithm 11 reads

\[
\| \tilde{q}_{i+1} \|_2 = v_{i+1} - \sum_{j=1}^{i} q_j^T v_{i+1} q_j.
\]

Multiply by \( q^T_{i+1} \) to obtain

\[
\| \tilde{q}_{i+1} \|_2 q^T_{i+1} v_{i+1} = q^T_{i+1} v_{i+1} - \sum_{j=1}^{i} q_j^T v_{i+1} q^T_{i+1} q_j.
\]
Hence, $\|\tilde{q}_{i+1}\|_2 = v_{i+1}^T q_{i+1}$. This identity and (6) give

$$v_{i+1} = \sum_{j=1}^{i+1} q_j^T v_{i+1} q_j = \begin{bmatrix} q_1, \ldots, q_{i+1} \end{bmatrix}_{n \times (i+1)} \begin{pmatrix} q_1^T v_{i+1} \\ \vdots \\ q_{i+1}^T v_{i+1} \end{pmatrix} = \begin{bmatrix} q_1, \ldots, q_{i+1}, q_{i+2}, \ldots, q_k \end{bmatrix}_{n \times k} \begin{pmatrix} q_1^T v_{i+1} \\ \vdots \\ q_{i+1}^T v_{i+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$  \hspace{1cm} (7)

Let $\tilde{q}_{i+1} \neq 0$ for all $i = 0, \ldots, k-1$. Equation (7) is the identity for the $(i + 1)$st column in

$$\begin{bmatrix} v_1, \ldots, v_{i+1}, v_{i+2}, \ldots, v_k \end{bmatrix}_{n \times k} = V \in \mathbb{R}^{n \times k} = Q \in \mathbb{R}^{n \times k} = R \in \mathbb{R}^{k \times k}$$

A decomposition

$$V = QR$$  \hspace{1cm} (9)

where

$$Q \in \mathbb{R}^{n \times k} \quad \text{with} \quad Q^T Q = I$$

and

$$R \in \mathbb{R}^{k \times k} \quad \text{is upper triangular}$$

is called a $QR$-decomposition (of $V$).

**Example 12** Example 9 leads to the $QR$ decomposition

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 4 & -4 \\ 1 & 5 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \sqrt{\frac{2}{10}} & \sqrt{\frac{16}{91}} \\ 1 & 1 & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{91}} \\ \frac{1}{2} & 1 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{14}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \sqrt{\frac{1}{91}} \end{pmatrix} \begin{pmatrix} 2 & 6 & -4 \\ \sqrt{10} & 0 & \sqrt{910} \\ -7 & 0 & \sqrt{910} \\ 0 & 0 & \sqrt{910} \end{pmatrix}. \quad \diamond$$
So far we have assumed that $\tilde{q}_{i+1} \neq 0$ for all $i = 0, \ldots, k - 1$. If $i$ is the first index such that $\tilde{q}_{i+1} = 0$, then the identity in step 2a of the Gram-Schmidt Algorithm 11 reads

$$0 = v_{i+1} - \sum_{j=1}^{i} q_j^T v_{i+1} q_j,$$

i.e.,

$$v_{i+1} = \sum_{j=1}^{i} q_j^T v_{i+1} q_j = \begin{bmatrix} q_1, \ldots, q_i \end{bmatrix}_{n \times i} \begin{pmatrix} q_1^T v_{i+1} \\ \vdots \\ q_i^T v_{i+1} \end{pmatrix}.$$

If $\tilde{q}_{i+1} = 0$ and all other $\tilde{q}_{j+1} \neq 0$ for all $i = 0, \ldots, k - 1$, $j \neq i$, then

$$v_{i+1} = \begin{bmatrix} q_1, \ldots, q_i \end{bmatrix}_{n \times i} \begin{pmatrix} q_1^T v_{i+1} \\ \vdots \\ q_i^T v_{i+1} \end{pmatrix} = \begin{bmatrix} q_1, \ldots, q_i, q_{i+2}, \ldots, q_k \end{bmatrix}_{n \times (k-1)} \begin{pmatrix} q_1^T v_{i+1} \\ \vdots \\ q_i^T v_{i+1} \end{pmatrix} = \begin{pmatrix} q_1^T v_{i+1} \\ \vdots \\ q_i^T v_{i+1} \end{pmatrix}.$$

We can now use (7) and (11) to decompose $V \in \mathbb{R}^{n \times k}$ as

$$V = QR$$

where now

$$Q \in \mathbb{R}^{n \times (k-1)}$$

with $Q^T Q = I$

and

$$R \in \mathbb{R}^{(k-1) \times k}$$

is upper triangular.

More generally, if the Gram-Schmidt Algorithm 11 generates $l \leq k$ nonzero vectors $q_i$, then $V \in \mathbb{R}^{n \times k}$ can be decomposed as

$$V = QR$$

where now

$$Q \in \mathbb{R}^{n \times l}$$

with $Q^T Q = I$

and

$$R \in \mathbb{R}^{l \times k}$$

is upper triangular.

**Example 13** Example 10 leads to the $QR$ decomposition

$$
\begin{pmatrix}
1 & 1 & 0 & 1 \\
1 & 2 & 1 & -3 \\
1 & 4 & 3 & -4 \\
1 & 5 & 4 & -2
\end{pmatrix}
= 
\begin{pmatrix}
1/2 & -1/\sqrt{10} & 1/\sqrt{910} & -7/\sqrt{910} \\
1/2 & -\sqrt{10}/17 & 1/17 & 0 \\
1/2 & 10/\sqrt{910} & 1/\sqrt{910} & 0 \\
1/2 & -\sqrt{10}/14 & 1/14 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 6 & 4 & -4 \\
6 \sqrt{10} & 10 \sqrt{10} & 7 \sqrt{910} \\
0 & 0 \sqrt{910} & 0 \\
0 & 0 & 0 & \sqrt{910}
\end{pmatrix}.
$$
3.3 Using the $QR$–Decomposition to Solve Linear Least Squares Problems

The crucial property for using the $QR$–decomposition to solve linear least squares problems, is the fact that an orthogonal matrix does not change the length (=2-norm) of a vector. Let

$$Q \in \mathbb{R}^{m \times n} \quad \text{with} \quad Q^T Q = I.$$ 

If $z \in \mathbb{R}^n$ then

$$\|Qz\|^2_2 = (Qz)^T Qz = z^T Q^T Qz = z^T z = \|z\|^2_2,$$

i.e., $z$ and $Qz$ have the same norm (length). Note that $z \in \mathbb{R}^n$ and $Qz \in \mathbb{R}^m$ have different number of entries.

Now, let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and consider the least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2_2. \quad (12)$$

We assume that $m \geq n$ (more measurements than unknowns) and we assume that rank$(A) = n$. We can use the QR decomposition for other cases as well.

We compute the $QR$-decomposition of $A$ (apply the Gram-Schmidt Algorithm 11 to the columns of $A$) to get

$$A = QR \quad \quad (13)$$

where

$$Q \in \mathbb{R}^{m \times n} \quad \text{with} \quad Q^T Q = I_{n \times n}$$

and $\mathcal{R}(A) = \mathcal{R}(Q)$, and where

$$R \in \mathbb{R}^{n \times n} \quad \text{is upper triangular.}$$

We will see that the $QR$-decomposition of $A$, (13), allows us to transform the least squares problem (12) into a simpler one that can be solved directly.

First we write

$$b = QQ^T b + (I - QQ^T)b. \quad (14)$$

Since $Q$ has orthonormal columns, $QQ^T b$ is the projection of $b$ onto $\mathcal{R}(Q) = \mathcal{R}(A)$ and $(I - QQ^T)b$ is the projection of $b$ onto $\mathcal{R}(Q)^\perp = \mathcal{R}(A)^\perp$. Now we insert (13), (14) into (12).

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2_2 = \min_{x \in \mathbb{R}^n} \|Q(Rx - Q^T b) - (I - QQ^T)b\|^2_2.$$
3.3 Using the QR–Decomposition to Solve Linear Least Squares Problems

Since $Q(Rx - Q^Tb) \in \mathcal{R}(Q)$ and $I - QQ^T)b$ are orthogonal,

\[
\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 = \min_{x \in \mathbb{R}^n} \|Q(Rx - Q^Tb) - (I - QQ^T)b\|_2^2 \\
= \min_{x \in \mathbb{R}^n} \|Q(Rx - Q^Tb)\|_2^2 + \|(I - QQ^T)b\|_2^2 \\
= \min_{x \in \mathbb{R}^n} \|Rx - Q^Tb\|_2^2 + \|(I - QQ^T)b\|_2^2, \tag{15}
\]

where in the last equation we have used the property $\|Qz\|_2^2 = \|z\|_2^2$ for all $z \in \mathbb{R}^n$. Since $R \in \mathbb{R}^{n \times n}$ is upper triangular,

\[
Rx = Q^Tb \tag{16}
\]

can be solved by back substitution. If $x_*$ is the solution of (15), then $x_*$ solves the least squares problem (15) and

\[
\|Ax_* - b\|_2^2 = \min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 = \|(I - QQ^T)b\|_2^2.
\]

**Example 14** Consider the linear least squares problem

\[
\min \|Ax - b\|_2
\]

with

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & -3 \\
1 & 4 & -4 \\
1 & 5 & -2
\end{pmatrix}, \quad b = \begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix}.
\]

See Example 12. The $QR$ decomposition of $A$ computed using Gram-Schmidt is (only first 4 digits shown)

\[
Q = \begin{pmatrix}
0.5000 & -0.6325 & 0.5304 \\
0.5000 & -0.3162 & -0.5635 \\
0.5000 & 0.3162 & -0.4309 \\
0.5000 & 0.6325 & 0.4641
\end{pmatrix}, \quad R = \begin{pmatrix}
2.0000 & 6.0000 & -4.0000 \\
0 & 3.1623 & -2.2136 \\
0 & 0 & 3.0166
\end{pmatrix}.
\]

We compute $Q^Tb$

\[
Q^Tb = \begin{pmatrix}
5.0000 \\
2.2136 \\
-0.0331
\end{pmatrix}.
\]

Solving the triangular linear system $Rx = Q^Tb$ yields the solution

\[
x = \begin{pmatrix}
0.4011 \\
0.6923 \\
-0.0110
\end{pmatrix}
\]

of the linear least squares problem.

\[ \diamond \]
3.4  QR–Decomposition in MATLAB

MATLAB ’s QR–Decomposition

This section is about the QR decomposition computed by MATLAB and many other software packages. The QR decomposition of a matrix $V$ is not unique and a $QR$-decomposition can be computed with other methods than Gram-Schmidt. In fact, because only a small subset of the real numbers can be represented in a computer and therefore rounding takes place when computing, the Gram-Schmidt method can suffer severely from this rounding. Instead MATLAB uses so-called Householder transformations to compute the $QR$-decomposition. You will learn about those in a course on Numerical Analysis, such as CAAM 453. For a matrix $V \in \mathbb{R}^{n \times k}$ MATLAB computes a $QR$-decomposition

$$V = QR$$

where now

$$Q \in \mathbb{R}^{n \times n} \quad \text{with} \quad Q^T Q = I$$

(note $Q$ is square) and

$$R \in \mathbb{R}^{n \times k} \quad \text{is upper triangular.}$$

**Example 15** We apply MATLAB ’s qr to the matrix in Example 12.

```matlab
>> V = [1 1 1; 1 2 -3; 1 4 -4; 1 5 -2];
>> [Q,R] = qr(V)
```

$Q =$

-0.5000  -0.6325  0.5304  -0.2621
-0.5000  -0.3162 -0.5635  0.5766
-0.5000  0.3162 -0.4309 -0.6814
-0.5000  0.6325  0.4641  0.3669

$R =$

-2.0000  -6.0000  4.0000
 0  3.1623 -2.2136
 0  0  3.0166
 0  0  0

Note that the first three columns in MATLAB ’s $Q$ are $\lq q_1, q_2, q_3 \rq$ in Example 12, and the upper $3 \times 3$ submatrix in MATLAB ’s $R$ is related to $R$ in Example 12.

Next, we apply MATLAB qr to the matrix in Example 13.

```matlab
>> V = [1 1 0 1; 1 2 1 -3; 1 4 3 -4; 1 5 4 -2];
>> [Q,R] = qr(V)
```

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3.4 \textit{QR–Decomposition in MATLAB}

\[
\begin{align*}
Q &= \begin{bmatrix}
-0.5000 & -0.6325 & 0.3990 & 0.4368 \\
-0.5000 & -0.3162 & -0.7103 & -0.3814 \\
-0.5000 & 0.3162 & 0.5350 & -0.6031 \\
-0.5000 & 0.6325 & -0.2237 & 0.5477
\end{bmatrix}
\end{align*}
\]

\[
R = \begin{bmatrix}
-2.0000 & -6.0000 & -4.0000 & 4.0000 \\
0 & 3.1623 & 3.1623 & -2.2136 \\
0 & 0 & -0.0000 & 0.8371 \\
0 & 0 & 0 & 2.8982
\end{bmatrix}
\]

These matrices (their last two columns) are different than the matrices \(Q\) and \(R\). Example 13.

In general the Matrices \(Q\) and \(R\) in MATLAB’s \(QR\)-decomposition are different from the one we compute by hand using the Gram-Schmidt method. However, the fundamental structure, and how we use either \(QR\)-decomposition is the same. We will discuss how to use the \(QR\)-decomposition to solve linear least squares problems.

\textbf{Using MATLAB’s \(QR\–Decomposition to Solve Linear Least Squares Problems}

Let \(A \in \mathbb{R}^{m \times n}\) and \(b \in \mathbb{R}^m\). Let \(Q \in \mathbb{R}^{m \times m}\) be an orthogonal matrix and let \(R \in \mathbb{R}^{n \times n}\) be an upper triangular matrix \(R \in \mathbb{R}^{n \times n}\) such that

\[
A = Q \begin{pmatrix} R \\ 0 \end{pmatrix} \begin{pmatrix} \{n \\ m-n \end{pmatrix}
\]

Note that since \(Q \in \mathbb{R}^{m \times m}\) and \(Q^TQ = I_{m \times m}\), \(Q^{-1} = Q^T\). Note that \(Q \in \mathbb{R}^{m \times m}\), not \(Q \in \mathbb{R}^{m \times n}\). Since \(Q^TQ = I\) we have that

\[
\|Q^Ty\|_2 = \|y\|_2 \quad \forall y \in \mathbb{R}^m.
\]

With this observation and the decomposition (17) we obtain that

\[
\|Ax - b\|_2^2 = \|Q^T(Ax - b)\|_2^2 = \| \begin{pmatrix} R \\ 0 \end{pmatrix} \begin{pmatrix} x \\ -Q^Tb \end{pmatrix} \|_2^2.
\]

Now we partition \(Q^Tb\) as

\[
Q^Tb = \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} \{n \\ m-n \end{pmatrix}
\]

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This yields
\[ \| Ax - b \|_2^2 = \left\| \begin{pmatrix} R \\ 0 \end{pmatrix} y - \begin{pmatrix} c \\ d \end{pmatrix} \right\|_2^2 = \left\| \begin{pmatrix} Rx - c \\ -d \end{pmatrix} \right\|_2^2 = \| Rx - c \|_2^2 + \| d \|_2^2. \]

Using this transformation, the least squares problem can be rewritten as
\[ \min_x \| Ax - b \|_2^2 = \min_x \| Rx - c \|_2^2 + \| d \|_2^2. \]

Now we observe that the term \( \| d \|_2^2 \) does not depend on \( y \) and that \( \| Rx - c \|_2^2 \geq 0 \) for all \( x \) and \( \| Rx_* - c \|_2^2 = 0 \) if and only if \( x_* \) solves
\[ Rx = c. \tag{19} \]

Thus,
\[ \| Ax - b \|_2^2 \geq \| d \|_2^2 \]
for all \( x \). Moreover, if \( x_* \) is the solution of (19), then
\[ \| Ax_* - b \|_2^2 = \| Ry_* - c \|_2^2 + \| d \|_2^2 = \| d \|_2^2. \]

Hence,
\[ \| Ax_* - b \|_2^2 = \| d \|_2^2 = \min_x \| Ax - b \|_2^2, \]
i.e. \( x_* \) is the solution of the linear least squares problem. Note that \( R \) is a nonsingular upper triangular matrix. Therefore (19) can be easily solved (backward solve).

---

**Algorithm 16** Compute the Solution of a Linear Least Squares Problem using the MATLAB QR–Decomposition

Let \( A \in \mathbb{R}^{m \times n} \) be a matrix with \( \text{rank}(A) = n \) and \( b \in \mathbb{R}^m \).

1. Compute an orthogonal matrix \( Q \in \mathbb{R}^{m \times m} \) and an upper triangular matrix \( R \in \mathbb{R}^{n \times n} \) such that
   \[ Q^T A = \begin{pmatrix} R \\ 0 \end{pmatrix}. \]

2. Compute
   \[ Q^T b = \begin{pmatrix} c \\ d \end{pmatrix}. \]

3. Solve \( Rx = c. \)
Remark 17 If you use the \texttt{MATLAB} backslash command $A \backslash b$ to compute the solution of the linear least squares problem $\min \frac{1}{2} \|Ax - b\|_2^2$, where $A \in \mathbb{R}^{m \times n}$, $m > n$, has $\text{rank}(A) = n$, then \texttt{MATLAB} essentially uses Algorithm 16.

Example 18 Consider the linear least squares problem

$$\min \|Ax - b\|_2$$

with

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 4 & -4 \\ 1 & 5 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$ 

See Example 14. Using \texttt{MATLAB}'s \texttt{qr} gives (only first 4 digits shown)

$$Q = \begin{pmatrix} -0.5000 & -0.6325 & 0.5304 & -0.2621 \\ -0.5000 & -0.3162 & -0.5635 & 0.5766 \\ -0.5000 & 0.3162 & -0.4309 & -0.6814 \\ -0.5000 & 0.6325 & 0.4641 & 0.3669 \end{pmatrix}, \quad R = \begin{pmatrix} -2.0000 & -6.0000 & 4.0000 \\ 0 & 3.1623 & -2.2136 \\ 0 & 0 & 3.0166 \end{pmatrix}.$$ 

We compute $Q^T b$ and determine $d, c$:

$$Q^T b = \begin{pmatrix} -5.0000 \\ 2.2136 \\ -0.0331 \\ 0.3145 \end{pmatrix}, \quad c = \begin{pmatrix} -5.0000 \\ 2.2136 \\ -0.0331 \end{pmatrix}, \quad d = \begin{pmatrix} 0.3145 \end{pmatrix}.$$ 

Solving the triangular linear system $Rx = d$ yields the solution

$$x = \begin{pmatrix} 0.4011 \\ 0.6923 \\ -0.0110 \end{pmatrix}$$

of the linear least squares problem. The corresponding \texttt{MATLAB} code is given below.

\begin{verbatim}
>> A = [1 1 1; 1 2 -3; 1 4 -4; 1 5 -2];
>> b = [1; 2; 3; 4];
>> [Q,R] = qr(A)

Q =
    -0.5000   -0.6325    0.5304   -0.2621
    -0.5000  -0.3162   -0.5635    0.5766
    -0.5000    0.3162   -0.4309   -0.6814
    -0.5000    0.6325    0.4641    0.3669

R =
    -2.0000   -6.0000    4.0000
       0     3.1623  -2.2136
       0       0     3.0166
       0       0       0
\end{verbatim}
3.4  QR–Decomposition in MATLAB

\[
\begin{bmatrix}
-0.5000 & 0.3162 & -0.4309 & -0.6814 \\
-0.5000 & 0.6325 & 0.4641 & 0.3669
\end{bmatrix}
\]

\[
R = \\
\begin{bmatrix}
-2.0000 & -6.0000 & 4.0000 \\
0 & 3.1623 & -2.2136 \\
0 & 0 & 3.0166 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
>> \text{cd} = Q' \ast b
\]

\[
\text{cd} = \\
\begin{bmatrix}
-5.0000 \\
2.2136 \\
-0.0331 \\
0.3145
\end{bmatrix}
\]

\[
>> \text{x} = R(1:3,1:3) \backslash \text{cd}(1:3)
\]

\[
\text{x} = \\
\begin{bmatrix}
0.4011 \\
0.6923 \\
-0.0110
\end{bmatrix}
\]
4 Complex Numbers

A complex number is given by \( z = x + iy \), where \( x \) and \( y \) are real numbers and where \( i \) is the imaginary unit, which satisfies \( i^2 = -1 \). Given \( z = x + iy \), \( x = \text{Re}(z) \) is the real part of \( z \) and \( y = \text{Im}(z) \) is the imaginary part of \( z \). The set of all complex numbers is denoted by \( \mathbb{C} \).

![Figure 7: Illustration of complex numbers](image)

The complex conjugate of a complex number \( z = x + iy \) is given by \( \overline{z} = x - iy \). The absolute value (or modulus or magnitude) of a complex number \( z = x + iy \) is

\[
|z| = \sqrt{x^2 + y^2}.
\]

The addition and multiplication of complex numbers \( z = a + ib \) and \( w = c + id \) is defined as follows.

\[
(a + ib) + (c + id) = a + c + i(b + d),
\]
\[
(a + ib)(c + id) = ac - bd + i(ad + bc).
\]

Note that

\[
|z|^2 = \overline{z}z = z\overline{z} = (a + ib)(a - ib) = a^2 + b^2.
\]

The division of complex numbers \( z = a + ib \) and \( w = c + id \) is defined as follows.

\[
\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd + i(-ad + bc)}{c^2 + d^2}.
\]
Polar Form

Every complex number $z = x + iy$ can be written as

$$z = r(\cos \theta + i \sin \theta),$$

(20)

where $r = |z|$ is the length of $z$ and $\theta$ is such that $x = r \cos \theta$ and $y = r \sin \theta$. To make $\theta$ unique we require also that $\theta \in (-\pi, \pi].$\footnote{Alternatively, we could have required that $\theta \in [0, 2\pi)$.} The representation (20) of a complex number is called the polar form.

To compute the polar form of $z = x + iy$, we first compute

$$r = |z| = \sqrt{x^2 + y^2}$$

The computation of the angle $\theta$ requires a bit more thought. The function $\tan(\theta) = \sin(\theta)/\cos(\theta)$ is not invertible. If we restrict $\tan$ to $(-\pi/2, \pi/2)$, then for every $y \in \mathbb{R}$ there exists a unique $\theta \in (-\pi/2, \pi/2)$ with $\tan(\theta) = y$. See Figure 9. The function $\arctan(y)$ returns this $\theta$.\footnote{Alternatively, we could have required that $\theta \in [0, 2\pi)$.}
Figure 9: The functions $\sin(\theta)$ and $\cos(\theta)$ (left plot), the function $\tan(\theta) = \sin(\theta)/\cos(\theta)$ (middle plot), and $\arctan(y)$ (right plot). The function $\arctan(y)$ is computed as the inverse of the branch of $\tan(\theta)$ for $\theta \in (-\pi/2, \pi/2)$ (red curve in middle plot).

Since we require $\theta \in (-\pi, \pi]$, the angle in the polar form is given by

$$\theta = \begin{cases} 
\pi/2 & \text{if } x = 0, y > 0 \\
-\pi/2 & \text{if } x = 0, y < 0, \\
\arctan(y/x) & \text{if } x > 0, \\
\arctan(y/x) + \pi & \text{if } x < 0, y \geq 0 \\
\arctan(y/x) - \pi & \text{if } x < 0, y < 0.
\end{cases}$$

Example 19 Consider the complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = -1 + \sqrt{3}i$. See Figure 10.

To compute the polar form of $z_1 = \sqrt{3} - i$, compute $r = 2$ and $\arctan(-1/\sqrt{3}) = -\pi/6$. Since $x = \sqrt{3} > 0$, $z_1 = 2(\cos(-\pi/6) + i \sin(-\pi/6))$.

To compute the polar form of $z_2 = -1 + \sqrt{3}i$, compute $r = 2$ and $\arctan(\sqrt{3}/(-1)) = -\pi/3$. Since $x = -1 < 0$ and $y = \sqrt{3} > 0$, $z_2 = 2(\cos(2\pi/3) + i \sin(2\pi/3))$.

If we require $\theta \in [0, 2\pi]$, then the angle in the polar form is given by

$$\theta = \begin{cases} 
\pi/2 & \text{if } x = 0, y > 0 \\
3\pi/2 & \text{if } x = 0, y < 0, \\
\arctan(y/x) & \text{if } x > 0, y \geq 0 \\
\arctan(y/x) + \pi & \text{if } x < 0, y \geq 0 \\
\arctan(y/x) + 2\pi & \text{if } x > 0, y < 0 \\
\arctan(y/x) + \pi & \text{if } x < 0, y < 0.
\end{cases}$$
Complex Vectors and Matrices

Given the vector

$$z = (\zeta_1, \ldots, \zeta_n)^T \in \mathbb{C}^n.$$ 

its conjugate transpose is denoted by $z^*$ and is the vector

$$z^* = (\overline{\zeta}_1, \ldots, \overline{\zeta}_n)$$

The 2-norm of the vector $z$ is given by

$$\|z\|_2 = \sqrt{z^* z} = \sqrt{\sum_{j=1}^{n} |\zeta_j|^2}.$$

Given the complex $m \times n$ matrix

$$Z = (\zeta_{ij}) \in \mathbb{C}^{m \times n}$$

its conjugate transpose is denoted by $Z^*$ and is given by

$$Z^* = Z^T = (\overline{\zeta}_{ji}) \in \mathbb{C}^{n \times m}.$$
5 Dynamical Systems

Given a square matrix $A \in \mathbb{R}^{n \times n}$ and a vector valued function $f : [0, \infty) \to \mathbb{R}^n$, we are interested in the solution of the dynamical system

$$
x'(t) = Ax(t) + f(t), \quad t > 0, \quad (21a)
$$
$$
x(0) = x_0. \quad (21b)
$$

We will later study dynamical systems in the context of circuits and trusses, but for now consider the generic dynamical system (21).

If $n = 1$, we obtain the scalar differential equation

$$
\xi'(t) = \lambda \xi(t) + \gamma(t), \quad t > 0, \quad (22a)
$$
$$
\xi(0) = \xi_0. \quad (22b)
$$

The solution of (22) is given by

$$
\xi(t) = e^{\lambda t} \xi_0 + \int_0^t e^{\lambda(t-\tau)} \gamma(\tau) d\tau. \quad (23)
$$

(Insert (23) into (22) to verify that this is true.) A similar expression holds for the solution of the dynamical system (21) (see (27) below) and we will derive in the following.

The Matrix Exponential

Recall that the scalar exponential function is given by

$$
e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}.
$$

We can use this representation to define the matrix exponential of a square matrix $A \in \mathbb{R}^{n \times n}$ as follows.

$$
\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \quad (24)
$$

It can be shown that the series on the right hand side of (24) converges for any square matrix $A$. Note that the matrix exponential is a matrix, $\exp(A) \in \mathbb{R}^{n \times n}$. In MATLAB the matrix exponential can be computed using $\expm$. The MATLAB command $\exp(A)$, however, returns a matrix in which the scalar exponential is applied to each entry of $A$. This is not the matrix exponential of $A$.
$\begin{equation*}
\begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\end{equation*}$

\begin{equation*}
\begin{bmatrix}
2.7183 & 0 \\
0 & 7.3891
\end{bmatrix}
\end{equation*}

\begin{equation*}
\begin{bmatrix}
2.7183 & 1.0000 \\
1.0000 & 7.3891
\end{bmatrix}
\end{equation*}

If

\begin{equation*}
A = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\end{equation*}

is a diagonal matrix, then

\begin{equation*}
A^k = \begin{pmatrix}
\lambda_1^k & 0 \\
0 & \lambda_2^k
\end{pmatrix}
\end{equation*}

and

\begin{equation*}
\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix}
\lambda_1^k & 0 \\
0 & \lambda_2^k
\end{pmatrix} = \begin{pmatrix}
\sum_{k=0}^{\infty} \frac{1}{k!} \lambda_1^k & 0 \\
0 & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_2^k
\end{pmatrix} = \begin{pmatrix}
e^{\lambda_1} & 0 \\
0 & e^{\lambda_2}
\end{pmatrix}.
\end{equation*}

The matrix exponential of a diagonal matrix is a diagonal matrix. If one computes $\exp(A)$, MATLAB just computes the (scalar) exponential of each entry, which explains the ones in the off-diagonals in the result of $\exp(A)$ above.

The simple example also shows that the matrix exponential of a diagonal matrix

\begin{equation*}
D = \text{diag}(d_1, \ldots, d_n)
\end{equation*}

can be easily computed and is given by

\begin{equation*}
\exp(D) = \text{diag}(e^{d_1}, \ldots, e^{d_n}).
\end{equation*}

In the next sections we will discuss the diagonalization of matrices, which will allow us (among many other things) to easily compute the matrix exponential.

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The Matrix Exponential and the Solution of the Dynamical System (21)

If we define the matrix valued function

\[ t \mapsto E(t) \stackrel{\text{def}}{=} \exp(At) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k, \]

then one can show that

\[ E'(t) = A E(t) = E(t) A \quad \text{(25a)} \]

and

\[ E(0) = I. \quad \text{(25b)} \]

In fact,

\[
\frac{d}{dt} E(t) = \sum_{k=0}^{\infty} \frac{d}{dt} \frac{1}{k!} A^k t^k = \sum_{k=1}^{\infty} \frac{1}{k!} A^k k t^{k-1}
\]

\[ = \sum_{k=1}^{\infty} A \frac{1}{(k-1)!} A^{k-1} t^{k-1} = A \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = A E(t) \]

(note that in the first equality above the order of summation and differentiation can be switched because the series converges uniformly) and

\[
\frac{d}{dt} E(t) = \sum_{k=0}^{\infty} \frac{d}{dt} \frac{1}{k!} A^k t^k = \sum_{k=1}^{\infty} \frac{1}{k!} A^k k t^{k-1}
\]

\[ = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} A^{k-1} t^{k-1} A = \left( \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k \right) A = E(t) A. \]

The properties (25) imply that

\[ x(t) = E(t)x_0 = \exp(At)x_0 \quad \text{(26)} \]

solves the dynamical system (21) with \( f = 0 \). Moreover, the solution of the dynamical system (21) is given by

\[ x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau)) f(\tau) d\tau. \quad \text{(27)} \]

We will return to the solution of dynamical systems and the matrix exponential in the following sections, where we use the so-called diagonalization of the matrix \( A \) to gain more insight into the structure of the solution dynamical systems.
Overview of Eigenvectors and Eigenvalues

Let \( A \) be a square \( n \times n \) matrix. We are interested in non-zero vectors \( v \) such that \( Av \) is a multiple of \( v \), that is
\[
Av = v\lambda, \tag{28}
\]
for some scalar \( \lambda \). A scalar \( \lambda \) and non-zero vector \( v \) that satisfy \( (28) \) are called an \textit{eigenvalue (of A)} and \textit{eigenvector (of A)}, respectively.

Why are we interested in eigenvalues and eigenvectors? If \( \lambda \) and \( v \) are an eigenvalue and corresponding eigenvector of \( A \), then
\[
A^2 v = A(Av) = A(v\lambda) = v\lambda^2.
\]
More generally, for any positive integer \( k \),
\[
A^k v = v\lambda^k.
\]
This implies, for example,
\[
\exp(At)v = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k v = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k t^k v = e^{\lambda t} v.
\]
Thus, if the initial data \( x_0 = v \) in the dynamical system \( (21) \) with \( f \equiv 0 \) is an eigenvector of \( A \) with eigenvalue \( \lambda \), then the solution is
\[
x(t) = \exp(At)v = e^{\lambda t} v.
\]
We can read off the behavior of the solution immediately. For example, if the eigenvalue is a negative real number, then \( x(t) = e^{\lambda t} v \to 0 \) as \( t \to \infty \). We will return to dynamical systems later and also study other examples where the knowledge of eigenvalues and eigenvectors of \( A \) allow us to study properties of the solution of linear systems, dynamical systems, difference equations, and quadratic optimization problems. First we need to learn more about eigenvalues and eigenvectors of a square matrix \( A \).

Hopefully you are convinced that eigenvalues and eigenvectors of square matrices \( A \) are useful. However, does any square matrix have eigenvalues and eigenvectors? To answer this question, we recall that there exists a non-zero vector \( v \) such that \( (28) \) holds if and only if \( \lambda I - A \) has a non-trivial nullspace, \( \mathcal{N}(\lambda I - A) \neq \{0\} \). In other words, there exists a non-zero vector \( v \) such that \( (28) \) holds if and only if \( \lambda I - A \) is not invertible (\( \lambda I - A \) is singular).\(^5\) We consider three examples that show what can happen.

\(^5\)Note that \( \lambda I - A \) is not invertible if and only if \( -(\lambda I - A) = A - \lambda I \) is not invertible. We consider the matrix \( \lambda I - A \), but could have also considered \( A - \lambda I \), which is what is used in other books.
Example 20  Let

\[ A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}. \]

Applying Gaussian elimination to \( \lambda I - A \) gives

\[ \lambda I - A = \begin{pmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 4 \end{pmatrix} \rightarrow \begin{pmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 4 - \frac{4}{\lambda - 1} \end{pmatrix}. \]

Note that we subtracted \(-2/(\lambda - 1)\) times the first row from the second row, to arrive at the row reduced form. This operation is only possible if \( \lambda \neq 1 \). Therefore, to study the invertibility of \( \lambda I - A \) we cannot look at the diagonal entries of its row reduced form individually, but look at the product of the diagonal entries.

The matrix \( \lambda I - A \) is singular if and only if

\[ (\lambda - 1) \left( \lambda - 4 - \frac{4}{\lambda - 1} \right) = (\lambda - 1)(\lambda - 4) - 4 = \lambda^2 - 5\lambda = 0. \]

The product of the diagonal entries of its row-reduced form of \( \lambda I - A \) is called the determinant of \( \lambda I - A \) and we denote the determinant by \( \det(\lambda I - A) \). In this example,

\[ \det(\lambda I - A) = \lambda^2 - 5\lambda. \]

We have shown that \( \lambda \) is an eigenvalue of \( A \) if and only if \( \det(\lambda I - A) = 0 \). Note that for an \( n \times n \) matrix \( A \), the determinant \( \det(\lambda I - A) \) is a polynomial of degree \( n \) in \( \lambda \). This polynomial is called the characteristic polynomial of \( A \) and often denoted by \( p_A(\lambda) = \det(\lambda I - A) \). In our example \( \det(\lambda I - A) = \lambda^2 - 5\lambda = 0 \) if and only if \( \lambda = 0 \) or \( \lambda = 5 \). Thus \( \lambda = 0 \) and \( \lambda = 5 \) are eigenvalues of \( A \).

The MATLAB function `poly` can be used to compute the coefficients of the characteristic polynomial of \( A \) and `roots` can be used to compute the roots of the characteristic polynomial.

>> A = [ 1 2; 2 4 ]

A =

1 2
2 4

>> p = poly(A)

\[ \text{Our definition of the determinant may differ from the definitions of the determinant you will find in books by a factor of } -1. \text{ Since we are interested in scalars } \lambda \text{ for which } \det(\lambda I - A) = 0, \text{ this factor is not important in our context.} \]

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\[ p = \begin{pmatrix} 1 & -5 & 0 \end{pmatrix} \]

>> roots(p)

ans =

0

5

>>

The eigenvectors corresponding to \( \lambda = 0 \) and \( \lambda = 5 \) can be computed by finding a basis for \( \mathcal{N}(\lambda I - A) \).

For \( \lambda = 0 \) we find that

\[
\lambda I - A = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(\lambda I - A).
\]

Hence, \( v = (-2, 1)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda = 0 \). (Note \( v = \alpha (-2, 1)^T \) for any nonzero scalar \( \alpha \) is also an eigenvector.)

For \( \lambda = 5 \) we find that

\[
\lambda I - A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(\lambda I - A).
\]

Hence, \( v = (1/2, 1)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda = 5 \). \( \diamond \)

**Example 21** Let

\[ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Applying Gaussian elimination to \( \lambda I - A \) gives

\[
\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \rightarrow \begin{pmatrix} \lambda & -1 \\ 0 & \lambda + \frac{1}{\lambda} \end{pmatrix}.
\]

In this example,

\[ \det(\lambda I - A) = \lambda^2 + 1. \]

The eigenvalues of \( A \) are the roots of \( p_A(\lambda) = \det(\lambda I - A) = \lambda^2 + 1. \) There are two roots of this polynomial, but they are complex. For this matrix the eigenvalues are the complex numbers

\[ \lambda = i \quad \text{and} \quad \lambda = -i. \]
The corresponding eigenvectors can be computed by finding a basis for $N(\lambda I - A)$. This is done using Gaussian elimination, but we now have to work with complex valued matrix entries, rather than real numbers.

For $\lambda = i$ we find that

$$\lambda I - A = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \rightarrow \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix} \text{ (multiply row 1 by } i \text{ and add result to row 2)}$$

Hence

$$\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$$

is a basis for $N(iI - A)$. The vector $v = (-i, 1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda = i$.

For $\lambda = -i$ we find that

$$\lambda I - A = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightarrow \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \text{ (multiply row 1 by } i \text{ and subtract result from row 2)}$$

Hence

$$\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$$

is a basis for $N(-iI - A)$. The vector $v = (i, 1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda = -i$.

\[\diamond\]

**Example 22** Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

Since

$$\lambda I - A = \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{pmatrix}$$

we find that

$$\det(\lambda I - A) = (\lambda - 1)^2.$$ 

The eigenvalues of $A$ are the roots of $p_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)^2$. The characteristic polynomial has a double root $\lambda = 1$. We say that $\lambda = 1$ is an eigenvalue with *algebraic multiplicity* 2. When we count (list) eigenvalues, we count (list) $\lambda = 1$ twice.

The corresponding eigenvectors for $\lambda = 1$ are computed by finding a basis for $N(\lambda I - A)$. We find that

$$I - A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $N(I - A)$.
Note that the dimension of $\mathcal{N}(I - A)$ is only one. The dimension of $\mathcal{N}(\lambda I - A)$ is called the \textit{geometric multiplicity} of $\lambda$. In this example, $\lambda = 1$ has algebraic multiplicity 2, but geometric multiplicity 1. (We will show later that the geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.)

In this example, the vector $v = (1, 0)^T$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$. All other eigenvectors are nonzero multiples of $v = (1, 0)^T$, i.e., are linearly dependent from $v = (1, 0)^T$.

To compute eigenvalues and corresponding eigenvectors for a square matrix $A \in \mathbb{R}^{n \times n}$ we have to perform the following steps.

1. Compute the determinant $\det(\lambda I - A)$. That is compute the row reduced form of $\lambda I - A$. The determinant $\det(\lambda I - A)$ is equal to the product of the diagonals in the row reduced form.

2. The determinant $\det(\lambda I - A)$ is a polynomial of degree $n$ in $\lambda$. This polynomial is called the \textit{characteristic polynomial} (of $A$). The eigenvalues are the roots of the characteristic polynomial. Since any polynomial of degree $n$ has exactly $n$ roots (including multiplicities; some of the roots may be complex), any $n \times n$ matrix $A$ has exactly $n$ eigenvalues (including multiplicities). Even if the matrix is real, it may have complex eigenvalues.

3. For each of the (distinct) eigenvalues $\lambda$ of $A$, any nonzero vector in $\mathcal{N}(\lambda I - A)$ is an an eigenvector of $A$ (corresponding to the eigenvalue $\lambda$). The subspace $\mathcal{N}(\lambda I - A)$ is called the \textit{eigenspace} of $A$ corresponding to the eigenvalue $\lambda$.

Since the subspace $\mathcal{N}(\lambda I - A)$ can be described by a basis, we compute a basis for $\mathcal{N}(\lambda I - A)$. In particular, any basis vector of $\mathcal{N}(\lambda I - A)$ is an eigenvector corresponding to the eigenvalue $\lambda$.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A \in \mathbb{R}^{n \times n}$ and let $v_1, \ldots, v_n$ be corresponding eigenvectors. We have

$$Av_1 = v_1\lambda_1, \ldots, Av_n = v_n\lambda_n.$$ 

If we put these vectors as columns of a matrix, then

$$(Av_1, \ldots, Av_n) = (v_1\lambda_1, \ldots, v_n\lambda_n).$$

If we define the matrix

$$V = (v_1, \ldots, v_n) \in \mathbb{C}^{n \times n}$$

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and the diagonal matrix

\[ \Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \in \mathbb{C}^{n \times n}, \]

then

\[ AV = (Av_1, \ldots, Av_n) = (v_1 \lambda_1, \ldots, v_n \lambda_n) = V \Lambda. \] (29)

If the matrix \( V \) is invertible, that is if we can find \( n \) linearly independent eigenvectors \( v_1, \ldots, v_n \), then

\[ A = V \Lambda V^{-1}. \] (30)

If there exists an invertible matrix \( V \) and a diagonal matrix \( \Lambda \) such that (30) holds, we say that the matrix \( A \) is diagonalizable. Example 22 has already shown that it is not always possible to find an invertible matrix \( V \) with (29).

Multiplication by the matrices \( V \) and \( V^{-1} \) transforms back and forth between regular Cartesian coordinates, and coordinates with respect to the basis of eigenvectors. In the eigenvector coordinates, the transformation \( A \) is represented by the diagonal matrix \( \Lambda \). This is illustrated in Figure 11.

Figure 11: Consider the vectors \( x \) and \( Ax \), shown in the left plot left. When \( x \) is expressed as a linear combination \( x = \sum_{j=1}^{n} y_j v_j \) of the eigenvectors \( v_1, \ldots, v_n \), i.e., in matrix form, \( x = Vy \) for some vector \( y \), the vector \( y \) is the coordinates of \( x \) with respect to the basis of eigenvectors. Since \( A = V \Lambda V^{-1} \), the vector \( x \) transforms to \( y = V^{-1}x \), while the vector \( Ax \) transforms to \( V^{-1}Ax = \Lambda V^{-1}Vy = \Lambda y \).

If it is possible to find an invertible matrix \( V \) with (29), then (30) holds. The identity (30) is extremely useful. For example, we can compute

\[ A^2 = V \Lambda V^{-1} V \Lambda V^{-1} = V \Lambda^2 V^{-1} \]

and, more generally

\[ A^k = V \Lambda V^{-1} V \Lambda V^{-1} \ldots V \Lambda V^{-1} = V \Lambda^k V^{-1}. \]
Thus, essentially, we can take powers of a diagonalizable matrix by taking powers of the diagonal matrix, whose diagonal entries are the eigenvalues of the matrix. We will use (30) to the study linear systems, linear least squares problems, as well as properties of solutions to dynamical systems and differential equations. Unfortunately, as we have seen in Example 22 (see also Example 25 below), not every square matrix is diagonalizable and we will show that an identity like (30) can be derived where $\Lambda$ is no longer a diagonal matrix, but has nonzero entries on the superdiagonal.

**Example 23** Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}. $$

from Example 20. We have shown that

$$(1 \ 2) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad (1 \ 2) \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}.$$

Hence,

$$(1 \ 2) \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}.$$

Since $V$ is invertible,

$$\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \begin{pmatrix} 2/5 & 4/5 \end{pmatrix}.$$

\[\diamondsuit\]

**Example 24** Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

from Example 21. We have shown that

$$(0 \ 1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad (0 \ 1) \begin{pmatrix} i \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} 0 \ 1 \\ -1 \ 0 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}.$$
Since $V$ is invertible,

$$
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} =
\begin{bmatrix}
-i & i \\
1 & i
\end{bmatrix}
\begin{bmatrix}
i & 0 \\
0 & -i
\end{bmatrix}
\begin{bmatrix}
i/2 & 1/2 \\
-i/2 & 1/2
\end{bmatrix}.
$$

$= A$
$= V$
$= \Lambda$
$= V^{-1}$

\[\diamond\]

**Example 25** Consider the matrix

$$A = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}.$$

from Example 22. We have shown that

$$
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix} =
\begin{bmatrix}
1 \\
0
\end{bmatrix},
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix} =
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
$$

Hence,

$$
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix} =
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
$$

(31)

In this case all eigenvectors are multiples of $v = (1, 0)^T$. Hence we cannot find an invertible matrix $V$ such that (31) holds.

\[\diamond\]

In the following lectures we will pursue a more in-depth study of eigenvalues and eigenvectors. In particular we will answer the following questions.

- What can be said about the properties of the eigenvalues and eigenvectors of a matrix. For example, what matrices have all real eigenvalues?

- For which classes of matrices can we find $n$ linearly independent eigenvectors, i.e., which classes of matrices are diagonalizable?

- How can eigenvalues and eigenvectors be used in the study of linear systems, linear least squares problems, dynamical systems and differential equations?
7 Eigenvectors and Eigenvalues

Let $A$ be a (possibly complex) square $n \times n$ matrix. A (possibly complex) scalar $\lambda$ is called an eigenvalue of $A$ if there exists a non-zero vector $v$ such that

$$Av = v\lambda. \quad (32)$$

In this section we prove several properties of eigenvalues and eigenvectors that we already observed in the examples studied in the previous section. For the first result, revisit Example 21.

**Theorem 26** If $A$ is a real $n \times n$ matrix and $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ with corresponding eigenvector $v \in \mathbb{C}^n$, then $\overline{\lambda}$ is an eigenvalue of $A$ with corresponding eigenvector $\overline{v} \in \mathbb{C}^n$.

**Proof:** If $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ with corresponding eigenvector $v \in \mathbb{C}^n$, then

$$Av = v\lambda.$$

Taking the complex conjugate on either side gives

$$\overline{A}v = \overline{v}\overline{\lambda}$$

For complex numbers $z_1$ and $z_2$ we have $z_1z_2 = \overline{z_1z_2} = \overline{z_1}\overline{z_2} = \overline{z_2z_1}$. Therefore,

$$\overline{Av} = \overline{A}v = \overline{v}\overline{\lambda}.$$

Since $A$ is real, $\overline{A} = A$ and we find that

$$A\overline{v} = \overline{v}\overline{\lambda}.$$

\[\square\]

For the next result, revisit Example 20.

A complex $n \times n$ matrix $A$ is called *Hermitian* if $A^* = A$. Real Hermitian matrices are symmetric matrices.

**Theorem 27** If $A$ is an $n \times n$ Hermitian matrix then all eigenvalues are real. If $A$ is a real $n \times n$ symmetric matrix then all eigenvalues and also the corresponding eigenvectors are real.

**Proof:** Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$ with corresponding eigenvector $v \in \mathbb{C}^n$, i.e, let

$$Av = v\lambda.$$

We have

$$\lambda\|v\|^2_2 = v^*v\lambda = v^*Av = v^*A^*v \quad \text{(since $A^* = A$)}$$

$$= (Av)^*v = (\lambda v)^*v = \overline{\lambda} \|v\|^2_2.$$
Since $v$ is a nonzero vector, $\|v\|^2 > 0$ and we obtain $\lambda = \overline{\lambda}$, i.e., that $\lambda$ is a real number.

If $A$ is a (real) symmetric matrix, then its eigenvalues are real (since $A$ is also Hermitian), and given an eigenvector $\lambda \in \mathbb{R}$, the null space of $\lambda I - A$ contains nonzero real vectors. \hfill \Box

The eigenvalues of $A$ are those scalars $\lambda$ for which $\lambda I - A$ is singular and these scalars can be determined by computing the roots of the characteristic polynomial

$$p_A(\lambda) = \det(\lambda I - A).$$

Since $p_A(\lambda)$ is a polynomial of degree $n$ if $A$ is an $n \times n$ matrix, and every polynomial of degree $n$ has exactly $n$ roots (counting multiplicities), every $n \times n$ matrix has $n$ eigenvalues. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A \in \mathbb{R}^{n \times n}$ and let $v_1, \ldots, v_n$ be corresponding eigenvectors. We have

$$Av_1 = v_1 \lambda_1, \ldots, Av_n = v_n \lambda_n.$$

If we put these vectors as columns of a matrix, then

$$AV = (Av_1, \ldots, Av_n) = (v_1 \lambda_1, \ldots, v_n \lambda_n) = V \Lambda.$$  

where

$$V = (v_1, \ldots, v_n) \in \mathbb{C}^{n \times n} \quad \text{and} \quad \Lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

We are interested in characterizing matrices $A$ for which we can find $n$ linearly independent eigenvectors $v_1, \ldots, v_n$, so that the matrix $V$ is invertible.

**Theorem 28 (Eigenvectors corresponding to distinct eigenvalues)** Eigenvectors $v_1, v_2, \cdots, v_m$ of $A \in \mathbb{C}^{n \times n}$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ (i.e., if $j \neq k$, then $\lambda_j \neq \lambda_k$) are linearly independent.

**Proof:** Suppose that the eigenvectors $v_1, v_2, \cdots, v_k$ are linearly independent for some $k < m$, but that $v_1, v_2, \cdots, v_k, v_{k+1}$ are linearly dependent. Then there exist constants $c_1, c_2, \cdots, c_k$ not all zero such that

$$v_{k+1} = \sum_{j=1}^{k} c_j v_j. \quad (33)$$

Multiplication of the left-hand side of (33) by $A$ yields

$$Av_{k+1} = \lambda_{k+1} v_{k+1} = \lambda_{k+1} \sum_{j=1}^{k} c_j v_j,$$
while multiplication of the right-hand side of (33) by $A$ gives

\[ A \sum_{j=1}^{k} c_j v_j = \sum_{j=1}^{k} c_j A v_j = \sum_{j=1}^{k} c_j \lambda_j v_j. \]

Hence $\lambda_{k+1} \sum_{j=1}^{k} c_j v_j = \sum_{j=1}^{k} c_j \lambda_j v_j$, which implies

\[ \sum_{j=1}^{k} c_j (\lambda_j - \lambda_{k+1}) v_j = 0. \]

Since $\lambda_j \neq \lambda_{k+1}$ for any $j \in \{1, 2, \cdots, k\}$, and the coefficients $c_j$ are not all zero, we conclude that $v_1, v_2, \cdots, v_k$ are linearly dependent, in contradiction to our assumption. Therefore, the vectors $v_1, v_2, \cdots, v_k, v_{k+1}$ are linearly independent.

Using induction, the vectors $v_1, v_2, \cdots, v_m$ are linearly independent.

\[ \square \]

**Example 29** Consider

\[ A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}. \]

We compute the row reduced form of $\lambda I - A$,

\[ \begin{pmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & -2 & \lambda \end{pmatrix} \rightarrow \begin{pmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 2/(\lambda - 1) \end{pmatrix} \]

to determine the characteristic polynomial of $A$,

\[ p_A(\lambda) = \det(\lambda I - A) = (\lambda - 2)((\lambda - 1)\lambda - 2) = (\lambda - 2)(\lambda^2 - \lambda - 2). \]

Hence the eigenvalues are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = -1$.

To compute eigenvectors we compute bases for $\mathcal{N}(2I - A)$ and $\mathcal{N}(-I - A)$:

\[ \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(2I - A) \]

and

\[ \left\{ \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(-I - A). \]
Each of the basis vectors $(1, 0, 0)^T$ and $(0, 1, 1)^T$ is linearly independent from $(0, -1/2, 1)^T$, as guaranteed by Theorem 28.

In this case 2 is an eigenvalue with algebraic multiplicity two and the dimension of $\mathcal{N}(2I - A)$ is two. We have

\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 1 & 1 \\
0 & 2 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1/2 \\
0 & 1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1/2 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1
\end{pmatrix}
= A
\]

and $V$ is invertible. Hence,

\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 1 & 1 \\
0 & 2 & 0
\end{pmatrix}
= A
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1/2 \\
0 & 1 & 1
\end{pmatrix}
= V
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2/3 & 1/3 \\
0 & -2/3 & 2/3
\end{pmatrix}
= \Lambda
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & -1/2 & 1)
\end{pmatrix}
= V^{-1}
\]

\[\blacksquare\]

**Example 30** Consider

\[
A = \begin{pmatrix}
2 & 1 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

We compute the row reduced form of $\lambda I - A$,

\[
\begin{pmatrix}
\lambda - 2 & -1 & -1 \\
0 & \lambda - 1 & -1 \\
0 & -1 & \lambda - 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\lambda - 2 & -1 & -1 \\
0 & \lambda - 1 & -1 \\
0 & 0 & \lambda - 1 - 1/((\lambda - 1))
\end{pmatrix}
\]

to determine the characteristic polynomial of $A$,

\[
p_A(\lambda) = \det(\lambda I - A) = (\lambda - 2)((\lambda - 1)^2 - 1) = (\lambda - 2)^2 \lambda.
\]

Hence the eigenvalues are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 0$.

To compute eigenvectors we compute bases for $\mathcal{N}(2I - A)$ and $\mathcal{N}(-A)$:

\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}
\text{is a basis for } \mathcal{N}(2I - A)
\]

and

\[
\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}
\text{is a basis for } \mathcal{N}(-A).
\]
Clearly, the eigenvectors \((1, 0, 0)^T\) and \((0, -1, 1)^T\) are linearly independent. Note that in this example, \(\lambda_1 = \lambda_2 = 2\) is an eigenvalue with algebraic multiplicity two but the dimension of \(N(2I - A)\) is one (the geometric multiplicity of the eigenvalue 2 is only one. In this case there does not exist an invertible matrix \(V\) with \(AV = A\Lambda\), where \(\Lambda = \text{diag}(2, 2, 0)\).

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial \(p_A(\lambda) = \det(\lambda I - A)\). The **geometric multiplicity** of an eigenvalue is the dimension of the null space \(N(\lambda I - A)\). In Example 30 the eigenvalue \(\lambda = 2\) has algebraic multiplicity two and geometric multiplicity one; the eigenvalue \(\lambda = 0\) has algebraic multiplicity one and geometric multiplicity one. One can show that the algebraic multiplicity of an eigenvalue is always greater than or equal to its geometric multiplicity. We can find \(n\) linearly independent eigenvectors if and only if for all eigenvalues the algebraic multiplicity is equal to the geometric multiplicity.

A simple consequence of Theorem 28 is that if all \(n\) eigenvalues of \(A \in \mathbb{R}^{n \times n}\) are distinct (i.e., are all simple roots of the characteristic polynomial \(p_A(\lambda)\)), then the matrix \(A\) has a complete set of \(n\) linearly independent eigenvectors. This is the case in Example 20 (see also Example 23) and in the following example.

**Example 31** Consider the matrix

\[
A = \begin{pmatrix}
-1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

The matrix \((\lambda I - A)\) is a lower triangular matrix and

\[
p_A(\lambda) = \det(\lambda I - A) = \lambda(\lambda + 1)(\lambda - 1).
\]

The eigenvalues of \(A\) are \(\lambda_1 = 0\), \(\lambda_2 = 1\), \(\lambda_3 = -1\).

To compute eigenvectors corresponding to an eigenvalue \(\lambda_j\) of \(A\), we have to compute the null-space \(N(\lambda_j I - A)\). We obtain

\[
0I - A = \begin{pmatrix}
1 & 0 & 0 \\
-1 & -1 & 0 \\
0 & -1 & 0
\end{pmatrix} \Rightarrow N(0I - A) = \text{span}\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \},
\]

\[
1I - A = \begin{pmatrix}
2 & 0 & 0 \\
-1 & 0 & 0 \\
0 & -1 & 1
\end{pmatrix} \Rightarrow N(1I - A) = \text{span}\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \},
\]

\[
(-1)I - A = \begin{pmatrix}
0 & 0 & 0 \\
-1 & -2 & 0 \\
0 & -1 & -1
\end{pmatrix} \Rightarrow N((-1)I - A) = \text{span}\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \}.
\]
Thus, $v_1 = (0, 0, 1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 0$, $v_2 = (0, 1, 1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$, and $v_3 = (2, -1, 1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = -1$.

Note that if $v$ is an eigenvector corresponding to the eigenvalue $\lambda$, then for any scalar $t \neq 0$ the vector $tv$ is also an eigenvector corresponding to the eigenvalue $\lambda$. Therefore we often normalize eigenvectors $v$ such that they satisfy $v^*v = 1$, i.e., given a nonzero vector $\tilde{v} \in \mathcal{N}(\lambda I - A)$ we set $v = \tilde{v}/\|\tilde{v}\|_2$, where $\|\tilde{v}\|_2 = \sqrt{\sum_{j=1}^{n} \tilde{v}_j \tilde{v}_j}$. Note that in general eigenvalues are complex and eigenvectors are vectors in $\mathbb{C}^n$, even if the matrix $A$ is real.

If we normalize the previously computed eigenvectors we find that $v_1 = (0, 0, 1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 0$, $v_2 = (0, 1, 1)^T/\sqrt{2}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$, and $v_3 = (2, -1, 1)^T/\sqrt{6}$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = -1$.

We can compute eigenvalues and corresponding eigenvectors using MATLAB function `eig`.

\[
\begin{bmatrix}
-1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
V, \Lambda\\n\end{bmatrix} = \text{eig}(A)
\]

\[
V =
\begin{bmatrix}
0 & 0 & 0.8165 \\
0 & 0.7071 & -0.4082 \\
1.0000 & 0.7071 & 0.4082
\end{bmatrix}
\]

\[
\Lambda =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

In this example, $[V, \Lambda] = \text{eig}(A)$ returns a $3 \times 3$ matrix $V$ and $3 \times 3$ diagonal matrix $\Lambda$ such that the diagonal entries in $\Lambda$ are the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$ of $A$ and the columns of $V$ are the corresponding normalized eigenvectors $v_1 = (0, 0, 1)^T$, $v_2 = (0, 1, 1)^T/\sqrt{2}$, $v_3 = (2, -1, 1)^T/\sqrt{6}$.
The eigenvalues - eigenvector relationships give

\[
\begin{pmatrix}
-1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \frac{2}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\
1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & \frac{2}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

(We use the normalized eigenvectors, but the same relation hold if do not normalize.) The matrix \(V\) is invertible. If we multiple both sides by \(V^{-1}\) from the left, we obtain

\[
\begin{pmatrix}
-1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \frac{2}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\
1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & \frac{2}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
0 & 0 & \frac{2}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\
1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
-1 & -1 & 1 \\
\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{6}} & 0 \\
\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{6}} & 0
\end{pmatrix}.
\]

\[\diamond\]
8 Diagonalization of Symmetric Matrices

8.1 Diagonalization of Symmetric Matrices

The matrices that arise in many physical applications are symmetric. We have already shown in Theorem 27 that eigenvalues and eigenvectors of symmetric matrices are real. We next show a few more important properties of symmetric matrices. First, we show that eigenvectors corresponding to distinct eigenvalues of symmetric matrices are not only linearly independent but even orthogonal. We have observed this already in Example 20. Then we will show that for symmetric matrices we can always find \( n \) linearly independent eigenvectors. In fact, we will show that for symmetric matrices we can always find \( n \) orthogonal eigenvectors.

**Theorem 32** Let \( \lambda \) and \( \mu \) be two distinct eigenvalues of a symmetric matrix. If \( x \) and \( y \) are eigenvectors corresponding to \( \lambda \) and \( \mu \), respectively, then \( x \) and \( y \) are orthogonal, \( x^T y = 0 \).

**Proof:** Since \( \lambda \) and \( \mu \) are eigenvalues of \( A \) with eigenvectors \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \),

\[
Ax = \lambda x, \quad Ay = \mu y.
\]

We multiply the first equation by \( y^T \) and the second by \( x^T \).

\[
y^T Ax = \lambda y^T x, \quad x^T Ay = \mu x^T y.
\]

Since \( A \) is symmetric, we have

\[
y^T Ax = (y^T Ax)^T = x^T A^T y = x^T Ay.
\]

Hence, \( \lambda y^T x = \mu x^T y \), which implies

\[
(\lambda - \mu) y^T x = 0.
\]

Since \( \lambda \neq \mu \) this proves \( y^T x = 0 \).

\[\square\]

**Example 33** Consider the matrix

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}.
\]

The row reduced form of \( \lambda I - A \) is given by

\[
(\lambda I - A) = \begin{pmatrix}
\lambda - 1 & -1 & -1 \\
-1 & \lambda - 1 & -1 \\
-1 & -1 & \lambda - 1
\end{pmatrix} \rightarrow \begin{pmatrix}
\lambda - 1 & -1 & -1 \\
0 & \lambda - 1 - \frac{1}{\lambda - 1} & -1 - \frac{1}{\lambda - 1} \\
0 & -1 - \frac{1}{\lambda - 1} & \lambda - 1 - \frac{1}{\lambda - 1}
\end{pmatrix}
\]

\[
\rightarrow \begin{pmatrix}
\lambda - 1 & -1 & -1 \\
0 & \lambda(\lambda - 2)/(\lambda - 1) & -1 - \frac{1}{\lambda - 1} \\
0 & 0 & \lambda(\lambda - 3)/(\lambda - 2)
\end{pmatrix}.
\]
Hence,

\[ p_A(\lambda) = \det(\lambda I - A) = \lambda^2(\lambda - 3). \]

The eigenvalues of \( A \) are \( \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 3 \).

To compute eigenvectors corresponding to an eigenvalue \( \lambda_j \) of \( A \), we have to compute the null-space \( \mathcal{N}(\lambda_j I - A) \). We obtain

\[
0I - A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \Rightarrow \mathcal{N}(0I - A) = \text{span}\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \},
\]

\[
3I - A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \Rightarrow \mathcal{N}(3I - A) = \text{span}\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \}.
\]

Thus, \( v_1 = (-1, 1, 0)^T \) and \( v_2 = (-1, 0, 1)^T \) are eigenvectors corresponding to the eigenvalue \( \lambda_1 = \lambda_2 = 0 \) and \( v_3 = (1, 1, 1)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda_3 = 3 \). It is easily verified that \( v_1^Tv_3 = 0 \) and \( v_2^Tv_3 = 0 \).

The eigenvalue - eigenvector relationships give

\[
\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = V, \quad \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \Lambda,
\]

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.
\]

The matrix \( V \) is invertible. If we multiply both sides by \( V^{-1} \) from the left, we obtain

\[
\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = V, \quad \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \Lambda,
\]

\[
= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.
\]

\[ \diamond \]

We now state the result on diagonalizability of symmetric matrices \( A \in \mathbb{R}^{n \times n} \). The proof of this result requires a few more preparations and will be given later.

**Theorem 34 (Diagonalizability of Symmetric Matrices)** *For any symmetric matrix \( A \in \mathbb{R}^{n \times n} \), \( A^T = A \), there exists an orthogonal matrix \( Q \in \mathbb{R}^{n \times n} \) (\( Q^T = Q^{-1} \)) and a diagonal matrix \( \Lambda \in \mathbb{R}^{n \times n} \) such that*

\[ A = Q\Lambda Q^T. \]
The extension of Theorem 34 to complex Hermitian matrices is stated in Theorem 39 below. Note that in Example 33 we have computed three linearly independent eigenvectors \( v_1, v_2, v_3 \). The eigenvectors \( v_1, v_2 \) corresponding to the eigenvalue 0 are both orthogonal to the eigenvector \( v_3 \) corresponding to the eigenvalue 3. However, \( v_1 \) and \( v_2 \) are not orthogonal. Thus we need to find a way to extract from the eigenvectors \( v_1, v_2 \) two orthogonal eigenvectors. This can be accomplished using Gram-Schmidt orthogonalization.

We use the Gram-Schmidt method to compute orthonormal bases for the eigenspaces \( N(\lambda_j I - A) \).

**Example 35** In Example 33 we have shown that the matrix

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

has eigenvalues \( \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 3 \). The corresponding eigenspaces, i.e, the nullspaces \( N(0 I - A) \) and \( N(3 I - A) \) are

\[
N(0 I - A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\},
\]

\[
N(3 I - A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.
\]

The matrix

\[
V = \begin{pmatrix}
-1 & -1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

of eigenvectors is invertible, but not orthonormal.

We can now compute orthonormal bases using Gram-Schmidt. To compute an orthonormal basis for \( N(0 I - A) \) we proceed as follows. Let \( v_1 = (-1, 1, 0)^T \) and \( v_2 = (-1, 0, 1)^T \). An orthonormal basis is obtained by computing

\[
q_1 = v_1 / \| v_1 \|_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} / \sqrt{2},
\]

\[
\tilde{q}_2 = v_2 - (v_2^T q_1) q_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix},
\]

\[
q_2 = \tilde{q}_2 / \| \tilde{q}_2 \| = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.
\]
To compute an orthonormal basis for $\mathcal{N}(3I - A)$ we just need to normalize the original basis vector. The normalized basis vector is

$$q_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$ 

It is easy to verify that the vectors $q_1, q_2, q_3$ are orthogonal.

The vectors $q_1, q_2$ are eigenvectors corresponding to the eigenvalue 0 and the vector $q_3$ is an eigenvectors corresponding to the eigenvalue 3. Hence

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Since the matrix $Q$ is orthogonal, $Q^{-1} = Q^T$ and we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}.$$

If we use Matlab to compute the eigendecomposition of $A$ we obtain

```matlab
>> A = ones(3,3);
>> [Q,Lambda]=eig(A);
>> Q
Q =
0.4082 0.7071 0.5774
0.4082 -0.7071 0.5774
-0.8165 0 0.5774

>> Lambda
Lambda =
-0.0000 0 0
0 0 0
0 0 3.0000
```
Note that the first column in the matrix $Q$ computed by Matlab is $-q_2$ and the second column in the matrix $Q$ computed by Matlab is $-q_1$. Thus, the first two columns in the matrix $Q$ computed by Matlab are just another orthonormal basis for $\mathcal{N}(0I - A)$.

For any symmetric matrix $A \in \mathbb{R}^{n \times n}$, $A^T = A$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ ($Q^T = Q^{-1}$) and a diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that

$$A = Q\Lambda Q^T.$$  

If the columns of $Q$ are denoted $q_1, \ldots, q_n$, and the diagonal entries of $\Lambda$, (the eigenvalues of $A$) are denoted by $\lambda_1, \ldots, \lambda_n$, then the rules of matrix-matrix multiplication imply

$$A = Q\Lambda Q^T = \sum_{j=1}^n \lambda_j q_j q_j^T.$$  \hspace{1cm} (34)

**Example 36** In Example 35 we have shown that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix},$$  

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix},$$  

$$\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2}, 0 \end{pmatrix} + \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix}.$$  

$$= 0 \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix} + 3 \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}.  

The matrices

$$\begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix},$$

are the orthogonal projections onto the eigenspaces $\mathcal{N}(0I - A)$ and $\mathcal{N}(3I - A)$, respectively.
Example 37 In Example 23 we have shown that
\[
\begin{pmatrix}
1 & 2 \\
2 & 4 \\
\end{pmatrix}
\begin{pmatrix}
-2 & 1/2 \\
1 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & 5 \\
\end{pmatrix}.
\]

Since the eigenspaces \( \mathcal{N}(0I - A) \) and \( \mathcal{N}(5I - A) \) are both one dimensional, we only need to normalize the eigenvectors \( v_1, v_2 \) to obtain an orthonormal basis:
\[
\begin{pmatrix}
1 & 2 \\
2 & 4 \\
\end{pmatrix}
= A \\
\begin{pmatrix}
-2/\sqrt{5} & 1/\sqrt{5} \\
1/\sqrt{5} & 2/\sqrt{5} \\
\end{pmatrix}
= Q \\
\begin{pmatrix}
0 & 0 \\
0 & 5 \\
\end{pmatrix}
= \Lambda \\
\begin{pmatrix}
-2/\sqrt{5} & 1/\sqrt{5} \\
1/\sqrt{5} & 2/\sqrt{5} \\
\end{pmatrix}
= Q^T.
\]

Figure 12: Orthonormal eigenvectors of the matrix \( A \) in Example ?? (note that \( Aq_1 = 0q = 0 \) in this example)
Example 38 The matrix

\[
A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
\]

has eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = 1 \). The corresponding normalized eigenvectors are \( q_1 = (-1, 1)^T / \sqrt{2} \) and \( q_2 = (1, 1)^T / \sqrt{2} \). Thus,

\[
\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = Q^T.
\]

From (34) we can also write

\[
A = \lambda_1 \, q_1 q_1^T + \lambda_2 \, q_2 q_2^T, \quad \text{i.e.,}
\]

\[
\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3 \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} + 1 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = P_1 + P_2.
\]

\( \diamond \)
8.2 Proof of Theorem 34

For completeness, we re-state Theorem 34.

**Theorem 34 (Diagonalizability of Symmetric Matrices)** For any symmetric matrix $A \in \mathbb{R}^{n \times n}$, $A^T = A$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ ($Q^T = Q^{-1}$) and a diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that

$$A = Q\Lambda Q^T$$

**Proof:** The proof uses induction over the size of the matrix.

The result is trivially true for $1 \times 1$ matrices.

Assume that for every symmetric matrix $A_n \in \mathbb{R}^{n \times n}$ there exists an orthogonal matrix $Q_n \in \mathbb{R}^{n \times n}$ ($Q_n^T = Q_n^{-1}$) and a diagonal matrix $\Lambda_n \in \mathbb{R}^{n \times n}$ such that

$$A_n = Q_n\Lambda_n Q_n^T.$$

We will show that for every symmetric matrix $A \in \mathbb{R}^{(n+1) \times (n+1)}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{(n+1) \times (n+1)}$ ($Q^T = Q^{-1}$) and a diagonal matrix $\Lambda \in \mathbb{R}^{(n+1) \times (n+1)}$ such that

$$A = Q\Lambda Q^T.$$

Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ be any matrix. Let $\lambda$ be an eigenvalue of $A$ with to eigenvector $q \in \mathbb{R}^{n+1}$, such that $q^T q = 1$. Consider the subspace

$$S = \{ x \in \mathbb{R}^n : q^T x = 0 \}.$$

Note that $S = \mathcal{N}(q^T)$. Since $q^T \in \mathbb{R}^{1 \times (n+1)}$, $\dim(S) = n$.

Let $\{p_1, \ldots, p_n\}$ be an orthonormal basis for $S$ (it can be computed using Gram-Schmidt). Then $\{q, p_1, \ldots, p_n\}$ is an orthonormal basis for $\mathbb{R}^{n+1}$. We define $P = (p_1, \ldots, p_n) \in \mathbb{R}^{(n+1) \times n}$. The matrix

$$(q \ p_1 \ \ldots \ p_n) = (q \ P)$$

is an orthogonal matrix.

$$(q \ P)^T A (q \ P) = \begin{pmatrix} q^T A q & q^T A P \\ P^T A q & P^T A P \end{pmatrix}.$$

Since $Aq = \lambda q$ and $q^T q = 1$, we have $q^T A q = \lambda$. Furthermore $q^T A P = (Aq)^T P = \lambda q^T P = 0$, since the columns of $P$ are basis vectors of $S$ and therefore satisfy $q^T p_j$. Thus, we obtain

$$(q \ P)^T A (q \ P) = \begin{pmatrix} \lambda & 0 \\ 0 & P^T A P \end{pmatrix}.$$
8.2 Proof of Theorem 34

The matrix $P^TAP$ is of size $n \times n$. Therefore by induction hypothesis, there exists an orthogonal matrix $\tilde{Q}_n \in \mathbb{R}^{n \times n}$ ($\tilde{Q}_n^T = \tilde{Q}_n^{-1}$) and a diagonal matrix $\Lambda_n \in \mathbb{R}^{n \times n}$ such that

$$P^TAP = \tilde{Q}_n \Lambda_n \tilde{Q}_n^T.$$ 

Moreover,

$$\begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix}^T \begin{pmatrix} \lambda & 0 \\ 0 & P^TAP \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \tilde{Q}_n^T P^TAP \tilde{Q}_n \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \Lambda_n \end{pmatrix}.$$

and

$$\begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix}^T \begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix}^T \begin{pmatrix} \lambda & 0 \\ 0 & P^TAP \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \Lambda_n \end{pmatrix} = \Lambda.$$

The matrix

$$Q = \begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix}$$

is orthogonal, since

$$\begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix}^T \begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix}^T \begin{pmatrix} q & P \\ q & P \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & \tilde{Q}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 0 & \tilde{Q}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}_n \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 0 & \tilde{Q}_n \end{pmatrix}.$$

Thus we have shown that $A = Q \Lambda Q^T$ with an orthogonal matrix $Q \in \mathbb{R}^{(n+1) \times (n+1)}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{(n+1) \times (n+1)}$. \qed

Theorem 34 can be extended to complex valued Hermitian matrices. Recall that a matrix $A \in \mathbb{C}^{n \times n}$ is called Hermitian if it is equal to its conjugate transpose, i.e., if $A^* = A$. A matrix $U \in \mathbb{C}^{n \times n}$ is called unitary if $U^* U = I$, i.e., $U^{-1} = U^*$.

**Theorem 39 (Diagonalizability of Hermitian Matrices)** For any Hermitian matrix $A \in \mathbb{C}^{n \times n}$, $A^* = A$, there exists an unitary matrix $Q \in \mathbb{R}^{n \times n}$ ($Q^T = Q^{-1}$) and a real diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that

$$A = U \Lambda U^*.$$
Example 40 Consider the Hermitian matrix

\[
A = \begin{pmatrix}
\frac{3}{2} & -\frac{i}{2} \\
\frac{i}{2} & \frac{3}{2}
\end{pmatrix}.
\]

Its diagonalization is given

\[
\begin{align*}
\begin{pmatrix}
\frac{3}{2} & -\frac{i}{2} \\
\frac{i}{2} & \frac{3}{2}
\end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix}
i & -i \\
1 & 1
\end{pmatrix} \begin{pmatrix}1 & 0 \ \ \ \ \ \ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}-i & 1 \\
i & 1
\end{pmatrix}.
\end{align*}
\]

If we use Matlab to compute the eigendecomposition of \(A\) we obtain

\[
A =
\begin{pmatrix}
1.5000 + 0.0000i & 0.0000 - 0.5000i \\
0.0000 + 0.5000i & 1.5000 + 0.0000i
\end{pmatrix}
\]

\[
[U,\Lambda] = \text{eig}(A)
\]

\[
U =
\begin{pmatrix}
0.0000 - 0.7071i & 0.0000 - 0.7071i \\
-0.7071 + 0.0000i & 0.7071 + 0.0000i
\end{pmatrix}
\]

\[
\Lambda =
\begin{pmatrix}1 & 0 \\
0 & 2
\end{pmatrix}
\]

\[
[U^*U =
\text{ans} =
\begin{pmatrix}1.0000 & 0 \\
0 & 1.0000
\end{pmatrix}
\]

\[
\diamondsuit
\]
9 Diagonalization of Matrices and the Solution of Linear Systems

9.1 Symmetric Matrices

Let $A$ be a real symmetric matrix, $A \in \mathbb{R}^{n \times n}$, $A = A^T$. We are interested in the solution of the linear system

$$Ax = b.$$ 

For any symmetric matrix $A \in \mathbb{R}^{n \times n}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ ($Q^{-1} = Q^T$) and a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n \times n}$ such that

$$A = Q\Lambda Q^T.$$ 

Let $q_1, \ldots, q_n$ be the columns of $Q$ and let $\lambda_1, \ldots, \lambda_n$ be the diagonal entries of $\Lambda$ (the eigenvalues of $A$). Then

$$A = Q\Lambda Q^T = \sum_{j=1}^{n} \lambda_j q_j q_j^T.$$ 

implies that the range space of $A$ is the span of all orthonormal eigenvectors corresponding to non-zero eigenvalues,

$$\mathcal{R}(A) = \text{span}\{q_j : \lambda_j \neq 0\},$$ 

and the null space of $A$ is the span of all orthonormal eigenvectors corresponding to zero eigenvalues,

$$\mathcal{N}(A) = \text{span}\{q_j : \lambda_j = 0\}.$$ 

Note that this representation of the range and the null space shows that $\mathcal{R}(A) \perp \mathcal{N}(A)$, which we of course already know from the Fundamental Theorem of Linear Algebra.

Using the diagonalization $A = Q\Lambda Q^T$, the linear system $Ax = b$ becomes

$$Q\Lambda Q^T x = b.$$ 

If we multiply both sides by $Q^T$ from the left and define the new unknown $z = Q^T x$ we obtain

$$\Lambda z = Q^T b.$$ 

While in $Ax = b$ every equation generally depends on every unknown $x_1, \ldots, x_n$, the equations $\Lambda z = Q^T b$ are

$$\lambda_1 z_1 = q_1^T b,$$

$$\vdots$$

$$\lambda_n z_n = q_n^T b.$$
If an eigenvalue $\lambda_j$ is zero, the corresponding equation is solvable if and only if $q_j^T b = 0$.

The matrix $A$ is invertible if and only if all eigenvalues are nonzero. In this case the inverse is

$$A^{-1} = Q\Lambda^{-1}Q^T.$$ 

If an eigenvalue of $A$ is zero, we can define the pseudo inverse of $A$. For a real symmetric matrix $A$ the pseudo inverse is given by

$$A^\dagger = Q\Lambda^\dagger Q^T,$$ 

where for a diagonal matrix

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

the pseudo inverse is defined as

$$\Lambda^\dagger = \text{diag}(\lambda_1^\dagger, \ldots, \lambda_n^\dagger)$$

with

$$\lambda_j^\dagger = \begin{cases} 1/\lambda_j & \text{if } \lambda_j \neq 0, \\ 0 & \text{if } \lambda_j = 0 \end{cases}, \quad j = 1, \ldots, n.$$ 

That is,

$$A^\dagger = Q\Lambda^\dagger Q^T = \sum_{\lambda_j \neq 0} \frac{1}{\lambda_j} q_j q_j^T.$$ 

**Example 41** In Example 36 we have shown that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 0 \left[ \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} + \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix} \right]$$

$$+ 3 \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}.$$ 

The pseudoinverse of the matrix above is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^\dagger = \frac{1}{3} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix}.$$ 

In MATLAB the pseudoinverse of a square matrix $A$ can be computed using `pinv(A)`.
9.2 Nonsymmetric Matrices

Let $A$ be a real matrix, $A \in \mathbb{R}^{n \times n}$ and assume that $A$ is diagonalizable, i.e., assume there exists a nonsignular matrix $V \in \mathbb{C}^{n \times n}$ and a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^{n \times n}$ such that

$A = V \Lambda V^{-1}$.

Note that not every non-symmetric matrix is diagonalizable (see, e.g., Examples 22 and 25).

We can use the diagonalization of $A$ to solve the linear system $Ax = b$. Using the diagonalization $A = V \Lambda V^{-1}$, the linear system $Ax = b$ becomes

$V \Lambda V^{-1} x = b$.

If we multiply both sides by $V^{-1}$ from the left and define the new unknown $z = V^{-1}x$ we obtain

$\Lambda z = V^{-1}b$.

Define $d = V^{-1}b$. While in $Ax = b$ every equation generally depends on every unknown $x_1, \ldots, x_n$, the equations $\Lambda z = V^{-1}b$ are

$\lambda_1 z_1 = d_1,$

$\vdots$

$\lambda_n z_n = d_n.$

If an eigenvalue $\lambda_j$ is zero, the corresponding equation is solvable if and only if the $j$ component of $d = V^{-1}b$ satisfies $d_j = 0$.

The matrix $A$ is invertible if and only if all eigenvalues are nonzero. In this case the inverse is

$A^{-1} = V \Lambda^{-1} V^{-1}$.

If an eigenvalue of $A$ is zero, we can define the pseudo inverse of $A$. We will define the pseudo inverse of non-symmetric (and even non-square) later in Section 14.4.

Example 42 The eigenvalues of

$A = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

are $\lambda_1 = 1$, $\lambda_2 = 2i$, $\lambda_3 = -2i$ with corresponding eigenvectors $v_1 = (0, 0, 1)^T$, $v_3 = (-2, -2i, -1)^T$, and $v_3 = (-2, 2i, -1)^T$. The eigenvectors are linearly independent. Hence $A$ is diagonalizable

$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 \\ 0 & -2i & 2i \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix} \begin{pmatrix} 0 & -2 & -2 \\ 0 & -2i & 2i \\ 1 & -1 & -1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 0 & -2 & -2 \\ 0 & -2i & 2i \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix} \begin{pmatrix} -1/2 & 0 & 1 \\ -3/4 & 3i/4 & 0 \\ -3/4 & -3i/4 & 0 \end{pmatrix}.$
We can also write
\[
\begin{pmatrix}
0 & 2 & 0 \\
-2 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
= 1 \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\left( -\frac{1}{2}, 0, 1 \right)
+ 2i \begin{pmatrix}
-2 \\
-2i \\
-1
\end{pmatrix}
\left( -\frac{3}{4}, 3i/4, 0 \right)
- 2i \begin{pmatrix}
-2 \\
2i \\
-1
\end{pmatrix}
\left( -\frac{3}{4}, -3i/4, 0 \right).
\]

The inverse is given by
\[
\begin{pmatrix}
0 & 2 & 0 \\
-2 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
0 & -2 & -2 \\
0 & -2i & 2i \\
1 & -1 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1/2i & 0 \\
0 & 0 & -1/2i
\end{pmatrix}
\begin{pmatrix}
-1/2 & 0 & 1 \\
-3/4 & 3i/4 & 0 \\
-3/4 & -3i/4 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 1/2 & 0 \\
-1/2 & 0 & 0 \\
-1/2 & 1/4 & 1
\end{pmatrix}.
\]
10 Diagonalization of Matrices and the Solution of Dynamical Systems

Given $A \in \mathbb{R}^{n \times n}$, we are interested in the solution of the dynamical system

\[
\begin{align*}
  x'(t) &= Ax(t), \quad t > 0, \quad (36a) \\
  x(0) &= x_0. \quad (36b)
\end{align*}
\]

The solution of the scalar differential equation

\[
\begin{align*}
  \xi'(t) &= \lambda \xi(t), \quad t > 0, \quad (37a) \\
  \xi(0) &= \xi_0. \quad (37b)
\end{align*}
\]

is given by

\[
\xi(t) = e^{\lambda t} \xi_0. \quad (38)
\]

We will show that the diagonalization of $A$, i.e., the spectral decomposition of $A$ can be used to transform the dynamical system (36) into a set of $n$ independent scalar differential equations of the form (37).

Suppose there exists a nonsingular matrix $V \in \mathbb{C}^{n \times n}$ of eigenvectors and a diagonal matrix $\Lambda \in \mathbb{C}^{n \times n}$ of eigenvalues. (Note that we are guaranteed to find $n$ linearly independent eigenvectors if $A$ is symmetric or if $A$ has $n$ distinct eigenvalues. Note also that there exist matrices $A$ that cannot be diagonalized and we will will discuss later what to do in that case.) Since $V$ is invertible, we have

\[
A = V \Lambda V^{-1}. \quad (39)
\]

If we insert (39) into the dynamical system we obtain

\[
x'(t) = Ax(t) = \underbrace{VAV^{-1}x(t)}_{\text{def} = z(t)}.
\]

If we multiply this equation by $V^{-1}$ and use $\frac{d}{dt} (V^{-1}x(t)) = V^{-1}x'(t)$, we obtain

\[
\begin{align*}
  z'(t) &= \frac{d}{dt} (V^{-1}x(t)) = V^{-1}x'(t) = V^{-1}V\Lambda z(t) = \Lambda z(t).
\end{align*}
\]

At $t = 0$ we have

\[
z_0 \overset{\text{def}}{=} z(0) = V^{-1}x(0) = V^{-1}x_0.
\]

Using the diagonalization of $A$ we have transformed (36) into the system

\[
\begin{align*}
  z'(t) &= \Lambda z(t), \quad t > 0, \quad (40a) \\
  z(0) &= z_0. \quad (40b)
\end{align*}
\]
Since $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix, (40) is equivalent to $n$ independent scalar differential equations.

$$z_1'(t) = \lambda_1 z_1(t), \quad t > 0,$$
$$z_1(0) = z_{1,0},$$
$$\vdots$$
$$z_n'(t) = \lambda_n z_n(t), \quad t > 0,$$
$$z_n(0) = z_{n,0}.$$

Each of these scalar differential equations is of the form (37) and the solutions of these scalar equations are

$$z_1(t) = e^{\lambda_1 t} z_{1,0},$$
$$\vdots$$
$$z_n(t) = e^{\lambda_n t} z_{n,0}.$$

In vector notation,

$$z(t) = \begin{pmatrix} e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{pmatrix} z_0 = \begin{pmatrix} e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{pmatrix} V^{-1} x_0.$$

Since $z(t) = V^{-1} x(t)$, the solution of the system (36) is given by

$$x(t) = V z(t) = V \begin{pmatrix} e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{pmatrix} V^{-1} x_0.$$

Note that for a diagonalizable matrix $A$,

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_{k=0}^{\infty} \frac{1}{k!} (V \Lambda V^{-1})^k = \sum_{k=0}^{\infty} \frac{1}{k!} V \Lambda^k V^{-1} = V \left( \sum_{k=0}^{\infty} \frac{1}{k!} \Lambda^k \right) V^{-1} = V \begin{pmatrix} e^{\lambda_1} \\ \vdots \\ e^{\lambda_n} \end{pmatrix} V^{-1}. \quad (41)$$

Hence,

$$x(t) = V \begin{pmatrix} e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{pmatrix} V^{-1} x_0 = \exp(A t) x_0. \quad (42)$$
If we define
\[ V^{-1} = W = \begin{pmatrix} w_1^T \\ \vdots \\ w_n^T \end{pmatrix}, \]
i.e., \( w_j^T \) is the \( j \)th row of the inverse of \( V \), then
\[ x(t) = V \begin{pmatrix} e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{pmatrix} V^{-1} x_0 = V \begin{pmatrix} e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{pmatrix} W x_0 = \sum_{j=1}^n e^{\lambda_j t} v_j w_j^T x_0. \tag{43} \]

Note the similarity between the solution formulas (38) for scalar equations and (42) for systems. In MATLAB the matrix exponential defined in (41) can be computed using \( \text{expm} \( A \) \). Note that this is very different from the output of the MATLAB command \( \text{exp} \( A \) \) which simply applies the scalar exponential function to every matrix entry of \( A \).

We can extend our technique to dynamical systems
\[ x'(t) = Ax(t) + f(t), \quad t > 0, \tag{44a} \]
\[ x(0) = x_0. \tag{44b} \]
with inhomegeneous right hand sides \( f : [0, \infty) \to \mathbb{R}^n \). The solution of the scalar differential equation
\[ \xi'(t) = \lambda \xi(t) + g(t), \quad t > 0, \tag{45a} \]
\[ \xi(0) = \xi_0 \tag{45b} \]
is given by
\[ \xi(t) = e^{\lambda t} \xi_0 + \int_0^t e^{\lambda(t-\tau)} g(\tau) d\tau. \tag{46} \]

If we insert (39) into the dynamical system with inhomegeneous right hand side, then
\[ x'(t) = Ax(t) + f(t) = V \Lambda V^{-1} x(t) + f(t) = z(t) \]
\[ \overset{\text{def}}{=} z(t) \]
If we multiply this equation by \( V^{-1} \) and use \( \frac{d}{dt} (V^{-1} x(t)) = V^{-1} x'(t) \), we obtain
\[ z'(t) = \frac{d}{dt} (V^{-1} x(t)) = V^{-1} x'(t) = V^{-1} \Lambda z(t) + V^{-1} f(t) = \Lambda z(t) + g(t). \]
\[ \overset{\text{def}}{=} g(t) \]
At $t = 0$ we have

$$z_0 \overset{\text{def}}{=} z(0) = V^{-1}x(0) = V^{-1}x_0.$$  

$$z'(t) = \Lambda z(t) + g(t), \quad t > 0,$$

$$z(0) = z_0.$$  \hfill (47a) \hfill (47b)

Since $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix, (47) is equivalent to $n$ independent scalar differential equations.

$$z_1'(t) = \lambda_1 z_1(t) + g_1(t), \quad t > 0,$$

$$z_1(0) = z_{1,0},$$

$$\vdots$$

$$z_n'(t) = \lambda_n z_n(t) + g_n(t), \quad t > 0,$$

$$z_n(0) = z_{n,0},$$

and the solutions are

$$z_j(t) = e^{\lambda_j t}z_{j,0} + \int_0^t e^{\lambda_j (t-\tau)}g_j(\tau)d\tau.$$  

If we insert this into $x(t) = Vz(t)$, we arrive at

$$x(t) = Vz(t) = V \begin{pmatrix} e^{\lambda_1 t} & \cdots & e^{\lambda_n t} \end{pmatrix} V^{-1}x_0 + V \int_0^t \begin{pmatrix} e^{\lambda_1 (t-\tau)} & \cdots \end{pmatrix} \begin{pmatrix} e^{\lambda_1 (t-\tau)} & \cdots \end{pmatrix} g(\tau)d\tau$$

$$= V \begin{pmatrix} e^{\lambda_1 t} & \cdots \end{pmatrix} V^{-1}x_0 + \int_0^t V \begin{pmatrix} e^{\lambda_1 (t-\tau)} & \cdots \end{pmatrix} V^{-1}f(\tau)d\tau$$

$$= \exp(At)x_0 + \int_0^t \exp(A(t-\tau))f(\tau)d\tau.$$  \hfill (48)

Again, note the similarity between the solution formulas (46) for scalar equations and

$$x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau))f(\tau)d\tau$$  \hfill (48)

for systems.

Before we return to circuit and truss examples, we consider a few simple dynamical systems.
Example 43 The symmetric matrix  

\[
A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}
\]

has eigenvalues \( \lambda_1 = -2, \lambda_2 = -4 \) and corresponding orthonormal eigenvectors \( v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), \( \sqrt{2} \), and \( v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \), \( \sqrt{2} \). In this case  

\[
\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} = A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

We have  

\[
\exp(At) = V \exp(\Lambda t) V^{-1} = \frac{1}{2} \begin{pmatrix} \exp(-2t) + \exp(-4t) & \exp(-2t) - \exp(-4t) \\ \exp(-2t) - \exp(-4t) & \exp(-2t) + \exp(-4t) \end{pmatrix}
\]

Let \( x_0 = (2, 3)^T \). The solutions \( z(t) = \text{diag}(-2, -4)^t z(t) \) and \( x(t) = Ax(t) \) decay exponentially. See Figure 13. Since \( V \) is orthogonal,  

\[
\|x(t)\|_2 = \|Vz(t)\|_2 = z(t)^T V^T V z(t) = z(t)^T z(t) = \|z(t)\|_2^2.
\]

Since \( t \mapsto \|z(t)\|_2^2 \) decays exponentially, \( t \mapsto \|x(t)\|_2^2 \) also decays exponentially. \( \diamond \)

Example 44 The matrix  

\[
A = \begin{pmatrix} 8 & -10 \\ 12 & -14 \end{pmatrix}
\]

has eigenvalues \( \lambda_1 = -2, \lambda_2 = -4 \) with corresponding to eigenvectors \( v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \). Since the eigenvectors are linearly independent,  

\[
\begin{pmatrix} 8 & -10 \\ 12 & -14 \end{pmatrix} = A = \begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix} \frac{-2}{0} \begin{pmatrix} 6 & -5 \\ -1 & 1 \end{pmatrix}
\]

We have  

\[
\exp(At) = V \exp(At) V^{-1} = \begin{pmatrix} 6 \exp(-2t) - 5 \exp(-4t) & -5 \exp(-2t) + 5 \exp(-4t) \\ 6 \exp(-2t) - 6 \exp(-4t) & -5 \exp(-2t) + 6 \exp(-4t) \end{pmatrix}
\]

Let \( x_0 = (2, 3)^T \). The solution \( z(t) = \text{diag}(-2, -4)^t z(t) \) decays exponentially. Since the eigenvectors \( v_1 \) and \( v_2 \) are not orthogonal,  

\[
\|x(t)\|_2^2 \neq \|z(t)\|_2^2.
\]

The solution \( x \) of \( x'(t) = Ax(t) \) no longer decreases monotonically for small \( t \), For large \( t \) the solution \( x \) decays exponentially. See Figure 14. \( \diamond \)
Figure 13: The solutions $z$ of $z'(t) = \Lambda z(t)$ (top left plot) and $x$ of $x'(t) = Ax(t)$ (top right plot) for the matrix in Example 43 decay exponentially since all eigenvalues of $A$ are negative. Since the matrix $V$ of eigenvectors is orthonormal $\|x(t)\|_2 = \|z(t)\|_2$ (bottom plot).
Figure 14: The solution $z$ of $z'(t) = \Lambda z(t)$ (top left plot) for the matrix in Example 44 decay exponentially since all eigenvalues of $A$ are negative. Since the eigenvectors of $A$ are not orthogonal, the solution $x$ of $x'(t) = Ax(t)$ (top right plot) is not monotonically decreasing. For large $t$ the solution $x$ decays exponentially, at the same rate as the solution $z$. The norms of the solutions $z$ and $x$ are shown in the plot in the center of the second row.
**Example 45** The matrix 

\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

has complex eigenvalues \( \lambda_1 = i, \lambda_2 = -i \) and corresponding eigenvectors \( v_1 = (1 -i)^T, v_2 = (1 \ i)^T \). See Example 24. Since \( V \) is invertible,

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = A = \begin{pmatrix} -i & i \\ i & 0 \end{pmatrix} \begin{pmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{pmatrix} = V = \Lambda = V^{-1}.
\]

The complex exponential is 

\[ e^{x+iy} = e^x (\cos(y) + i \sin(y)). \]

Using this expression followed by matrix multiplications gives

\[
\exp(At) = V \exp(\Lambda t)V^{-1} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.
\]

Let \( x_0 = (2, 3)^T \). Since the eigenvalues of \( A \) are purely imaginary (the real parts of the eigenvalues are zero), the solution to the dynamical system \( x'(t) = Ax(t) \) is oscillatory. See Figure 15.

![Figure 15](image-url)

**Figure 15:** The solution \( x \) of \( x'(t) = Ax(t) \) for the matrix in Example 45 oscillates since all eigenvalues of \( A \) are complex and have zero real parts.
Example 46  The matrix

\[ A = \begin{pmatrix} -3/2 & 1 \\ -5/4 & -5/2 \end{pmatrix} \]

has complex eigenvalues \( \lambda_1 = -2 + i \), \( \lambda_2 = -2 - i \) and corresponding eigenvectors \( v_1 = \begin{pmatrix} -2 + i \\ 5 \end{pmatrix} \), \( v_2 = \begin{pmatrix} -2 - i \\ 5 \end{pmatrix} \). Since \( V \) is invertible,

\[
\begin{pmatrix} 3/2 & 1 \\ 5/4 & -5/2 \end{pmatrix} = A = \begin{pmatrix} -2 + i & -2 - i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 + i & 0 \\ 1 & -2 - i \end{pmatrix} = \Lambda = \begin{pmatrix} i/8 & 1/10 + 5i/100 \\ -i/8 & 1/10 - 5i/100 \end{pmatrix} = V^{-1}.
\]

We have

\[
\exp(At) = V \exp(\Lambda t) V^{-1} = \begin{pmatrix} \cos(t) + \frac{1}{2} \sin(t) e^{-2t} \\ -(5/4) \sin(t) e^{-2t} \end{pmatrix} \begin{pmatrix} \sin(t) e^{-2t} \\ \cos(t) - \frac{1}{2} \sin(t) e^{-2t} \end{pmatrix}.
\]

Let \( x_0 = (2, 3)^T \). Since the eigenvalues of \( A \) are complex, the the solution to the dynamical system \( x'(t) = Ax(t) \) is oscillatory. Since the real parts of the eigenvalues of \( A \) are negative, the solution decays. See Figure 16.

![Figure 16](image.png)

Figure 16: The solution \( x \) of \( x'(t) = Ax(t) \) for the matrix in Example 46 oscillates since all eigenvalues of \( A \) are complex and decays, since the real parts of the eigenvalues are negative.
Example 47  The matrix

\[
A = \begin{pmatrix}
-17 & 15 \\
-18 & 16
\end{pmatrix}
\]

has eigenvalues \(\lambda_1 = -2, \lambda_2 = 1\) with corresponding to eigenvectors \(v_1 = (1 \ 1)^T\) and \(v_2 = (5 \ 6)^T\). Since the eigenvectors are linearly independent,

\[
\begin{pmatrix}
-17 & 15 \\
-18 & 16
\end{pmatrix} = A = \begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -5 \\ -1 & 1 \end{pmatrix} = V = \Lambda = V^{-1} = W
\]

The solution of \(x'(t) = Ax(t), \ t > 0\), with initial condition \(x(0) = x_0\) is given by

\[
x(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (6 -5) x_0 + e^t \begin{pmatrix} 5 \\ 6 \end{pmatrix} (-1 \ 1) x_0.
\]

See (43).

If \(x_0 \perp ( -1 \ 1)^T\), then

\[
x(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (6 -5) x_0 + e^t \begin{pmatrix} 5 \\ 6 \end{pmatrix} (-1 \ 1) x_0
\]

\[
= e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (6 -5) x_0 \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ as } t \rightarrow \infty.
\]

For example, if \(x_0 = (1 \ 1)^T\), then the solution \(x(t) \rightarrow 0\) as \(t \rightarrow \infty\).

If \(( -1 \ 1)^T x_0 \neq 0\), then

\[
x(t) = e^{-2t} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) (6 -5) x_0 + e^t \left( \begin{array}{c} 5 \\ 6 \end{array} \right) \left( \begin{array}{c} -1 \\ 1 \end{array} \right) x_0
\]

and \(\|x(t)\| \rightarrow \infty\) as \(t \rightarrow \infty\). Thus even a small perturbation of the initial value \((1 \ 1)^T\) will generate a solution whose norm increases exponentially. This is illustrated in Figure 17. \(\diamond\)
Figure 17: Given the matrix $A$ in Example 47, the solution of $x'(t) = Ax(t)$ with initial value $x_0 = (1, 1)^T$ decays exponentially (left plot) while the solution of $x'(t) = Ax(t)$ with initial value $x_0 = (1.1, 1)^T$ increases (in norm) exponentially (right plot).
11 Dynamical Systems: Mechanical Systems

Earlier, we have derived equations for static mechanical systems such as planar trusses. The vector \( x \in \mathbb{R}^n \) of displacements of the nodes is the solution of the system

\[
Sx = f
\]

of linear equations, where \( S \) is a symmetric matrix and \( f \) is the force vector. Now we allow the displacements to vary with time. The governing equations are derived from ‘force equal mass times acceleration’ and are given by

\[
Mx''(t) + Sx(t) = f(t),
\]

where \( M \) is a diagonal matrix with diagonal entries given by the masses \( m_j > 0, j = 1, \ldots, n \), of the nodes. The right hand side \( f(t) \) is the vector of external forces, which can vary with time. To compute the displacements, we need to specify the initial displacements \( x(0) \) and the initial velocities \( x'(0) \). The displacements of a time varying truss are computed as the solution of the second order dynamical system

\[
\begin{align*}
Mx''(t) + Sx(t) &= f(t), & t > 0, \\
x(0) &= x_0, \\
x'(0) &= x_1.
\end{align*}
\]

Since \( M \) is invertible, (49) is equivalent to

\[
\begin{align*}
x''(t) &= -M^{-1}Sx(t) + M^{-1}f(t), & t > 0, \\
x(0) &= x_0, \\
x'(0) &= x_1.
\end{align*}
\]

We will use the diagonalization of \(-M^{-1}S\) to solve (50). Although \( S \) and \( M \) are symmetric, the matrix \(-M^{-1}S\) is in general not symmetric. Therefore, it is not obvious that \(-M^{-1}S\) can be diagonalized. We will show next that \(-M^{-1}S\) is similar to a symmetric matrix and therefore can be diagonalized.

The matrix \( M \) is diagonal with diagonal entries \( m_j > 0, j = 1, \ldots, n \). We define

\[
M^{1/2} = \begin{pmatrix}
m_1^{1/2} \\
\vdots \\
m_n^{1/2}
\end{pmatrix}.
\]

Clearly, \( M^{1/2}M^{1/2} = M \). We set \( M^{-1/2} = (M^{1/2})^{-1} = (M^{-1})^{1/2} \).
If $\lambda$ is an eigenvalue of $M^{-1}S$ with corresponding eigenvector $v$, then

$$M^{-1}Sv = v\lambda.$$  

This is equivalent to

$$M^{-1/2}SM^{-1/2}v = M^{1/2}v.$$  

Thus, $\lambda$ is an eigenvalue of $M^{-1}S$ with corresponding eigenvector $v$ if and only if $\lambda$ is an eigenvalue of $M^{-1/2}SM^{-1/2}$ with corresponding eigenvector $M^{1/2}v$. (Note that $M^{1/2}$ is invertible and therefore $M^{1/2}v \neq 0$ if and only if $v \neq 0$.) Since $S$ and $M^{-1/2}$ are symmetric the matrix $M^{-1/2}SM^{-1/2}$ is symmetric. Hence, $M^{-1/2}SM^{-1/2}$ has $n$ real eigenvalues and $n$ orthonormal eigenvectors. Consequently, $-M^{-1}S$ has $n$ real eigenvalues and $n$ linearly independent eigenvectors. That is there exists a nonsingular matrix $V \in \mathbb{R}^{n \times n}$ of eigenvectors and a diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that

$$M^{-1}S = V\Lambda V^{-1}. \quad (51)$$

We will show that for the matrices $M$ and $S$ arising in trusses, the eigenvalues of $M^{-1}S$ are positive.

If we insert this into (50a) and define $z(t) = V^{-1}x(t)$ we obtain

$$x''(t) = -V\Lambda V^{-1}x(t) + M^{-1}f(t), \quad t > 0$$

$$z''(t) = \frac{d^2}{dt^2} (V^{-1}x(t)) = V^{-1}x''(t) = -\Lambda z(t) + \underbrace{V^{-1}M^{-1}f(t)}_{= g(t)}, \quad t > 0$$

Hence (50) is equivalent to

$$z''(t) = -\Lambda z(t) + g(t), \quad t > 0, \quad (52a)$$
$$z(0) = z_0 \overset{\text{def}}{=} V^{-1}x_0, \quad (52b)$$
$$z'(0) = z_1 \overset{\text{def}}{=} V^{-1}x_1. \quad (52c)$$

Since the the eigenvalues of $M^{-1}S$ are positive, $\Lambda$ is a diagonal matrix with positive diagonal entries.

The system (52) is a collection of scalar second order differential equations of the type

$$z''_j(t) = -\lambda_j z_j(t) + g_j(t), \quad t > 0, \quad (53a)$$
$$z_j(0) = z_{j,0}, \quad (53b)$$
$$z'_j(0) = z_{j,1}. \quad (53c)$$

where $\lambda_j > 0$. If $g_j(t) = 0$, the solution of (53) is given by

$$z_j(t) = \sin(\sqrt{\lambda_j}t) \frac{1}{\sqrt{\lambda_j}} z_{j,1} + \cos(\sqrt{\lambda_j}t) z_{j,0}. \quad (54)$$
We can now apply the solution formula (54) to obtain \( z(t) \). The solution \( x \) of (50) is then given by 
\[
x(t) = Vz(t).
\]

Alternatively, we can convert (50) into a system of first order equations and then apply the techniques introduced in the previous section. We convert (50) into a system of first order equations by introducing the auxiliary unknown

\[
y(t) = x'(t).\]

Inserting this variable into (50) we arrive at

\[
\begin{align*}
x'(t) &= y(t), & t > 0, \\
y'(t) &= -M^{-1}Sx(t) + M^{-1}f(t), & t > 0, \\
x(0) &= x_0, \\
y(0) &= x_1,
\end{align*}
\]

or, in matrix-vector notation,

\[
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}S & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}f(t) \end{pmatrix}, \quad t > 0, \tag{56a}
\]

\[
\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}. \tag{56b}
\]

To solve (56) with the techniques we have discussed, we need to diagonalize \( B \in \mathbb{R}^{2n \times 2n} \). We will show that the eigenvalues and eigenvectors of \( B \) are closely related to those of \( M^{-1}S \). Let \( \mu \) be an eigenvalue of \( B \). Since \( B \) is a block \( 2 \times 2 \) matrix, it is useful to write the eigenvector of \( B \) corresponding to \( \mu \) as

\[
\begin{pmatrix} w \\ z \end{pmatrix}
\]

where \( w \) and \( z \) are vectors of length \( n \). By definition of eigenvalues and eigenvectors we obtain

\[
\begin{pmatrix} z \\ -M^{-1}Sw \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}S & 0 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \mu \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} \mu w \\ \mu z \end{pmatrix}.
\]

If we insert the first equation \( z = \mu w \) into the second equation \( -M^{-1}Sw = \mu z \) we obtain

\[
-M^{-1}Sw = \mu^2 w.
\]

Thus we have shown that if \( \mu \) is an eigenvalue of \( B \) with eigenvector \( \begin{pmatrix} w \\ z \end{pmatrix} \), then \( \mu^2 \) is an eigenvalue of \( -M^{-1}S \) with eigenvector \( w \). Since \( -M^{-1}S \) is diagonalizable and has negative eigenvalues \( -\lambda_1, \ldots, -\lambda_n \) \( (\lambda_1, \ldots, \lambda_n > 0) \) with corresponding eigenvectors \( v_1, \ldots, v_n \) (see (51)), the
eigenvalues of $B$ are

$$
\mu_1 = \sqrt{\lambda_1} i, \ldots, \mu_n = \sqrt{\lambda_n} i, \mu_{n+1} = -\sqrt{\lambda_1} i, \ldots, \mu_{2n} = -\sqrt{\lambda_n} i,
$$

with corresponding eigenvectors

$$
\begin{pmatrix}
v_1 \\
\sqrt{\lambda_1} i \ v_1 \\
\end{pmatrix}, \ldots, 
\begin{pmatrix}
v_n \\
\sqrt{\lambda_1} i \ v_n \\
\end{pmatrix},
\begin{pmatrix}
-v_1 \\
\sqrt{\lambda_1} i \ v_1 \\
\end{pmatrix}, \ldots, 
\begin{pmatrix}
-v_n \\
\sqrt{\lambda_1} i \ v_n \\
\end{pmatrix}.
$$

**Example 48** As an example we consider the mass-spring system studied in Section 2.2 of the lecture notes. For this system

$$
S = \begin{pmatrix}
k_1 + k_2 & -k_2 & 0 \\
-k_2 & k_2 + k_3 & -k_3 \\
0 & -k_3 & k_3 + k_4
\end{pmatrix}, \quad M = \begin{pmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3
\end{pmatrix}.
$$

We assume that at rest, the masses are positioned at 1, 2 and 3, respectively. The solution of (49) for masses $m_1 = 2, m_2 = 1, m_3 = 1$, spring stiffnesses $k_1 = \ldots, k_4 = 1$, initial displacements $x_0 = (0, 0.5, 0)^T$ and initial velocities $x_1 = (0, 0, 0)^T$ is shown in Figure 18.

![Figure 18: The positions $x_1(t) + 1, x_2(t) + 2, x_3(t) + 3$ of the three masses over time, where the displacements are computed by solving (49) with initial displacements $x_0 = (0, 0.5, 0)^T$ and initial velocities $x_1 = (0, 0, 0)^T$.](image-url)
We can extend the techniques above to damped systems with certain damping matrices. A damped system corresponding to (49) is given by

\[ Mx''(t) + Dx'(t) + Sx(t) = f(t), \quad t > 0, \]  
\[ x(0) = x_0, \]  
\[ x'(0) = x_1 \]  
(57a)

where \( D \in \mathbb{R}^{n \times n} \) represents the damping term. We assume that

\[ D = \alpha M + \beta S \]  
(58)

with some real coefficients \( \alpha, \beta > 0 \).

We can use the diagonalizability of \( M^{-1}S \) to transform (57) into \( n \) scalar differential equations of the type

\[ z''_j(t) = a_jz'_j(t) + b_jz_j(t), \quad t > 0, \]  
\[ z_j(0) = z_{j,0}, \]  
\[ z'_j(0) = z_{j,1} \]  
(59a)

where \( a_j \) and \( b_j \) are coefficients depending on \( \alpha, \beta \) and the eigenvalue \( \lambda_j > 0 \) of \( M^{-1}S \).

If \( \zeta_j = \mu_j \pm i\theta_j \) are the roots of \( \zeta^2 - a_j \zeta - b_j = 0 \), then solution of (59) is given by

\[ z_j(t) = e^{\mu_j t} \sin(\theta_j t) \frac{z_{j,1} \mu_j - b_j z_{j,0}}{\theta_j} + e^{\mu_j t} \cos(\theta_j t) z_{j,0}. \]  
(60)

Alternatively, we can write (57) as a system of first order differential equations

\[
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ * & * \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}f(t) \end{pmatrix}, \quad t > 0,
\]

\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \overset{\text{def}}{=} B \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.
\]

(61a)

(61b)

What is \( B \) for the damped system? Compute the eigenvalues and eigenvectors of \( B \).

**Example 49** We consider the mass spring system of Example 48 and add damping of the form \( D = 0.01M + 0.01S \). The solution of the damped system (57) is shown in Figure 19. \( \diamond \)
Figure 19: The positions $x_1(t)+1$, $x_2(t)+2$, $x_3(t)+3$ of the three masses over time, where the displacements are computed by solving (57) with damping $D = 0.01M + 0.01S$, initial displacements $x_0 = (0, 0.5, 0)^T$ and initial velocities $x_1 = (0, 0, 0)^T$. 
11.1 Positive (Semi-)Definite Matrices

Earlier, in this section we have claimed that the matrix $M^{-1}S$ has positive eigenvalues. This is due to the structure of the stiffness matrix $S$.

A symmetric matrix $S \in \mathbb{R}^{n \times n}$ is called symmetric positive definite if

$$v^T S v > 0 \quad \text{for all vectors } v \neq 0,$$

A symmetric matrix $S \in \mathbb{R}^{n \times n}$ is called symmetric positive semidefinite if

$$v^T S v \geq 0 \quad \text{for all vectors } v.$$

The positive [semi-]definiteness property of a symmetric matrix $S$ is closely related to the properties of the eigenvalues of $S$: If $\lambda$ is an eigenvalues of a symmetric positive [semi-]definite matrix $S$ with corresponding eigenvector $v$, then $Sv = \lambda v$. Hence

$$0 < \left| v^T S v \right| = \| v \|_2^2 \lambda,$$

which implies that the eigenvalues of a symmetric positive [semi-]definite matrix are positive [non-negative].

On the other hand if all eigenvalues $\lambda_1, \ldots, \lambda_n$ of the symmetric matrix are positive [non-negative], then there exists an orthonormal matrix $Q \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ such that

$$S = Q \Lambda Q^T.$$

Hence for a nonzero vector $v$,

$$v^T S v = v^T Q \Lambda Q^T v = \sum_{j=1}^{m} \lambda_j z_j^2 \geq 0.$$

That is, if all eigenvalues $\lambda_1, \ldots, \lambda_n$ of the symmetric matrix are positive [non-negative], then the matrix is positive [semi-]definite.

Thus we have shown that a symmetric matrix $S$ is positive [semi-]definite if and only if its eigenvalues are positive [non-negative].

The stiffness matrices arising in mechanical systems are given by

$$S = A^T K A,$$

where $K = \text{diag}(k_1, \ldots, k_m)$. For a vector $v$ we find

$$v^T S v = v^T A^T K (Av) = z^T K z = \sum_{j=1}^{m} k_j z_j^2 \geq 0.$$

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Thus, the stiffness matrix $S$ is always symmetric positive semidefinite. If the truss is stable, then $\mathcal{N}(A) = \{0\}$. In particular, $z = Av \neq 0$ for $v \neq 0$. Hence, for a stable truss

$$v^T S v = v^T A^T K A v = (Av)^T K Av = z^T K z = \sum_{j=1}^{m} k_j z_j^2 > 0 \text{ for all } v \neq 0.$$  

Thus, the stiffness matrix $S$ for a stable truss is always symmetric positive definite and therefore has only positive eigenvalues. If $S$ is symmetric positive definite, then $M^{-1/2} S M^{-1/2}$ is also positive definite and therefore has only positive eigenvalues. Since $M^{-1/2} S M^{-1/2}$ and $M^{-1} S$ have the same eigenvalues, all eigenvalues of $M^{-1} S$ are positive and the truss is stable.
12 Dynamical Systems: Electrical Circuits

Consider the RC circuit shown in figure 21 with two resistors and two capacitors. Ohm’s Law tells us that the current through a resistor is proportional to the potential drop across it, i.e. $y = e/R$. A capacitor consists of a pair of charged plates, separated by a gap, often filled with a dielectric material. The charges create an electric field between the plates, and a corresponding voltage drop $e = x_1 - x_2$, shown in figure 20. Since the time derivative of the charge $Q'(t)$ is equal to the current

$$y(t) = Ce'(t),$$

where $C$ is the capacitance of the capacitor, and the prime notation denotes differentiation with respect to time. Furthermore, the current stimulus $V(t)$ is now a function varying in time. Thus, we have moved from the static arena (constant in time) to the dynamic arena (varying in time). We now use the four steps outlined in Chapter 1 of the Course Notes to model the RC circuit.

Figure 20: The capacitor obeys the equation $Q = Ce$, where $e = x_1 - x_2$ is the voltage drop across the capacitor, and $C$ is the capacitance (measured in Farads), which depends on the geometry of the plates, and the properties of the material in the gap.

$y(t)$, the current through a capacitor is proportional to the time rate of change of the potential drop across it, i.e.

$\frac{y}{x_1} \frac{y}{x_2}$

Figure 21: A Simple RC Circuit

Step 1. Potential Drops
The convention is to measure the potential drops across each element in the circuit as “tail-minus-head”. Thus, we have the equations

\[ e_1 = V(t) - x_1 \]
\[ e_2 = x_1 - x_2 \]
\[ e_3 = x_1 - 0 \]
\[ e_4 = x_2 - 0 \]

These equations can be written in the form of \( e(t) = b(t) - Ax(t) \) where

\[
A = \begin{pmatrix}
1 & 0 \\
-1 & 1 \\
-1 & 0 \\
0 & -1
\end{pmatrix}
\text{ and } b(t) = \begin{pmatrix}
V(t) \\
0 \\
0 \\
0
\end{pmatrix}.
\]

**Step 2.** Resistance and Capacitance

For the resistors, we apply Ohm’s Law:

\[ y_1 = \frac{e_1}{R_1} \]
\[ y_2 = \frac{e_2}{R_2} \]

For the capacitors, we have:

\[ y_3 = C_1 e_3' \]
\[ y_4 = C_2 e_4' \]

Thus, we have \( y(t) = Ge(t) + Ce'(t) \) where

\[
G = \begin{pmatrix}
1/R_1 & 0 & 0 & 0 \\
0 & 1/R_2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\text{ and } C = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & C_1 & 0 \\
0 & 0 & 0 & C_2
\end{pmatrix}.
\]

**Step 3:** Kirchoff’s Current Law

At each node, we balance the current going in with the current going out. Thus,

\[ y_1 - y_2 - y_3 = 0 \]
\[ y_2 - y_4 = 0 \]

or \( A^T y(t) = 0 \).
Step 4: Assemble previous steps.

We have

\[ 0 = A^T y(t) = A^T [Ge(t) + Ce'(t)] = A^T [G(b(t) - Ax(t)) + C(b'(t) - Ax'(t))] = A^T Gb(t) - A^T GAx(t) + A^T Cb'(t) - A^T CAx'(t) \]

which is equivalent to

\[ A^T CAx'(t) = -A^T GAx(t) + A^T Gb(t) + A^T Cb'(t). \]

Simple matrix multiplication gives

\[ A^T GA = \begin{pmatrix} 1/R_1 + 1/R_2 & -1/R_2 \\ -1/R_2 & 1/R_2 \end{pmatrix}, \quad A^T Gb(t) = \begin{pmatrix} V(t)/R_1 \\ 0 \end{pmatrix}, \]

\[ A^T CA = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \quad A^T Cb'(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

We can easily compute

\[ (A^T CA)^{-1} = \begin{pmatrix} 1/C_1 & 0 \\ 0 & 1/C_2 \end{pmatrix}. \]

So

\[ x'(t) = Bx(t) + g(t), \quad (62) \]

where

\[ B = -(A^T CA)^{-1}(A^T GA) = \begin{pmatrix} \frac{1}{C_1} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) & \frac{1}{C_1} \frac{1}{R_2} \\ \frac{1}{C_2} \frac{1}{R_2} & -\frac{1}{C_2} \frac{1}{R_2} \end{pmatrix}, \]

and

\[ g(t) = (A^T CA)^{-1}A^T Gb(t) = \begin{pmatrix} \frac{V(t)}{C_1 R_1} \\ 0 \end{pmatrix}. \]

12.1 Inductors

The present framework can easily be extended to include inductors. An inductor is made with a coil of wire, often surrounding a magnetic material. With constant current, it creates a magnetic field. Changes in the current cause the magnetic flux through the coil to change with time, which induces a back EMF (a potential difference, measured in volts) in the wire due to Faraday’s law. It is shown in figure 22.

The steps for the Strang Quartet must be modified as follows to account for a circuit with inductors.
Figure 22: The inductor obeys the equation $e(t) = Ly'(t)$, where $e(t) = x_1(t) - x_2(t)$ is the voltage drop across the inductor, $y'(t)$ is the time derivative of the current through the inductor, and $L$ is the inductance (measured in Henrys), a property determined by the geometry of the inductor, the number of coils, and by the magnetic properties of the material enclosed by the coils.

**Step 1. Potential Drops**

No change from the previous description; one still computes the voltage drop across inductor elements using “tail-minus-head.” The result is $e(t) = b(t) - Ax(t)$, as usual.

**Step 2. Inductance**

Recall that we obtained for an RC circuit a matrix equation for the current of the form

$$y(t) = Ge(t) + Ce'(t).$$

If circuit element $j$ is an inductor, we have $e_j(t) = L_j y'_j(t)$. Since

$$y_j(t) = y_j(0) + \int_0^t y'_j(\tau) \, d\tau = y_j(0) + \frac{1}{L_j} \int_0^t e_j(\tau) \, d\tau.$$

Thus, we can write

$$y'(t) = Ge'(t) + Ce''(t) + Le(t),$$

where $L$ is a diagonal matrix whose $j$th diagonal entry is $1/L_j$ if circuit element $j$ is an inductor, and 0 otherwise.

**Step 3: Kirchoff’s Current Law**

Since $A^T y(t) = 0$, we also have that $A^T y'(t) = 0$.

**Step 4: Assemble previous steps**

We have

$$0 = A^T y'(t) = A^T [Ge'(t) + Ce''(t) + Le(t)] = A^T \left[ G (b'(t) - Ax'(t)) + C (b''(t) - Ax''(t)) + L (b(t) - Ax(t)) \right].$$

This is a second-order system in $x$. We can go about solving it using techniques similar to those used in section 11. See the next section for an example using the Laplace transform, a technique that we will learn later in the course.
12.2 Application to Power Transmission

Consider the following simple model of a transmission line:

The component on the far left is an AC voltage source. Its functional form is

\[ V(t) = V_0 \cos(\omega t). \]

In the U.S., \( \omega = 120\pi \text{ rad/s} \) since our AC power operates at a frequency of 60 Hz. \( R_1 \) and \( L_1 \) represent the impedance of the transmission line. Later we will assume that we have a 200 km transmission line, whose resistance is 0.05 \( \Omega/\text{km} \) and inductance is 0.001 H/km (so that \( R_1 = 10\Omega \) and \( L_1 = 0.2\text{H} \)). Finally, \( R_2 \) represents the load that the voltage source has been built to power. It could represent the electricity demands of a town, for instance.

The above system may be represented as

\[ x'(t) = Bx(t) + g(t), \]

with

\[ B = -(A^T GA)^{-1}(A^T LA), \quad g(t) = (A^T GA)^{-1}(A^T Gb'(t) + A^T Lb(t)). \]

As usual,

\[ e(t) = \begin{pmatrix} V(t) \\ 0 \\ 0 \\ b(t) \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} x(t). \]

Now with the currents we must account for the inductor along with the resistors. Since \( y_2(t) = y_2(0) + \frac{1}{L_1} \int_0^t e(t) \, dt \) we have:

\[ y(t) = \begin{pmatrix} \frac{1}{R_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R_2} \end{pmatrix} e(t) + \begin{pmatrix} 0 \\ y_2(0) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/L_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \int_0^t e(t) \, dt. \]

Now, Kirchoff’s current law gives

\[ A^T y(t) = 0. \]
Taking the derivative of both sides and substituting in for $y$ and then $e$ we get:

\[
\begin{align*}
A^T y'(t) &= 0 \\
A^T G e'(t) + A^T L e(t) &= 0 \\
A^T G b'(t) - A^T G A x'(t) + A^T L b(t) - A^T L A x(t) &= 0 \\
A^T G A x'(t) &= - A^T L A x(t) + A^T G b'(t) + A^T L b(t)
\end{align*}
\]

and we are done as long as $A^T G A$ is invertible. In fact,

\[
A^T G A = \begin{pmatrix} 1/R_1 & 0 \\ 0 & 1/R_2 \end{pmatrix}
\]

which is certainly invertible for physically reasonable values of $R_1$ and $R_2$ ($> 0$ and $< \infty$).

We will analyze this circuit using the Laplace transform; see sections 6.1-6.4 in the course notes. Upon application of the Laplace transform to this first-order system of ODE’s, we obtain the resolvent matrix $(sI - B)^{-1}$. The poles of the resolvent are the values $s$ for which the resolvent matrix does not exist. These values are the eigenvalues of $B$.

\[
B = -(A^T G A)^{-1} A^T L A = \begin{pmatrix} -R_1/L_1 & R_1/L_1 \\ R_2/L_1 & -R_2/L_1 \end{pmatrix}
\]

\[
sI - B = \begin{pmatrix} s + R_1/L_1 & -R_1/L_1 \\ -R_2/L_1 & s + R_2/L_1 \end{pmatrix}
\]

\[
(sI - B)^{-1} = \frac{1}{s(sL_1 + R_1 + R_2)} \begin{pmatrix} sL_1 + R_2 & R_1 \\ R_2 & sL_1 + R_1 \end{pmatrix}
\]

Here, we have

\[
\det(sI - B) = (s + R_1/L_1)(s + R_2/L_1) - R_1 R_2 / L_1^2 = s \left( s + \frac{R_1 + R_2}{L_1} \right),
\]

such that the eigenvalues of $B$ are 0 and $-R_1 + R_2 / L_1$.

Next, we calculate the Laplace transform $\mathcal{L} x$, assuming that

\[
\begin{align*}
x_i(0) &= \frac{V_p R_2}{R_1 + R_2}, \quad i = 1, 2. \\
\mathcal{L} x(s) &= (sI - B)^{-1} (\mathcal{L} g(s) + x(0))
\end{align*}
\]
We have everything except $Lg(s)$ ...

$$g(t) = \begin{pmatrix} V'(t) \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega V_p \sin(\omega t) \\ 0 \end{pmatrix} \Rightarrow Lg(s) = \begin{pmatrix} -\frac{\omega^2 V_p}{s^2 + \omega^2} \\ 0 \end{pmatrix}.$$

Now assemble ...

$$Lx(s) = \frac{1}{s(sL_1 + R_1 + R_2)} \begin{pmatrix} sL_1 + R_2 & R_1 \\ R_2 & sL_1 + R_1 \end{pmatrix} \begin{pmatrix} -\frac{\omega^2 V_p}{s^2 + \omega^2} \\ 0 \end{pmatrix} + \frac{V_p R_2}{R_1 + R_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{s(sL_1 + R_1 + R_2)} \begin{pmatrix} V_p R_2 (sL_1 + R_2) \\ s^2 V_p R_2 \end{pmatrix} \begin{pmatrix} \frac{1}{R_1 + R_2} \\ \frac{1}{R_1 + R_2} \end{pmatrix}$$

$$= \frac{V_p R_2}{s(R_1 + R_2)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{\omega^2 V_p}{s(sL_1 + R_1 + R_2)(s^2 + \omega^2)} \begin{pmatrix} sL_1 + R_2 \\ R_2 \end{pmatrix}$$

The solution $x(t)$ can be calculated using Matlab’s symbolic toolbox and the \texttt{ilaplace} function. See the code below:

```matlab
%% No capacitor system. Set-up.
syms R1 L1 R2 Vp om y20
syms t s
V = Vp*cos(om*t);
b = [V; 0; 0];
A = [1 0; -1 1; 0 -1];
G = diag([1/R1 0 1/R2]);
y0 = [0; y20; 0];
L = diag([0 1/L1 0]);
B = -inv(A'*G*A)*A'*L*A;
g = inv(A'*G*A)*(A'*G*diff(b,'t') + A*L*b);

%% Transform.
R = inv(s*eye(2) - B);
Lg = laplace(g);
x0 = (Vp*R2/(R2 + R1))*[1; 1];
Lx = R*(Lg+x0);
```

% Invert and analyze.
x1 = ilaplace(Lx);
e1 = b - A*x1;
y1 = G*e1 + y0 + L*int(e1,'t');
P1 = e1(3)*y1(3);
x1a = maple('eval',x1,'[R1 = 10, L1 = 0.2, R2 = 250, Vp = 100000, om = 2*pi*60]');
ezplot(x1a(2),[0,1]);

Now, consider adding a capacitor in parallel with the load:

This system may be represented as

$$Hz'(t) = Bz(t) + g(t),$$

where

$$z(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}, \quad H = \begin{pmatrix} I & 0 \\ 0 & A^TCA \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -A^TLA \end{pmatrix}$$

and

$$g(t) = \begin{pmatrix} 0 \\ A^TCb''(t) + A^TGb'(t) + A^TLb(t) \end{pmatrix}.$$ 

Now, we have four voltage drops:

$$e(t) = \begin{pmatrix} V(t) \\ 0 \\ 0 \\ b(t) \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} x(t),$$

and a capacitor in addition to our inductor and two resistances:

$$y(t) = \begin{pmatrix} 1/R1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/R2 \end{pmatrix} e(t) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e'(t) + \begin{pmatrix} 0 \\ y2(0) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/L1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int_0^t e(t)dt.$$
Again we use $A^T y'(t) = 0$:

$$A^T y'(t) = 0$$

$$A^T C e''(t) + A^T G e'(t) + A^T L e(t) = 0$$

$$A^T C b''(t) - A^T C A x''(t) + A^T G b'(t) - A^T G A x'(t) + A^T L b(t) - A^T L A x(t) = 0$$

$$A^T C A x''(t) = -A^T G A x'(t) - A^T L A x(t) + A^T C b''(t) + A^T G b'(t) + A^T L b(t)$$

To transform this into a first-order system, set $z_1(t) = x(t)$, $z_2(t) = x'(t)$. Then we can write

$$
\begin{pmatrix}
I & 0 \\
0 & A^T C A
\end{pmatrix}
\begin{pmatrix}
z_1'(t) \\
z_2'(t)
\end{pmatrix} =
\begin{pmatrix}
0 & I \\
-A^T L A & -A^T G A
\end{pmatrix}
\begin{pmatrix}
z_1(t) \\
z_2(t)
\end{pmatrix} +
\begin{pmatrix}
0 \\
A^T C b''(t) + A^T G b'(t) + A^T L b(t)
\end{pmatrix}
\begin{pmatrix}
z_1(t) \\
z_2(t)
\end{pmatrix}.$$

Note that

$$A^T C A = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix},$$

which is not invertible. This is what prohibits us from putting this system into the form $x'(t) = B x(t) + g(t)$.

Next, we compute the Laplace transform $\mathcal{L}(z(s))$, given that $H z'(t) = B z(t) + g(t)$. We start by taking the Laplace transform of both sides and applying the properties discussed in class:

$$\mathcal{L}(H z'(t)) = \mathcal{L}(B z(t) + g(t))$$

$$H(s\mathcal{L}(z(s)) - z(0)) = B \mathcal{L}(z(s)) + \mathcal{L}(g(s))$$

Now solve for $\mathcal{L}(z(s))$:

$$(sH - B) \mathcal{L}(z(s)) = \mathcal{L}(g(s)) + H z(0)$$

$$\mathcal{L}(z(s)) = (sH - B)^{-1}(\mathcal{L}(g(s)) + H z(0))$$

We can once again use Matlab’s symbolic toolbox to calculate the Laplace transform of $z$ and then the value of $z(t)$ for this system, assuming that

$$x_i(0) = \frac{V_p R_2}{R_1 + R_2}, \quad x_i'(0) = 0, \quad i = 1, 2.$$

See the code below:

```matlab
%% System with capacitor.

syms R1 L1 R2 C Vp om y20
syms t s

V = Vp*cos(om*t);
```

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12.2 Application to Power Transmission

\[ b = [V; 0; 0; 0]; \]
\[ A = [1 0; -1 1; 0 -1; 0 -1]; \]
\[ G = \text{diag}([1/R1 0 0 1/R2]); \]
\[ Cm = \text{diag}([0 0 C 0]); \]
\[ y0 = [0; y20; 0; 0]; \]
\[ L = \text{diag}([0 1/L1 0 0]); \]

\[ H = [\text{eye}(2) \ zeros(2,2); \ zeros(2,2) \ A' \ Cm \ A]; \]
\[ B = [\zeros(2,2) \ \text{eye}(2); \ -A' \ L \ A ; \ -A' \ G \ A]; \]
\[ g = [\zeros(2,1); \ A' \ Cm \ \text{diff} \ (\text{diff}(b,'t'),'t') + A' \ G \ \text{diff}(b,'t') + A' \ L \ b]; \]

%% Transform.
\[ R = \text{inv}(s \ H - B); \]
\[ Lg = \text{laplace}(g); \]
\[ z0 = [(Vp \ R2/(R2 + R1)); (Vp \ R2/(R2 + R1)); 0; 0]; \]
\[ Lz = R \*(Lg + H \* z0); \]

%% Invert and analyze.
\[ z = \text{ilaplace}(Lz); \]
\[ x2 = z(1:2); \]
\[ e2 = b - A' \* x2; \]
\[ y2 = G \* e2 + Cm \* \text{diff} \ (e2,'t') + y0 + L \* \text{int} \ (e2,'t'); \]
\[ P2 = e2(4) \* y2(4); \]

We will now analyze the power dissipated by \( R_2 \) (the amount consumed by the load we aim to supply), with some given values for the circuit parameters: \( V_p = 100 \text{ kV}, \ R_1 = 10 \Omega, \ L_1 = 0.2 \text{ H}, \ R_2 = 250 \Omega, \ \omega = 120\pi \text{ rad/sec}. \) We then plot the result for the first system for \( 5 \text{ sec} \leq t \leq (5 + 1/20) \text{ sec} \), and make three plots for the second system over the same interval, one each with \( C = 1 \mu F, \ C = 10 \mu F \) and \( C = 50 \mu F \). This is done by the following code:

\[
\text{x2a} = \text{maple}('\text{eval}',x2,'[R1 = 10, L1 = 0.2, R2 = 250, C = 1E-6, ... 
\text{Vp} = 100000, \ \text{om} = 2*\text{pi}*60']); \\
\text{ezplot(x2a(2),[0,1])}; \\
\]

\text{% Plot power}
\[
\text{P1a} = \text{maple}('\text{eval}',P1,'[R1 = 10, L1 = 0.2, R2 = 250, Vp = 100000, \ \text{om} = 2*\text{pi}*60']); \\
\text{P2a} = \text{maple}('\text{eval}',P2,'[R1 = 10, L1 = 0.2, R2 = 250, Vp = 100000, \ \text{om} = 2*\text{pi}*60']); \\
\text{figure} \\
\text{ezplot(P1a,[5,5+1/20]);} \\
\text{title('System 1 Power through R_2');} \\
\text{xlabel('t (s)');}
\]
ylabel('P_1 (W)');
figure
h = ezplot(maple('eval',P2a,'C=1E-6'),[5,5+1/20]);
title('System 2 Power through R_2, C = 1E-6 F');
xlabel('t (s)');
ylabel('P_2 (W)');
set(h,'Color','k');
figure
h = ezplot(maple('eval',P2a,'C=10E-6'),[5,5+1/20]);
title('System 2 Power through R_2, C = 10E-6 F');
xlabel('t (s)');
ylabel('P_2 (W)');
set(h,'Color','k');
figure
h = ezplot(maple('eval',P2a,'C=50E-6'),[5,5+1/20]);
title('System 2 Power through R_2, C = 50E-6 F');
xlabel('t (s)');
ylabel('P_2 (W)');
set(h,'Color','k');
Pavg = 20E6;
Pavg1 = int(60*P1a,'t',5,5+1/60);
Pavg2_1 = int(60*maple('eval',P2a,'C=1E-6'),'t',5,5+1/60);
Pavg2_10 = int(60*maple('eval',P2a,'C=10E-6'),'t',5,5+1/60);
Pavg2_50 = int(60*maple('eval',P2a,'C=50E-6'),'t',5,5+1/60);
Pavg1/Pavg
Pavg2_1/Pavg
Pavg2_10/Pavg
Pavg2_50/Pavg

with the plots shown below:
Average power dissipated by each system: $P_{1,\text{avg}} = 17.0\,\text{MW}$, $P_{2,1\mu F,\text{avg}} = 17.9\,\text{MW}$, $P_{2,10\mu F,\text{avg}} = 29.1\,\text{MW}$, and $P_{2,50\mu F,\text{avg}} = 51.9\,\text{MW}$.

Ratio of average power dissipated to $P_{\text{avg}}$, for each system: $\frac{P_{1,\text{avg}}}{P_{\text{avg}}} = 0.853$, $\frac{P_{2,1\mu F,\text{avg}}}{P_{\text{avg}}} = 0.896$, $\frac{P_{2,10\mu F,\text{avg}}}{P_{\text{avg}}} = 1.46$, $\frac{P_{2,50\mu F,\text{avg}}}{P_{\text{avg}}} = 2.59$.

Adding a capacitor in parallel with the load counteracts the effects of impedance (resistance and inductance) in the transmission line, at least in terms of the amount of power available to the load. In fact, if the capacitance is large enough, the amount of power available can be amplified to values significantly greater than what would be available if there was no impedance in the transmission line. However, we suspect that such amplification is not free (with regards to the input energy required to maintain the amplitude of $V(t)$)...


13  The Jordan Normal Form

13.1  Definition of the Jordan Normal Form

In the previous sections, we have studied several applications of the diagonalization of matrices $A \in \mathbb{R}^{n \times n}$. Unfortunately, not all matrices can be diagonalized as we have seen already in Example 25. This section briefly explains that we can always find vectors that put $A$ in an almost diagonal form if we are willing to relax our notion of the eigenvector. This leads to the so-called Jordan Normal Form, which can also be seen as a consequence of the spectral representation developed in Chapter 11 of the course notes *Linear Algebra in Situ*.

We begin with an example.

**Example 50**  The matrix

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

has eigenvalues $\lambda_1 = -2$, $\lambda_2 = \lambda_3 = 4$. The corresponding eigenspaces are

$$\mathcal{N}(-2I - A) = \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{N}(4I - A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

We can insert any eigenvector corresponding to the eigenvalue 4 into the third column of the matrix $V$ to obtain

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 1 & * \\ 1 & -1 & * \\ 1 & 1 & * \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

However, since the geometric multiplicity of the eigenvalue 4 is one, any eigenvector corresponding to the eigenvalue 4 is just a nonzero multiple of the second columns of $V$ and the resulting $V$ is not invertible. Thus, since the dimension of the eigenspace $\mathcal{N}(4I - A) (= \text{the geometric multiplicity of the eigenvalue 4})$ is less than the algebraic multiplicity of the eigenvalue 4, the matrix $A$ is not diagonalizable.

In Example 50, we had only two linearly independent eigenvectors, so it was impossible to come up with a diagonal form for $A$ via

$$A = V \Lambda V^{-1},$$
as we are “missing” a third vector for $V$. It turns out that we can always find vectors that can put $A$ in an almost diagonal form if we are willing to relax our notion of the eigenvector. The rest of this section briefly develops these ideas into the Jordan Normal Form.

Let $\lambda_j$ be an eigenvalue of $A$ with algebraic multiplicity $m_j$. Non-zero elements of the null-space

$$\mathcal{N}(A - \lambda_j I)^{m_j}$$

are called generalized eigenvectors of $A$, corresponding to the eigenvalue $\lambda_j$. Since

$$\mathcal{N}(A - \lambda_j I) \subset \mathcal{N}(A - \lambda_j I)^{m_j}$$
eigenvectors of $A$ are also generalized eigenvectors.

Let $\lambda$ be an eigenvalue of $A$. Suppose that $v \neq 0$ satisfies $(A - \lambda I)^m v = 0$, and that $m > 0$ is the smallest integer such that this equation holds. From the vector $v$, we obtain a cycle of generalized eigenvectors

$$v, (A - \lambda I)v, \ldots, (A - \lambda I)^{m-1}v.$$ 

Since $(A - \lambda I)^{m-1}v \neq 0$ (by definition of $m$) and $(A - \lambda I)\left((A - \lambda I)^{m-1}v\right) = (A - \lambda I)^{m}v = 0$, the vector $(A - \lambda I)^{m-1}v$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$. All other vectors $v, (A - \lambda I)v, \ldots, (A - \lambda I)^{m-2}v$ in the cycle are not eigenvector of $A$.

The next theorem gives linear independence of this cycle. These cycles will turn out to contain the missing basis vectors required to complete the matrix $V$, and render $A$ into an almost diagonal form.

**Theorem 51 (Linear Independence Generalized Eigenvector Cycles)** Let $(A - \lambda I)^{m}v = 0$ and $(A - \lambda I)^{m-1}v \neq 0$, i.e., let $v$ be a generalized eigenvector of $A$ with eigenvalue $\lambda$. The generalized eigenvectors in the cycle

$$v, (A - \lambda I)v, \ldots, (A - \lambda I)^{m-1}v$$

are linearly independent.

**Proof:** By induction. Suppose that $(A - \lambda I)^{k}, \ldots, (A - \lambda I)^{m-1}$ are linearly independent. We have to show that $(A - \lambda I)^{k+1}, \ldots, (A - \lambda I)^{m-1}$ are also linearly independent.

Let

$$\sum_{j=k-1}^{m-1} \alpha_j (A - \lambda I)^j v = 0.$$ 

Multiply by $(A - \lambda I)$ to obtain

$$0 = (A - \lambda I)\sum_{j=k-1}^{m-1} \alpha_j (A - \lambda I)^j v = \sum_{j=k-1}^{m-1} \alpha_j (A - \lambda I)^{j+1} v$$

$$= \left(\sum_{j=k}^{m-1} \alpha_{j-1} (A - \lambda I)^j v\right) + \alpha_{m-1} (A - \lambda I)^m v.$$
Thus, $\alpha_{k-1} = \ldots = \alpha_{m-2} = 0$, and therefore,

$$0 = \sum_{j=k-1}^{m-1} \alpha_j (A - \lambda I)^j v = \alpha_{m-1} (A - \lambda I)^{m-1} v \Rightarrow \alpha_{m-1} = 0.$$

For any matrix $A \in \mathbb{R}^{n \times n}$, it can be shown (see Chapter 11 of the course notes *Linear Algebra in Situ*) that we can find $n$ linearly independent generalized eigenvectors of $A$. When the generalized eigenvectors are appropriately chosen, the matrix

$$V^{-1}AV$$

can be reduced to *Jordan Canonical Form* or *Jordan Normal Form*.

**Theorem 52 (Jordan Normal Form)** For any square matrix $A \in \mathbb{R}^{n \times n}$ there exists a nonsingular matrix $V \in \mathbb{C}^{n \times n}$ such that

$$V^{-1}AV = \begin{pmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & J_k \end{pmatrix} \overset{\text{def}}{=} J,$$

(63)

where

$$J_j = \begin{pmatrix} \lambda_j & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_j & 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_j & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda_j \end{pmatrix} \in \mathbb{C}^{n_j \times n_j}, \quad j = 1, \ldots, k,$$

are the so-called Jordan blocks, $\lambda_1, \ldots, \lambda_k$ are eigenvalues of $A$, and $\sum_{j=1}^{k} n_j = n$. Moreover, let

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 & \ldots & 0 \\ 0 & \Lambda_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \Lambda_k \end{pmatrix},$$
where
\[
\Lambda_j = \begin{pmatrix}
\lambda_j & 0 & 0 & \ldots & 0 & 0 \\
0 & \lambda_j & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & \lambda_j & 0 \\
0 & 0 & 0 & \ldots & 0 & \lambda_j
\end{pmatrix} \in \mathbb{C}^{n_j \times n_j}, \quad j = 1, \ldots, k.
\]
Furthermore, we can split \(A = S + N\), where \(S = V \Lambda V^{-1}\) is diagonalizable, and the matrix \(N\) is nilpotent of order \(k\), i.e., satisfies \(N^k = 0, N^{k-1} \neq 0\), for some \(k \leq n\). In addition, \(NS = SN\).

The computation of the Jordan Normal Form can be complicated and, except for simple cases, we will not compute it. The next section gives a rough sketch of the computation of the Jordan Normal Form. However, as mentioned already, we will not compute the Jordan Normal Form except for simple cases. Instead, we use the existence of the Jordan Normal Form and basic properties of the Jordan blocks to learn about the behavior of solutions of dynamical systems and other problems.

### 13.2 Computing the Jordan Normal Form

From (63) we find that
\[
AV = V \begin{pmatrix}
J_1 & 0 & \ldots & 0 \\
0 & J_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & J_k
\end{pmatrix}.
\]
To see how we can use this to compute the columns of \(V\), we partition the columns of \(V\) according to the blocks in \(J\). We set
\[
V = \left( v_{1,1}, \ldots, v_{1,n_1}, \ldots, v_{k,1}, \ldots, v_{k,n_k} \right).
\]
Note that \(v_{1,1}\) is the first column of \(V\).

If we insert this and use the rules for matrix-matrix multiplication, we see that
\[
A\left( v_{j,1}, \ldots, v_{j,n_j} \right) = \left( v_{j,1}, \ldots, v_{j,n_j} \right) J_j
\]
for \(j = 1, \ldots, k\). Using the structure of the Jordan block \(J_j\) we find that
\[
Av_{j,1} = \lambda_j v_{j,1}, \quad Av_{j,2} = \lambda_j v_{j,2} + v_{j,1}, \quad \ldots \quad Av_{j,n_j} = \lambda_j v_{j,n_j} + v_{j,n_j-1}.
\]
The first identity states that \( v_{j,1} \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda_j \). The others are part of a generalized eigenvector cycle.

Computation of the Jordan Normal Form (in the case of real eigenvalues) can be summarized by the following procedure. We assume that \( \lambda_j \in \mathbb{R} \) is an eigenvalue of \( A \) with algebraic multiplicity \( m_j \). Define

\[ \delta_k = \dim \mathcal{N}(A - \lambda_j I)^k. \]

1. Compute a basis \( \{v_1, v_2, \ldots, v_{\delta_1}\} \) for \( \mathcal{N}(A - \lambda_j I) \). The geometric multiplicity of \( \lambda_j \) is \( \delta_1 \). Arrange the basis as the columns of the matrix \( V \):

\[ V = (v_1 \ v_2 \ \cdots \ v_{\delta_1}). \]

2. If \( \delta_1 < \delta_2 \), find a basis \( \{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{\delta_1}\} \) for \( \mathcal{N}(A - \lambda_j I) \), with

\[ \tilde{V} = (\tilde{v}_1 \ \tilde{v}_2 \ \cdots \ \tilde{v}_k) = VC, \]

\( (C \in \mathbb{R}^{k \times k} \) is an invertible matrix\), such that there are \( \delta_2 - \delta_1 \) linearly independent solutions \( w_1, \ldots, w_{\delta_2 - \delta_1} \) to

\[ (A - \lambda I)w_j = \tilde{v}_j. \]

This leads to a matrix of generalized eigenvectors

\[ \begin{pmatrix} \tilde{v}_1 & w_1 & \tilde{v}_2 & w_2 & \cdots & \tilde{v}_{\delta_2 - \delta_1} & w_{\delta_2 - \delta_1} | \tilde{v}_{\delta_2 - \delta_1 + 1} & \cdots & \tilde{v}_{\delta_1} \end{pmatrix}. \]

3. If \( \delta_2 < \delta_3 \), repeat the process...

4. Continue until the matrix \( V \) contains a complete set of \( m_j \) generalized eigenvectors. Note that at each step where \( \delta_k < \delta_{k+1} \), this will involve recombining the basis vectors in \( V \), as described in step 2.

We illustrate the computation of the Jordan Normal Form in the following two examples.

**Example 53** Consider the matrix \( A \) in Example 50. The eigenvalue \(-2\) is simple. The eigenvalue \( 4 \) has a geometric multiplicity of 1 that is less than its algebraic multiplicity of 2. Therefore, the Jordan normal form of \( A \) is given by

\[
\begin{pmatrix}
5 & 4 & 3 \\
-1 & 0 & -3 \\
1 & -2 & 1 \\
\end{pmatrix} = A \quad \begin{pmatrix}
-1 & 1 & * \\
1 & -1 & * \\
1 & 1 & * \\
\end{pmatrix} = V \quad \begin{pmatrix}
-2 & 0 & 0 \\
1 & -1 & * \\
-1 & 1 & * \\
0 & 4 & 1 \\
\end{pmatrix} = V \quad \begin{pmatrix}
-2 & 0 & 0 \\
1 & -1 & * \\
-1 & 1 & * \\
0 & 4 & 1 \\
\end{pmatrix} = J.
\]
The missing column, denoted by $v_{2,2}$ in the notation used above, is obtained from
\[
\begin{pmatrix}
5 & 4 & 3 \\
-1 & 0 & -3 \\
1 & -2 & 1
\end{pmatrix} v_{2,2} = 4 v_{2,2} + \begin{pmatrix}
1 \\
-1 \\
1
\end{pmatrix} = v_{2,1},
\]
i.e., $(4I - A)v_{2,2} = -v_{2,1}$.

Solving $(4I - A)v_{2,2} = -v_{2,1}$ gives
\[
v_{2,2} = \begin{pmatrix} 1 + \alpha \\ -\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha v_{2,1}, \quad \alpha \in \mathbb{R}.
\]

Note that $(4I - A)$ is singular and $\text{span}\{v_{2,1}\} = \mathcal{N}(4I - A)$, which can be seen from $v_{2,2} = (1, 0, 0)^T + \alpha v_{2,1}$. We find
\[
\begin{pmatrix}
5 & 4 & 3 \\
-1 & 0 & -3 \\
1 & -2 & 1
\end{pmatrix} = A
\begin{pmatrix}
-1 & 1 & 2 \\
1 & -1 & -1 \\
1 & 1 & 1
\end{pmatrix} = V
\begin{pmatrix}
-1 & 1 & 2 \\
1 & -1 & -1 \\
1 & 1 & 1
\end{pmatrix} = V
\begin{pmatrix}
-2 & 0 & 0 \\
0 & 4 & 1 \\
0 & 0 & 4
\end{pmatrix} = J.
\]

The second example indicates why the computation of the Jordan Normal Form is involved.

**Example 54** The matrix
\[
A = \begin{pmatrix}
2 & 1 & 0 \\
0 & 2 & 0 \\
0 & -1 & 2
\end{pmatrix}
\]
has eigenvalue $\lambda = 2$ with algebraic multiplicity 3. The eigenspace is
\[
\mathcal{N}(2I - A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]

Since the dimension of the eigenspace $\mathcal{N}(2I - A)$ (= the geometric multiplicity of the eigenvalue 2) is less than the algebraic multiplicity of the eigenvalue 2, the matrix $A$ is not diagonalizable.
From the geometric and algebraic multiplicity of the eigenvalue 2 we can deduce the structure of the matrix $J$ in the Jordan normal form

$$
\begin{pmatrix}
2 & 1 & 0 \\
0 & 2 & 0 \\
0 & -1 & 2
\end{pmatrix}
= A,
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & *
\end{pmatrix}
= V,
\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 1 \\
0 & 0 & 2
\end{pmatrix}
= J.
$$

We also know that the first column $v_{1,1}$ of $V$ is an eigenvector corresponding to the eigenvalue 2 and that the second column $v_{2,1}$ of $V$ is an eigenvector corresponding to the eigenvalue 2 that is linearly independent of $v_{1,1}$. However, the columns $v_{1,1}$ and $v_{2,1}$ are in general not equal to the basis vectors $\tilde{v}_1$ and $\tilde{v}_2$ for the eigenspace $\mathcal{N}(2I - A)$.

Since $v_{2,1}$ is an eigenvector corresponding to the eigenvalue 2, $v_{2,1} = c_1 \tilde{v}_1 + c_2 \tilde{v}_2$ for some scalars $c_1, c_2$. We need to find $c_1, c_2$ as well as a vector $v_{2,2}$ such that

$$(2I - A)v_{2,2} = c_1 \tilde{v}_1 + c_2 \tilde{v}_2,$$

i.e.,

$$
\begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
v_{2,2}
\end{pmatrix}
= \begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}.
$$

This system only has a solution if $c_1 = -c_2$ and in this case the solutions are

$$v_{2,2} = \begin{pmatrix}
0 \\
-c_1 \\
0
\end{pmatrix} + \alpha \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} + \beta \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}.
$$

Hence, we obtain

$$v_{2,1} = c_1 \tilde{v}_1 - c_1 \tilde{v}_2,$$

for any $c_1 \neq 0$ and

$$v_{2,2} = \begin{pmatrix}
0 \\
-c_1 \\
0
\end{pmatrix} + \alpha \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} + \beta \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

for any $\alpha, \beta \in \mathbb{R}$. We set $c_1 = 1$ and $\alpha = \beta = 0$, which gives

$$v_{2,1} = \begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix}, \quad v_{2,2} = \begin{pmatrix}
0 \\
-1 \\
0
\end{pmatrix}.$$

The vector $v_{1,1}$ is an eigenvector corresponding to the eigenvalue 2 that is linearly independent of $v_{2,1}$. That is we need to find

$$v_{1,1} \in \mathcal{N}(2I - A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$
with \(v_{1,1} \neq \gamma v_{2,1}\) for some \(\gamma \in \mathbb{R}\). The vector \(v_{1,1} = (1, 0, 0)^T\) satisfies these criteria. (There are other choices, such as \(v_{1,1} = (0, 0, 1)^T\).)

Thus, the Jordan Normal Form of \(A\) is given by

\[
\begin{bmatrix}
2 & 1 & 0 \\
0 & 2 & 0 \\
0 & -1 & 2
\end{bmatrix}
= A
\]

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{bmatrix}
= V
\]

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 1 \\
0 & 0 & 2
\end{bmatrix}
= J
\]

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{bmatrix}
= V^{-1}
\]

\[\diamond\]

### 13.3 Computing the Matrix Exponential using the Jordan Normal Form

In this section we study the matrix exponential of Jordan blocks \(J_j\) and of \(A\), which are important quantities for the solution of dynamical systems with constant coefficients.

The Jordan Normal Form

\[
A = V
\begin{pmatrix}
J_1 & 0 & \ldots & 0 \\
0 & J_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \\
0 & 0 & \ldots & J_k
\end{pmatrix}
V^{-1}
\]

can be used to compute powers of \(A\) and evaluate matrix functions, such as \(\exp(A)\). For example, the \(\ell\)th power of \(A\) is

\[
A^\ell = V J^\ell V^{-1} V J^\ell V^{-1} \ldots V J^\ell V^{-1} = V J^\ell V^{-1}
\]

Since \(J\) is block diagonal,

\[
J^\ell =
\begin{pmatrix}
J_1^\ell & 0 & \ldots & 0 \\
0 & J_2^\ell & \ldots & 0 \\
\vdots & \ddots & \ddots & \\
0 & 0 & \ldots & J_k^\ell
\end{pmatrix}
\]
To compute powers of a Jordan block we write

\[
J_j = \begin{pmatrix}
\lambda_j & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & \lambda_j & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & \lambda_j & 1 \\
0 & 0 & 0 & \ldots & 0 & \lambda_j \\
0 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

= \lambda_j I + N_j,

where

\[
N_j = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\in \mathbb{R}^{n_j \times n_j}.
\]

One computes

\[
N_j^2 = \begin{pmatrix}
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\in \mathbb{R}^{n_j \times n_j},
\]

and \(N_j^{n_j} = 0 \in \mathbb{R}^{n_j \times n_j}\). For example if \(n_j = 4\),

\[
N_j = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad N_j^2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad N_j^3 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad N_j^4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Therefore

\[
J_j^\ell = (\lambda_j I + N_j)^\ell = \sum_{i=0}^{\ell} \binom{\ell}{i} \lambda_j^{\ell-i} N_j^i = \sum_{i=0}^{\min\{\ell,n_j-1\}} \binom{\ell}{i} \lambda_j^{\ell-i} N_j^i,
\]  \tag{64}
where
\[
\binom{\ell}{i} = \frac{\ell!}{i!(\ell-i)!}.
\]

If we use the fact that for two matrixes \(A, B\) that commute, i.e., \(AB = BA\), it holds \(\exp(A + B) = \exp(A) \exp(B)\), then the expression for \(\exp(J_t)\) can be derived easily (of course we have avoided the difficulty of actually showing that \(\exp(A + B) = \exp(A) \exp(B)\) for two matrices \(A, B\) that commute). Since the matrices \(\lambda_j I\) and \(N_j\) commute we have
\[
\exp(J_t) = \exp((\lambda_j I + N_j)t) = \exp(\lambda_j tI) \exp(N_j t)
= \exp(\lambda_j t) \exp(N_j t) = \exp(\lambda_j t) \left( \sum_{i=0}^{\infty} \frac{t^i}{i!} N_j^i \right)
= \exp(\lambda_j t) \left( \sum_{i=0}^{n_j-1} \frac{t^i}{i!} N_j^i \right).
\]
\(\text{(65)}\)

(In Section 13.4 we will derive (65) without using the identity \(\exp(A + B) = \exp(A) \exp(B)\) for two matrices \(A, B\) that commute.)

For example if
\[
J_j = \begin{pmatrix}
\lambda_j & 1 & 0 & 0 \\
0 & \lambda_j & 1 & 0 \\
0 & 0 & \lambda_j & 1 \\
0 & 0 & 0 & \lambda_j
\end{pmatrix} \in \mathbb{R}^{4 \times 4},
\]
then
\[
\exp(J_t) = \exp(\lambda_j t) \begin{pmatrix}
1 & t & t^2/2 & t^3/6 \\
0 & 1 & t & t^2/2 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix} \in \mathbb{R}^{4 \times 4}.
\]
The functions that appear in the entries of \(\exp(J_t)\) are shown in Figure 23 for \(\lambda_j = -1\). The functions \(\exp(-t)\), \(\exp(-t)t^2/2\), and \(\exp(-t)t^3/6\) grow initially until the term \(\exp(-t)\) dominates the powers of \(t\) and the functions decrease monotonically.

If (63) holds, the matrix exponential of \(At\) for a scalar \(t\) is given by
\[
\exp(At) = V \begin{pmatrix}
\exp(J_1t) & 0 & \ldots & 0 \\
0 & \exp(J_2t) & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \exp(J_kt)
\end{pmatrix} V^{-1}.
\]
Figure 23: While the function $\exp(-t)$ decreases monotonically, the functions $\exp(-t)t$, $\exp(-t)t^2/2$, and $\exp(-t)t^3/6$ grow initially until the $\exp(-t)$ term dominates the powers of $t$ and the functions decrease monotonically.
Example 55 Suppose that the concentrations as a function of time for three chemical species are given by $x_1(t), x_2(t), x_3(t)$. Furthermore, suppose the rate of the reaction $x_1 \rightarrow x_2$ is proportional to $x_1$ and the rate of the reaction $x_2 \rightarrow x_3$ is proportional to $x_2$. Finally, suppose that the initial concentrations are $x_1(0) = 1, x_2(0) = x_3(0) = 1$.

The mathematical model for the chemical reaction is

$$x'_1(t) = -k_1 x_1(t),$$
$$x'_2(t) = k_1 x_1(t) - k_2 x_2(t),$$
$$x'_3(t) = k_2 x_2(t),$$

and $x(0) = (1, 0, 0)^T$, where $k_1, k_2$ are the rate constants for the reaction. In matrix vector form,

$$
\begin{pmatrix}
  x'_1(t) \\
  x'_2(t) \\
  x'_3(t)
\end{pmatrix} =
\begin{pmatrix}
  -k_1 & 0 & 0 \\
  k_1 & -k_2 & 0 \\
  0 & k_2 & 0
\end{pmatrix}
\begin{pmatrix}
  x_1(t) \\
  x_2(t) \\
  x_3(t)
\end{pmatrix},
$$

and $x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

The solution of the differential equation $x'(t) = Ax(t)x(t), t > 0$, with initial condition $x(0) = x_0$ is given by

$$x(t) = \exp(At)x_0.$$ 

We study two cases, $k_1 = k_2$ and $k_1 \neq k_2$.

Case $k_1 = k_2$: If $k_1 = k_2 \neq 0$, the eigenvalues of

$$A =
\begin{pmatrix}
  -k_1 & 0 & 0 \\
  k_1 & -k_1 & 0 \\
  0 & k_1 & 0
\end{pmatrix}
$$

are 0 with algebraic multiplicity 1 and $-k_1$ with algebraic multiplicity 2. The corresponding eigenspaces are

$$\mathcal{N}(-A) = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{N}(-k_1 I - A) = \text{span} \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$ 

Since the dimension of the eigenspace $\mathcal{N}(-k_1 I - A)$ (= the geometric multiplicity of the eigenvalue $-k_1$) is less than the algebraic multiplicity of the eigenvalue $-k_1$, the matrix $A$ is not diagonalizable.
The previous calculations show that the Jordan normal form is
\[
\begin{pmatrix}
-k_1 & 0 & 0 \\
k_1 & -k_1 & 0 \\
0 & k_1 & 0 \\
\end{pmatrix}
= A
\begin{pmatrix}
0 & 0 & \ast \\
0 & -1 & \ast \\
1 & 1 & \ast \\
\end{pmatrix}
= V
\begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 1 \\
0 & 0 & -k_1 \\
\end{pmatrix}
= J.
\]

The missing column, denoted by \(v_{1,2}\) in the notation used above, is obtained from
\[
\begin{pmatrix}
-k_1 & 0 & 0 \\
k_1 & -k_1 & 0 \\
0 & k_1 & 0 \\
\end{pmatrix}
v_{2,2} = -k_1 v_{2,2} + \begin{pmatrix}
0 \\
-1 \\
1 \\
\end{pmatrix},
\]

i.e., \((-k_1 I - A) v_{2,2} = -v_{2,1}\).

Solving \((-k_1 I - A) v_{2,2} = -v_{2,1}\) gives
\[
v_{1,2} = \begin{pmatrix}
-1/k_1 \\
1/k_1 - \alpha \\
\alpha \\
\end{pmatrix} = \begin{pmatrix}
-1/k_1 \\
1/k_1 \\
0 \\
\end{pmatrix} + \alpha \begin{pmatrix}
0 \\
-1 \\
1 \\
\end{pmatrix} = \begin{pmatrix}
-1/k_1 \\
1/k_1 \\
0 \\
\end{pmatrix} + \alpha v_{2,1}, \quad \alpha \in \mathbb{R}.
\]

Choosing \(\alpha = 0\), we find
\[
\begin{pmatrix}
-k_1 & 0 & 0 \\
k_1 & -k_1 & 0 \\
0 & k_1 & 0 \\
\end{pmatrix}
= A
\begin{pmatrix}
0 & 0 & -1/k_1 \\
0 & -1 & 1/k_1 \\
1 & 1 & 0 \\
\end{pmatrix}
= V
\begin{pmatrix}
0 & 0 & -1/k_1 \\
0 & -1 & 1/k_1 \\
1 & 1 & 0 \\
\end{pmatrix}
= J
\]
or, equivalently,
\[
\begin{pmatrix}
-k_1 & 0 & 0 \\
k_1 & -k_1 & 0 \\
0 & k_1 & 0 \\
\end{pmatrix}
= A
\begin{pmatrix}
0 & 0 & -1/k_1 \\
0 & -1 & 1/k_1 \\
1 & 1 & 0 \\
\end{pmatrix}
= V
\begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 1/k_1 \\
0 & 0 & -k_1 \\
\end{pmatrix}
= J
\]

We have
\[
\exp \begin{pmatrix}
-k_1 & 0 & 0 \\
k_1 & -k_1 & 0 \\
0 & k_1 & 0 \\
\end{pmatrix}
\]
\[
= \begin{pmatrix}
0 & 0 & -1/k_1 \\
0 & -1 & 1/k_1 \\
1 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \exp(-k_1 t) & \exp(-k_1 t) \\
0 & 0 & \exp(-k_1 t) \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & -1/k_1 \\
0 & -1 & 1/k_1 \\
1 & 1 & 0 \\
\end{pmatrix}
= V \exp(Jt) = V^{-1}.
\]
The right plot in Figure 24 shows the solution $x(t) = \exp(At)x_0$ of the differential equation $x'(t) = Ax(t)$, $t > 0$, where $A$ is constructed with $k_1 = k_2 = 1$, and with initial condition $x_0 = (1, 0, 0)^T$. The left plot shows the solution $z(t) = \exp(Jt)z_0$ of the differential equation $z'(t) = Jz(t)$, $t > 0$, with initial condition $z_0 = V^{-1}x_0$.

Figure 24: The solution $z(t) = Jz(t)$ (left plot) and the solution $x(t) = Ax(t)$ of with $k_1 = k_2 = 1$ and initial value $x_0 = (1, 0, 0)^T$.

Case $k_1 \neq k_2$: If $k_1 \neq k_2$ and $k_1 \neq 0, k_2 \neq 0$, then eigenvalues of

$$A = \begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{pmatrix}$$

are $0, -k_1, -k_2$, each with algebraic multiplicity 1. The corresponding eigenspaces are

$$\mathcal{N}(-A) = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{N}(-k_1I - A) = \text{span} \left\{ \begin{pmatrix} (k_1 - k_2)/k_2 \\ -k_1/k_2 \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{N}(-k_2I - A) = \text{span} \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$
Since $A$ has three distinct eigenvalues, it is diagonalizable. In particular,

$$
\begin{pmatrix}
-k_1 & 0 & 0 \\
 k_1 & -k_2 & 0 \\
 0 & k_2 & 0
\end{pmatrix}
= A
\begin{pmatrix}
0 & (k_1 - k_2)/k_2 & 0 \\
0 & -k_1/k_2 & -1 \\
1 & 1 & 1
\end{pmatrix}
= V
\begin{pmatrix}
1 & 1 & 1 \\
 k_2/(k_1 - k_2) & 0 & 0 \\
-k_1/(k_1 - k_2) & -1 & 0
\end{pmatrix}
= \Lambda
\begin{pmatrix}
-1 & 0 \\
0 & -k_1 & 0 \\
0 & 0 & -k_2
\end{pmatrix}
= V^{-1}.
$$

The right plot in Figure 25 shows the solution $x(t) = \exp(At)x_0$ of the differential equation $x'(t) = Ax(t)$, $t > 0$, where $A$ is constructed with $k_1 = k_2 = 1$, and with initial condition $x_0 = (1, 0, 0)^T$. The left plot shows the solution $z(t) = \exp(At)z_0$ of the differential equation $z'(t) = \Lambda z(t)$, $t > 0$, with initial condition $z_0 = V^{-1}x_0$.

Figure 25: The solution $z(t)$ of $z'(t) = \Lambda z(t)$ (left plot) and the solution $x(t)$ of $x'(t) = A x(t)$ of with $k_1 = 0.8$, $k_2 = 1$ and initial value $x_0 = (1, 0, 0)^T$.

13.4 Computing the Matrix Exponential of a Jordan Block

In the previous section we have computed the matrix exponential of a Jordan block, $\exp(J_i t)$, using the identity $\exp((\lambda_i I + N_i)t) = \exp(\lambda_i tI) \exp(N_i t)$. This identity is true, but it relies in the fact that for two matrixes $A, B$ that commute, i.e., $AB = BA$, it holds $\exp(A + B) = \exp(A) \exp(B)$, which we haven’t proven. For completeness, we show (65) using elementary computations.
Using the expression for $J^\ell_i$ derived in (64) we obtain

$$\exp(J_i) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} J^\ell_i = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j=0}^{\min(\ell, n_i-1)} \binom{\ell}{j} \lambda_i^{\ell-j} N_i^j$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( \lambda_i^{\ell} I + \sum_{j=1}^{\min(\ell, n_i-1)} \binom{\ell}{j} \lambda_i^{\ell-j} N_i^j \right)$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \lambda_i^{\ell} I + \sum_{j=1}^{\min(\ell, n_i-1)} \frac{1}{\ell!} \sum_{\ell=0}^{\ell} \binom{\ell}{j} \lambda_i^{\ell-j} N_i^j$$

$$= \sum_{\ell=0}^{\infty} \frac{\lambda_i^{\ell}}{\ell!} I + \sum_{j=1}^{n_i-1} \left( \frac{\lambda_i^{\ell-j}}{(\ell-j)! j!} \right) N_i^j$$

$$= \exp(\lambda_i) \sum_{j=0}^{n_i-1} \frac{1}{j!} N_i^j.$$
The Singular Value Decomposition

In this section we show that for every matrix $A \in \mathbb{R}^{m \times n}$ we can find orthogonal matrices $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ and a ‘diagonal’ matrix $\Sigma \in \mathbb{R}^{m \times n}$ with non-negative diagonal entries such that

$$A = U \Sigma V^T. \quad (66)$$

This is called the singular value decomposition of $A$. The diagonals of $\Sigma$ are called the singular values of $A$, the columns of $V$ are called the right singular vectors, and the columns of $U$ are called the left singular vectors.

Derivation of the Singular Value Decomposition

Let $A \in \mathbb{R}^{m \times n}$. To avoid case distinction in the presentation we assume that $m \geq n$, but everything can be generalized to the case $m < n$.

We have shown that

$$\mathcal{N}(A^T A) = \mathcal{N}(A), \quad \mathcal{R}(A^T A) = \mathcal{R}(A^T).$$

Exchanging $A$ and $A^T$ we also have $\mathcal{N}(AA^T) = \mathcal{N}(A^T)$ and $\mathcal{R}(AA^T) = \mathcal{R}(A)$.

The matrix $A^T A \in \mathbb{R}^{n \times n}$ is symmetric positive semidefinite, since

$$v^T A^T A v = (Av)^T Av = \|Av\|_2^2 \geq 0 \quad \text{for all } v \in \mathbb{R}^n.$$  

If $A \in \mathbb{R}^{m \times n}$ has rank $n$, i.e., if $\mathcal{N}(A) = \{0\}$, then we have that $Av \neq 0$ for all $v \neq 0$ and

$$v^T A^T A v = \|Av\|_2^2 > 0 \quad \text{for all } v \in \mathbb{R}^n, v \neq 0,$$

i.e., $A^T A \in \mathbb{R}^{n \times n}$ symmetric positive definite.

Since $A^T A \in \mathbb{R}^{n \times n}$ is symmetric positive semidefinite, it has real, non-negative eigenvalues.

Let $\lambda \geq 0$ be an eigenvalue of $A^T A$ with corresponding eigenvector $v \neq 0$, that is let

$$A^T A v = v \lambda. \quad (67)$$

Multiplying by $A$ gives

$$AA^T (Av) = Av \lambda. \quad (68)$$

If $\lambda > 0$, then by (67) $Av \neq 0$ (otherwise $A^T 0 = A^T Av = v \lambda \neq 0$). Thus (68) shows that $\lambda > 0$ is also an eigenvalue of $AA^T$ with corresponding eigenvector $Av$.

We have show that every positive eigenvalue $\lambda > 0$ of $A^T A$ is also an eigenvalue of $AA^T$ and if $v \in \mathbb{R}^n$ is the corresponding eigenvector for $A^T A$, then $Av \in \mathbb{R}^m$ is the corresponding eigenvector for $AA^T$. Replacing the roles of $A$ and $A^T$, we can also show that very positive eigenvalue $\lambda > 0$ of $AA^T$ is also an eigenvalue of $A^T A$ and if $u \in \mathbb{R}^m$ is the corresponding eigenvector for $AA^T$, then $A^T u \in \mathbb{R}^n$ is the corresponding eigenvector for $A^T A$.
Now consider the orthogonal diagonalization of $A^T A$,

$$A^T A = V \Lambda V^T,$$

(69)

with

$$V = (v_1, \ldots, v_n) \text{ orthogonal}, V^T V = I, \text{ and } \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n).$$

Let $\lambda_1, \ldots, \lambda_r > 0$ and $\lambda_{r+1} = \ldots = \lambda_n = 0$.

Since

$$(Av_i)^T(Av_j) = v_i^T A^T A v_j = v_i^T v_j \lambda_j = 0 \lambda_j = 0 \text{ for } i \neq j,$$

the orthonormal eigenvectors $v_1, \ldots, v_r$ of $A^T A$ corresponding to positive eigenvalues $\lambda_1, \ldots, \lambda_r > 0$ generate orthogonal eigenvectors $Av_1, \ldots, Av_r$ of $AA^T$. To normalize the latter eigenvectors, we define

$$u_j = \frac{1}{\sqrt{\lambda_j}} Av_j, \quad j = 1, \ldots, r.$$

We have

$$\left(\frac{1}{\sqrt{\lambda_i}} Av_i\right)^T \left(\frac{1}{\sqrt{\lambda_j}} Av_j\right) = \frac{1}{\sqrt{\lambda_i \sqrt{\lambda_j}}} v_i^T A^T A v_j = v_i^T v_j \frac{\lambda_j}{\sqrt{\lambda_i \sqrt{\lambda_j}}},$$

$$= \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \quad i, j \in \{1, \ldots, r\}. \end{cases}$$

Define

$$\sigma_j = \sqrt{\lambda_j}, \quad j = 1, \ldots, r,$$

$$\sigma_j = \sqrt{\lambda_j} = 0, \quad j = r + 1, \ldots, n.$$

(70)

We have shown that if $v_1, \ldots, v_r$ are orthonormal eigenvectors of $A^T A$ corresponding to positive eigenvalues $\lambda_1, \ldots, \lambda_r > 0$, then

$$u_1 = \frac{1}{\sigma_1} Av_1, \ldots, u_r = \frac{1}{\sigma_r} Av_r,$$

(71)

are orthonormal eigenvectors of $AA^T$ corresponding to positive eigenvalues $\lambda_1, \ldots, \lambda_r > 0$.

We can rearrange (71) as

$$Av_1 = \sigma_1 u_1, \ldots, Av_r = \sigma_r u_r.$$  

(72a)

Since for $j = r + 1, \ldots, n$ it holds that $Av_j = v_j 0$ and $\sigma_j = 0$, we have

$$Av_{r+1} = \sigma_{r+1} u_{r+1}, \ldots, Av_n = \sigma_n u_n,$$

(72b)
for any vectors $u_{r+1}, \ldots, u_n$. The identities (72) imply

$$A \begin{pmatrix} v_1, \ldots, v_n \end{pmatrix} = (Av_1, \ldots, Av_n) = (u_1 \sigma_1, \ldots, u_n \sigma_n) = V \in \mathbb{R}^{n \times r}$$

$$= \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \\ \cdots \\ 0 \end{pmatrix} = \begin{pmatrix} u_1, \ldots, u_n \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \\ \cdots \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \\ \cdots \\ 0 \end{pmatrix} \begin{pmatrix} u_1, \ldots, u_n \end{pmatrix} = \begin{pmatrix} u_1 \sigma_1, \ldots, u_n \sigma_n \end{pmatrix} = \begin{pmatrix} u_1, \ldots, u_n \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \\ \cdots \\ 0 \end{pmatrix} = \begin{pmatrix} u_1, \ldots, u_m \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \\ \cdots \\ 0 \end{pmatrix} = \Sigma \in \mathbb{R}^{m \times n}$$

(73)

for any vectors $u_{r+1}, \ldots, u_m$. If we can find vectors $u_{r+1}, \ldots, u_m$ such that $U = \begin{pmatrix} u_1, \ldots, u_m \end{pmatrix}$ is orthogonal, then the identity (73) is equivalent to (66).

How do we compute $u_{r+1}, \ldots, u_m$ such that $U = \begin{pmatrix} u_1, \ldots, u_m \end{pmatrix}$ is orthogonal? The SVD (66) implies

$$A^T U = V \Sigma^T. \tag{74}$$

Since the columns $r + 1, \ldots, m$ of $\Sigma^T$ are zero, (74) implies

$$A^T u_{r+1} = 0, \ldots, A^T u_m = 0,$$

i.e., the missing columns of $u_{r+1}, \ldots, u_m$ of $U$ are obtained by computing an orthonormal basis of $\mathcal{N}(A^T)$. Note that $u_1, \ldots, u_r \in \mathcal{R}(A)$ by (71) and $\mathcal{R}(A) \perp \mathcal{N}(A^T)$, i.e., the $m$ vectors $u_1, \ldots, u_r, u_{r+1}, \ldots, u_m$ are orthogonal, if \{u_{r+1}, \ldots, u_m\} orthonormal basis of $\mathcal{N}(A^T)$.

The computation of the SVD of $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ is summarized in Algorithm 56.
Algorithm 56 (SVD)

0. Given $A \in \mathbb{R}^{m \times n}$ with $m \geq n$.

1. Compute eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ of $A^T A$ and corresponding orthonormal eigenvectors $v_1, \ldots, v_n \in \mathbb{R}^n$. With $V = (v_1, \ldots, v_n)$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $A^T A = V \Lambda V^T$.

2. Compute the singular values $\sigma_j = \sqrt{\lambda_j}$, $j = 1, \ldots, n$.

Let $\sigma_1 \geq \ldots, \sigma_r > \sigma_{r+1} = \ldots, \sigma_n = 0$.

3. Compute $u_1 = \sigma_1^{-1} Av_1, \ldots, u_r = \sigma_r^{-1} Av_r$, and an orthonormal basis $\{u_{r+1}, \ldots, u_m\}$ orthonormal basis of $\mathcal{N}(A^T)$. Set $U = (u_1, \ldots, u_m)$

4. The matrix $A$ has the SVD $A = U \Sigma V^T$.

Note if $A \in \mathbb{R}^{m \times n}$ with $m < n$ we can apply Algorithm 56 to $A^T$.

Example 57 We compute the SVD of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}.$$ 

Orthogonal diagonalization of $A^T A$ gives

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \Lambda V^T.$$

The positive singular values of $A$ are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$. The first two columns of $U \in \mathbb{R}^{3 \times 3}$ are given by

$$u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$ 

and

$$u_2 = \frac{1}{\sigma_2} Av_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
It is easy to verify that $u_1^T u_1 = 1$, $u_2^T u_2 = 1$, and $u_2^T u_1 = 0$, which we have proven must hold in general.

The identities $Av_j = u_j \sigma_j$, $j = 1, 2$, give

$$A(v_1|v_2) = (Av_1|Av_2) = (u_1 \sigma_1|u_2 \sigma_2) = (u_1|u_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}.$$  

Since $V = (v_1|v_2)$ is orthogonal,

$$A = (u_1|u_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} (v_1|v_2)^T,$$

which is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} \\ -1/\sqrt{6} & -1/\sqrt{2} \\ 2/\sqrt{6} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \quad (75)$$

The decomposition (75) is known as the “economy size” SVD.

The missing third column of $U$ is a (normalized) basis vector for $\mathcal{N}(A^T)$. Since

$$\mathcal{N} \left( \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

we find

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \quad (76)$$

We can compute the SVD using MATLAB as follows.

```matlab
>> A = [1 0; 0 -1; -1 1];
>> [U,Sig,V]=svd(A)
```

```
U =
0.4082 0.7071 -0.5774
0.4082 -0.7071 -0.5774
-0.8165 0.0000 -0.5774

Sig =
```

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14.1 Derivation of the Singular Value Decomposition

\[
V = \begin{bmatrix}
0.7071 & 0.7071 \\
0.7071 & 0.7071 \\
-0.7071 & 0.7071
\end{bmatrix}
\]

Note that MATLAB rearranged the ordering of the singular values (and therefore also the singular vectors), which is fine. MATLAB always orders the singular values \( \sigma_1 \geq \sigma_2 \geq \ldots \sigma_{\min\{m,n\}} \geq 0 \).

The “economy size” SVD is computed as follows.

\[
\begin{align*}
&>> A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}; \\
&>> [U,Sig,V]=svd(A,'econ')
\end{align*}
\]

\[
U = \begin{bmatrix}
-0.4082 & 0.7071 \\
-0.4082 & -0.7071 \\
0.8165 & 0.0000
\end{bmatrix}
\]

\[
Sig = \begin{bmatrix}
1.7321 & 0 \\
0 & 1.0000
\end{bmatrix}
\]

\[
V = \begin{bmatrix}
-0.7071 & 0.7071 \\
0.7071 & 0.7071
\end{bmatrix}
\]

Example 58 We compute the SVD of

\[
A = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
0 & 0
\end{pmatrix}.
\]

Orthogonal diagonalization of \( A^T A \) gives

\[
\begin{pmatrix}
2 \\
2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
4 & 0 \\
0 & 0
\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}.
\]
The matrix \( A \) has only one positive singular value \( \sigma_1 = 2 \). The first column of \( U \in \mathbb{R}^{3 \times 3} \) is given by
\[
\frac{1}{\sigma_1} A v_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.
\]

The missing second and third column of \( U \) comes from an orthonormal basis for \( \mathcal{N}(A^T) \). We compute a basis for \( A^T \) in the usual way,
\[
\mathcal{N} \left( \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \right) = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right\}.
\]

The SVD of \( A \) is
\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = U \Sigma \Sigma = V^T.
\]

We can compute the SVD using MATLAB as follows.

\[
>> A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix};
\]
\[
>> [U,Sig,V]=\text{svd}(A)
\]
\[
U =
\begin{pmatrix}
-0.7071 & -0.7071 & 0 \\
-0.7071 & 0.7071 & 0 \\
0 & 0 & 1.0000
\end{pmatrix}
\]
\[
Sig =
\begin{pmatrix}
2 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]
\[
V =
\begin{pmatrix}
-0.7071 & -0.7071 \\
-0.7071 & 0.7071
\end{pmatrix}
\]
14.1 Derivation of the Singular Value Decomposition

\[
\begin{align*}
\text{>> } [U,\text{Sig},V] &= \text{svd}(A, 'econ') \\
U &= \\
&= \begin{bmatrix}
-0.7071 & -0.7071 \\
-0.7071 & 0.7071 \\
0 & 0
\end{bmatrix} \\
\text{Sig} &= \\
&= \begin{bmatrix}
2 & 0 \\
0 & 0
\end{bmatrix} \\
V &= \\
&= \begin{bmatrix}
-0.7071 & -0.7071 \\
-0.7071 & 0.7071
\end{bmatrix}
\end{align*}
\]

\[\Box\]

**Example 59** We compute the SVD of

\[
A = \begin{pmatrix}
1 & 2 & 2 \\
2 & 2 & 1
\end{pmatrix}
\]

Since \( A \) has fewer rows than columns, we apply Algorithm 56 to \( A^T \) and reverse the roles of \( U \) and \( V \).

Orthogonal diagonalization of \( AA^T \) gives

\[
\begin{bmatrix}
9 & 8 \\
8 & 9
\end{bmatrix} = \frac{1}{\sqrt{17}} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
17 & 0 \\
0 & 1
\end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} = AA^T = U^TU = \Lambda.
\]

The matrix \( A^T \) has singular values \( \sigma_1 = \sqrt{17} \) and \( \sigma_2 = 1 \). The first two columns of \( V \in \mathbb{R}^{3 \times 3} \) are given by

\[
\begin{align*}
v_1 &= \frac{1}{\sigma_1} A^T u_1 = \frac{1}{\sqrt{34}} \begin{pmatrix}
3 \\
4 \\
3
\end{pmatrix}, \\
v_2 &= \frac{1}{\sigma_2} A^T u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix}.
\end{align*}
\]
The missing third column of $V$ comes from an orthonormal basis for $\mathcal{N}(A)$,

$$\mathcal{N}\left(\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}\right) = \text{span}\left\{ \begin{pmatrix} 1 \\ -3/2 \\ 1 \end{pmatrix}\right\} = \text{span}\left\{ \begin{pmatrix} 2/\sqrt{17} \\ -3/\sqrt{17} \\ 2/\sqrt{17} \end{pmatrix}\right\}.$$  

The SVD of $A^T$ is

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{34} & -1/\sqrt{2} & 2/\sqrt{17} \\ 4/\sqrt{34} & 0 & -3/\sqrt{17} \\ 3/\sqrt{34} & 1/\sqrt{2} & 2/\sqrt{17} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.\quad (1)$$

Taking the transpose gives the SD of $A$,

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3/\sqrt{34} & 4/\sqrt{34} & 3/\sqrt{34} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 2/\sqrt{17} & -3/\sqrt{17} & 2/\sqrt{17} \end{pmatrix}.\quad (2)$$

We can compute the SVD using MATLAB as follows.

```matlab
>> A = [ 1 2 2; 2 2 1];
>> [U,Sig,V]=svd(A)
U =
   -0.7071  -0.7071
   -0.7071   0.7071

Sig =
   4.1231   0   0
   0 1.0000   0

V =
   -0.5145   0.7071   0.4851
   -0.6860   0  -0.7276
   -0.5145  -0.7071   0.4851

>> [U,Sig,V]=svd(A, ‘econ’)
U =
```

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\[
\begin{pmatrix}
-0.7071 & -0.7071 \\
-0.7071 & 0.7071
\end{pmatrix}
\]

\[
\text{Sig} =
\begin{pmatrix}
4.1231 & 0 \\
0 & 1.0000
\end{pmatrix}
\]

\[
\text{V} =
\begin{pmatrix}
-0.5145 & 0.7071 \\
-0.6860 & 0 \\
-0.5145 & -0.7071
\end{pmatrix}
\]

\[\triangleleft\]

Theorem 60 For any matrix \( A \in \mathbb{R}^{m \times n} \) there exist orthogonal matrices \( U \in \mathbb{R}^{m \times m} \), \( V \in \mathbb{R}^{n \times n} \) and a ‘diagonal’ matrix

\[
\Sigma =
\begin{pmatrix}
\sigma_1 & & & & \\
& \ddots & & & \\
& & \sigma_r & & \\
& & & \sigma_{r+1} & \\
& & & & \ddots
\end{pmatrix}
\in \mathbb{R}^{m \times n}
\text{ if } m \leq n
\]

or

\[
\Sigma =
\begin{pmatrix}
\sigma_1 & & & & \\
& \ddots & & & \\
& & \sigma_r & & \\
& & & \sigma_{r+1} & \\
& & & & \ddots
\end{pmatrix}
\in \mathbb{R}^{m \times n}
\text{ if } m > n,
\]

with diagonal entries

\[
\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_{\min\{m,n\}} = 0
\]

such that

\[
A = U \Sigma V^T.
\]
The decomposition (77) is called the singular value decomposition (SVD) of \( A \in \mathbb{R}^{m \times n} \). The scalars \( \sigma_1, \ldots, \sigma_{\min\{m,n\}} \) are called the singular values of \( A \).

If the columns of \( V \) are \( v_1, \ldots, v_n \) and the columns of \( U \) are \( u_1, \ldots, u_m \), then the SVD (77) can also be written as

\[
A = \sum_{i=1}^{r} \sigma_i u_i v_i^T. \tag{78}
\]

### 14.2 The SVD and the Fundamental Theorem of Linear Algebra

From the SVD we can easily determine the range and null-spaces of \( A \) and \( A^T \). A schematic of the SVD is given as follows

\[
A = \begin{pmatrix}
  u_1 & \cdots & u_r & u_{r+1} & \cdots & u_m \\
  | & | & | & | & | & | \\
  1 & & & & & \\
  \vdots & & & & & \\
  1 & & & & & \\
  \vdots & & & & & \\
  1 & & & & & \\
  | & | & | & | & | & |
\end{pmatrix}
\begin{pmatrix}
  \sigma_1 & & & & & \\
  & \ddots & & & & \\
  & & 0 & & & \\
  & & & \ddots & & \\
  & & & & 0 & \\
  & & & & & \ddots \\
  & & & & & & \ddots \\
  & & & & & & 1 \\
\end{pmatrix}
\begin{pmatrix}
  v_1^T & & & & & \\
  & \vdots & & & & \\
  & & v_r^T & & & \\
  & & & \vdots & & \\
  & & & & v_{r+1}^T & \\
  & & & & & \ddots \\
  & & & & & & v_n^T \\
\end{pmatrix}
\]

- The range space of \( A \) is \( \mathcal{R}(A) = \text{span}\{u_1, \ldots, u_r\} \).
  \( (\mathcal{R}(A) = \{0\} \text{ if } r = 0, \text{i.e., if all singular values of } A \text{ are zero (i.e., } A \text{ is the zero matrix)).} \)

**Proof:** Since \( Ax = \sum_{i=1}^{r} u_i(\sigma_i v_i^T x) \in \text{span}\{u_1, \ldots, u_r\} \) for any \( x \in \mathbb{R}^n \), we have \( \mathcal{R}(A) \subset \text{span}\{u_1, \ldots, u_r\} \). On the other hand, the orthogonality of the vectors \( v_1, \ldots, v_n \) implies \( Av_j/\sigma_j = \sum_{i=1}^{r} \sigma_i (\sigma_i v_i^T v_j/\sigma_j) = u_j \) for \( j = 1, \ldots, r \), i.e, \( u_j \in \mathcal{R}(A) \) for \( j = 1, \ldots, r \). Consequently, \( \text{span}\{u_1, \ldots, u_r\} \subset \mathcal{R}(A) \).

- The null space of \( A \) is \( \mathcal{N}(A) = \text{span}\{v_{r+1}, \ldots, v_n\} \).
  \( (\mathcal{N}(A) = \{0\} \text{ if } r = n, \text{i.e., if all singular values of } A \text{ are positive and } m \geq n \).)

**Proof:** If \( x \in \mathcal{N}(A) \), then \( Ax = \sum_{i=1}^{r} \sigma_i (\sigma_i v_i^T x) = 0 \). If we multiply by \( u_j^T, \) \( j \in \{1, \ldots, r\} \), then \( 0 = u_j^T Ax = \sum_{i=1}^{r} u_j^T u_i (\sigma_i v_i^T x) = \sigma_j v_j^T x \). Since \( \sigma_j > 0 \) for \( j \in \{1, \ldots, r\} \), we obtain \( v_j^T x = 0, j = 1, \ldots, r \). By orthogonality of the vectors \( v_1, \ldots, v_n, x \in \text{span}\{v_{r+1}, \ldots, v_n\} \). On the other hand, the orthogonality of the vectors \( v_1, \ldots, v_n \) implies that for \( j > r \), \( Av_j = \sum_{i=1}^{r} \sigma_i (\sigma_i v_i^T v_j) = 0 \) i.e, \( v_j \in \mathcal{N}(A) \) for \( j = r + 1, \ldots, n \). Consequently, \( \text{span}\{v_1, \ldots, v_r\} \subset \mathcal{N}(A) \).

- Since

\[
A^T = \left( \sum_{i=1}^{r} \sigma_i u_i v_i^T \right)^T = \sum_{i=1}^{r} (\sigma_i u_i v_i^T)^T = \sum_{i=1}^{r} \sigma_i v_i u_i^T
\]
the same arguments as above can be used to show that
\[ \mathcal{R}(A^T) = \text{span}\{v_1, \ldots, v_r\} \]
and
\[ \mathcal{N}(A^T) = \text{span}\{u_{r+1}, \ldots, u_m\}. \]

### 14.3 The SVD and Data Compression

A black and white image can be represented as a \( m \times n \) matrix of pixels, where each entry \( a_{ij} \) of the matrix is the grey value of the pixel \( ij \). (A color image can be stored as three \( m \times n \) matrices, where the \( ij \)th entry of the first matrix is the "red" value of the pixel \( ij \), the \( ij \)th entry of the second matrix is the "green" value of the pixel \( ij \), and \( ij \)th entry of the second matrix is the "blue" value of the pixel \( ij \).) If stored this way, the representation of a matrix requires \( mn \) reals.

We can compute the singular values decomposition
\[
A = U \Sigma V^T = \sum_{i=1}^{\min\{m,n\}} \sigma_i u_i v_i^T.
\]
Recall that the singular values are ordered
\[ \sigma_1 \geq \ldots \geq \sigma_{\min\{m,n\}} \geq 0. \]

If a singular value \( \sigma_{k+1} \) is much smaller than the first \( k \) singular values, then the contributions of \( \sigma_{k+1} u_{k+1} v_{k+1}^T, \sigma_{k+2} u_{k+2} v_{k+2}^T, \ldots \), are smaller than the contributions of \( \sigma_1 u_1 v_1^T, \ldots, \sigma_k u_k v_k^T \). Therefore we may approximate
\[
A = U \Sigma V^T \approx \sum_{i=1}^{k} \sigma_i u_i v_i^T \overset{\text{def}}{=} A_k
\]
Although \( A_k \) is also an \( m \times n \) matrix, we only need to store \( k \) vectors \( u_i \) is size \( m \), \( k \) vectors \( v_i \) is size \( n \), and \( k \) scalars \( \sigma_i \). Hence, the storage of \( A_k \) requires \((m+n+1)k\) reals. If \( k \) is small relative to \( m \) and \( n \), \((m+n+1)k \ll mn\).

See `implot.m` for an example.

How well does the compressed image approximate the original one? We can provide a simple expression for the error \( A - A_k \) in the Frobenius norm.

Recall that the Frobenius norm of an \( m \times n \) matrix \( A \) with entries \( a_{ij} \) is given by
\[
\|A\|_F = \left( \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}^2 \right)^{1/2}.
\]
Before we analyze the error \( \|A - A_k\|_F \), we prove a few properties of the Frobenius norm of a matrix.
Theorem 61 Let $A \in \mathbb{R}^{m \times n}$.

1. If $Q \in \mathbb{R}^{m \times m}$ is an orthogonal matrix, then $\|QA\|_F = \|A\|_F$.

2. If $P \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\|AP\|_F = \|A\|_F$.

3. If $\sigma_1, \ldots, \sigma_{\min\{m,n\}}$ are the singular values of $A \in \mathbb{R}^{m \times n}$, then

$$\|A\|_F = (\sigma_1^2 + \ldots + \sigma_{\min\{m,n\}}^2)^{1/2}.$$ 

Proof:

1. If $A \in \mathbb{R}^{m \times n}$ has columns $a_1, \ldots, a_n$, then the square of the Frobenius norm of $A$ is equal to the sum of the squares of the 2-norms of the columns of $A$, i.e.,

$$\|A\|_F = \left( \sum_{j=1}^{n} \|a_j\|_2^2 \right)^{1/2}.$$ 

Since the columns of $QA$ are $Qa_1, \ldots, Qa_n$ we have

$$\|QA\|_F = \left( \sum_{j=1}^{n} \|Qa_j\|_2^2 \right)^{1/2}.$$ 

Since $Q$ is orthogonal, $\|Qv\|_2^2 = v^T Q^T Q v = v^T v = \|v\|_2^2$ for any vector $v \in \mathbb{R}^m$. Hence,

$$\|QA\|_F = \left( \sum_{j=1}^{n} \|Qa_j\|_2^2 \right)^{1/2} = \left( \sum_{j=1}^{n} \|Qa_j\|_2^2 \right)^{1/2} = \|A\|_F.$$ 

2. Note that the Frobenius norm of the transpose of a matrix is equal to the Frobenius norm of the matrix, i.e., $\|M\|_F = \|M^T\|_F$ for any matrix $M$. Therefore,

$$\|AP\|_F = \|P^T A^T\|_F = \|A^T\|_F = \|A\|_F.$$ 

3. Let $A = U \Sigma V^T$ be the singular value decomposition of $A$. Then

$$\|A\|_F = \|U \Sigma V^T\|_F = \|\Sigma V^T\|_F = \|\Sigma\|_F = (\sigma_1^2 + \ldots + \sigma_{\min\{m,n\}}^2)^{1/2}.$$ 

\[\square\]
Now we can provide an estimate for the error $A - A_k$. We consider the case $m \geq n$, but $m < n$ can be handled the same way. We have

$$A = U \Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$

and

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T = U \Sigma_k V^T$$

where

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix}, \quad \Sigma_k = \begin{pmatrix} \sigma_1 & & \\ & \ddots & 0 \\ & & 0 \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

Hence,

$$\|A - A_k\|_F = \|U (\Sigma - \Sigma_k) V^T\|_F = \|\Sigma - \Sigma_k\|_F = (\sigma_{k+1}^2 + \ldots + \sigma_{\min\{m,n\}}^2)^{1/2}.$$
using the orthogonality of $U$ and $V$ we find that

\[
\|Ax - b\|_2^2 = \|U^T(Ax - b)\|_2^2 \\
= \|U^T(AV V^T x - b)\|_2^2 \\
= \|\Sigma V^T x - U^T b\|_2^2 \\
= \sum_{i=1}^{r} (\sigma_i z_i - u_i^T b)^2 + \sum_{i=r+1}^{m} (u_i^T b)^2,
\]

where we have set $z = V^T x$. Thus,

\[
\min_x \|Ax - b\|_2^2 = \min_{z = V^T x} \sum_{i=1}^{r} (\sigma_i z_i - u_i^T b)^2 + \sum_{i=r+1}^{m} (u_i^T b)^2.
\]

The solutions are obviously given by

\[
z_i = \frac{u_i^T b}{\sigma_i}, \quad i = 1, \ldots, r,
\]

\[
z_i = \text{arbitrary}, \quad i = r + 1, \ldots, n.
\]

and

\[
x = V z = \sum_{i=1}^{n} v_i z_i.
\]

Moreover,

\[
\min_x \|Ax - b\|_2^2 = \sum_{i=r+1}^{m} (u_i^T b)^2.
\]

Since $V$ is orthogonal, we find that

\[
\|x\|_2 = \|VV^T x\|_2 = \|V^T x\|_2 = \|z\|_2
\]

Hence, the minimum norm solution of the linear least squares problem is given by

\[
x^\dagger = V z^\dagger,
\]

where $z^\dagger \in \mathbb{R}^n$ is the vector with entries

\[
z_i^\dagger = \frac{u_i^T b}{\sigma_i}, \quad i = 1, \ldots, r,
\]

\[
z_i^\dagger = 0, \quad i = r + 1, \ldots, n.
\]
i.e.

\[ x^\dagger = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i. \quad (79) \]

Given the singular value decomposition \( A = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T \) of \( A \in \mathbb{R}^{m \times n} \), the pseudo inverse of \( A \) is given by

\[ A^\dagger = V \Sigma^\dagger U^T = \sum_{i=1}^{r} \sigma_i^{-1} v_i u_i^T, \quad (80) \]

where

\[
\Sigma^\dagger = \begin{pmatrix}
\sigma_1^{-1} \\
\vdots \\
\sigma_r^{-1}
\end{pmatrix} \\
0 \\
\vdots \\
0
\end{pmatrix} \in \mathbb{R}^{n \times m} \quad \text{if } m \leq n
\]

or

\[
\Sigma^\dagger = \begin{pmatrix}
\sigma_1^{-1} \\
\vdots \\
\sigma_r^{-1}
\end{pmatrix} \\
0 \\
\vdots \\
0
\end{pmatrix} \in \mathbb{R}^{n \times m} \quad \text{if } m > n
\]

**Example 62** In Example 58 we have computed the SVD

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{2} & 0 & -1/\sqrt{2} \\
1/\sqrt{2} & 0 & 1/\sqrt{2} \\
0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
2 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/\sqrt{2}
\end{pmatrix}.
\]

Hence the pseudoinverse of \( A \) is

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
0 & 0
\end{pmatrix}^\dagger = \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/\sqrt{2}
\end{pmatrix} \begin{pmatrix}
1/2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
0 & 0 & 0 \\
-1/\sqrt{2} & 1/\sqrt{2} & 1
\end{pmatrix} = \begin{pmatrix}
1/4 & 1/4 & 0 \\
1/4 & 1/4 & 0
\end{pmatrix}.
\]

We can compute the pseudoinverse using MATLAB as follows.
>> A = [ 1 1; 1 1; 0 0];
>> pinv(A)

ans =
0.2500 0.2500 0
0.2500 0.2500 0

Example 63 In Example 59 we have computed the SVD
\[
\begin{bmatrix}
1 & 2 & 2 \\
2 & 2 & 1
\end{bmatrix} = 
\begin{bmatrix}
1/\sqrt{2} & 1/\sqrt{2} & \sqrt{17} & 0 & 0 \\
1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
3/\sqrt{34} & 4/\sqrt{34} & 3/\sqrt{34} \\
-1/\sqrt{2} & 0 & 1/\sqrt{2} \\
2/\sqrt{17} & -3/\sqrt{17} & 2/\sqrt{17}
\end{bmatrix}
\]

Hence, the pseudoinverse of \( A \) is
\[
\begin{bmatrix}
1 & 2 & 2 \\
2 & 2 & 1
\end{bmatrix}^+ = 
\begin{bmatrix}
3/\sqrt{34} & -1/\sqrt{2} & 2/\sqrt{17} \\
4/\sqrt{34} & 0 & -3/\sqrt{17} \\
3/\sqrt{34} & 1/\sqrt{2} & 2/\sqrt{17}
\end{bmatrix}
\begin{bmatrix}
1/\sqrt{17} & 0 & 0 \\
0 & 1 & 0 \\
1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2}
\end{bmatrix}
\begin{bmatrix}
1/\sqrt{17} & 0 & 0 \\
0 & 1 & 0 \\
1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2}
\end{bmatrix}
\]

We can compute the pseudoinverse using MATLAB as follows.

>> A = [ 1 2 2; 2 2 1];
>> pinv(A)

ans =
-0.4118 0.5882
0.1176 0.1176
0.5882 -0.4118

The SVD approach to solving a linear least squares problem gives valuable insight into the solution (79) of the linear least squares problem. Suppose that the data \( b \) are
\[
b = b^\infty + \delta b,
\]
where $b^{ex}$ is the exact data and $\delta b$ represents the measurement error. In this case (79) can be written as

$$x^\dagger = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i = \sum_{i=1}^{r} \left( \frac{u_i^T b^{ex}}{\sigma_i} + \frac{u_i^T (\delta b)}{\sigma_i} \right) v_i.$$  

(81)

We are really interested in

$$x^{ex} = \sum_{i=1}^{r} \frac{u_i^T b^{ex}}{\sigma_i} v_i$$

but because of the presence of measurement errors can only compute (81). The error

$$\|x^\dagger - x^{ex}\|_2 = \left\| \sum_{i=1}^{r} \frac{u_i^T (\delta b)}{\sigma_i} v_i \right\|_2 = \sum_{i=1}^{r} \left( \frac{u_i^T (\delta b)}{\sigma_i} \right)^2 \sigma_i^2$$

depends not only on the size $\|\delta b\|_2$ of the measurement error, but also on the singular values. If a singular value $\sigma_i$ is small, then $u_i^T (\delta b)/\sigma_i$ could be large, even if $u_i^T (\delta b)$ is small. This shows that errors $\delta b$ in the data can be magnified by small singular values $\sigma_i$. 

Example 64

% Compute A
\[ t = 10^{0:-1:-10}; \]
\[ A = [\text{ones(size}(t)) \ t \ t.^2 \ t.^3 \ t.^4 \ t.^5]; \]

% compute SVD of A
\[ [U,D,V] = \text{svd}(A); \]
\[ \text{sigma} = \text{diag}(D); \]

% compute exact data
\[ x_{\text{ex}} = \text{ones}(6,1); \]
\[ b_{\text{ex}} = A * x_{\text{ex}}; \]

for \( i = 1:10 \)
    % data perturbation
    \[ \text{deltab} = 10^{-i} \times (0.5 - \text{rand(size}(b_{\text{ex}})) \times b_{\text{ex}}; \]
    \[ b = b_{\text{ex}} + \text{deltab}; \]
    \[ w = U' * b; \]
    % solution of perturbed linear least squares problem
    \[ x = V \times (w(1:6) ./ \text{sigma}); \]
    \[ \text{errx}(i+1) = \text{norm}(x - x_{\text{ex}}); \]
    \[ \text{errb}(i+1) = \text{norm}(\text{deltab}); \]
end

loglog(errb,errx,'*')
ylabel('\| x^{\text{ex}} - x \|_2 ')
xlabel('\| \delta b \|_2 ')

The singular values of \( A \) are given by

\[
\begin{align*}
\sigma_1 & \approx 3.4 \\
\sigma_2 & \approx 2.1 \\
\sigma_3 & \approx 8.2 \times 10^{-2} \\
\sigma_4 & \approx 7.2 \times 10^{-4} \\
\sigma_5 & \approx 6.6 \times 10^{-7} \\
\sigma_6 & \approx 5.5 \times 10^{-11}
\end{align*}
\]

The error \( \| x^{\text{ex}} - x \|_2 \) for different values of \( \| \delta b \|_2 \) are shown in Figure 26. Figure 26 shows that small perturbations \( \delta b \) in the measurements can lead to large errors in the solution \( x \) of the linear least squares problem if the singular values of \( A \) are small.

\[ \diamond \]
Remark 65 The pseudo-inverse of any matrix $A \in \mathbb{R}^{m \times n}$ is defined via (80). If the matrix $A$ is square and symmetric, then its spectral decomposition is

$$A = Q\Lambda Q^T = \sum_{i=1}^{n} \lambda_i q_i q_i^T = \sum_{i=1}^{n} |\lambda_i| (\text{sign}(\lambda_i) q_i) q_i^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T = U\Sigma V^T,$$

where $V = Q$, $U \in \mathbb{R}^{n \times n}$ is the matrix with columns $u_i = \text{sign}(\lambda_i) q_i$, $i = 1, \ldots, n$, and $\Sigma = \text{diag}(|\lambda_1|, \ldots, |\lambda_n|)$. In particular, for symmetric matrices the formula (35) introduced earlier for the pseudo-inverse of a symmetric matrix coincides with (80).
15 Laplace Transform

We return to the solution of dynamical systems

\[ x'(t) = Ax(t) + f(t), \quad t > 0, \quad (82a) \]
\[ x(0) = x_0. \quad (82b) \]

where \( A \in \mathbb{R}^{n \times n} \) and \( f : [0, \infty) \to \mathbb{R}^n \) is a vector valued function. We begin with the scalar case

\[ x'(t) = ax(t) + f(t), \quad t > 0, \quad (83a) \]
\[ x(0) = x_0. \quad (83b) \]

We multiply (83a) by \( e^{-st} \) and integrate each side over \([0, T]\) to get

\[ \int_0^T e^{-st} x'(t) dt = a \int_0^T e^{-st} x(t) dt + \int_0^T e^{-st} f(t) dt, \quad (84) \]

We recall the integration by parts formula. For two continuously differentiable functions \( u \) and \( v \),

\[ \int_0^T u'(t)v(t) dt = u(t)v(T) \bigg|_0^T - \int_0^T u(t)v'(t) dt. \quad (85) \]

Applying the integration by parts formula on the left hand side of (84) gives

\[ (e^{-sT} x(T) - x(0)) + s \int_0^T e^{-st} x(t) dt = a \int_0^T e^{-st} x(t) dt + \int_0^T e^{-st} f(t) dt. \quad (86) \]

If the function \( x \) is such that

\[ \lim_{T \to \infty} e^{-sT} x(T) = 0 \quad (87a) \]

and the limits

\[ \lim_{T \to \infty} \int_0^T e^{-st} x(t) dt \quad \text{and} \quad \lim_{T \to \infty} \int_0^T e^{-st} f(t) dt \quad \text{exist}, \quad (87b) \]

then (86) leads to

\[ (s - a) \int_0^\infty e^{-st} x(t) dt = x_0 + \int_0^\infty e^{-st} f(t) dt. \quad (88) \]

Thus for any \( s \) such that (87) holds, the integral of the solution of (83) multiplied by \( e^{-st} \) can be computed.

\[ \int_0^\infty e^{-st} x(t) dt = (s - a)^{-1} x_0 + (s - a)^{-1} \int_0^\infty e^{-st} f(t) dt. \quad (89) \]

It turns out that once we know \( \int_0^\infty e^{-st} x(t) dt \) for all \( s \), we can recover \( x \).
The Laplace transform of the function $x$ is defined as
\[ \mathcal{L}(x)(s) \overset{\text{def}}{=} \int_0^\infty e^{-st}x(t)dt = \lim_{T \to \infty} \int_0^T e^{-st}x(t)dt, \] (90)
whenever the limit exists (as a finite number). The argument $s$ in the definition of the Laplace transform (90) is a complex number. We recall the definition of the complex exponential function
\[ e^{x+iy} = e^x \left( \cos(y) + i \sin(y) \right) \] (91)
and
\[ e^{x+iy} \to 0 \quad \text{if} \quad x \to -\infty. \]

The Laplace transform (90) does not exist for any function $x$ and not for any complex number $s$.

Before, we discuss a few examples on how to compute the Laplace transform, we restate (83) using the Laplace transform: The Laplace transform of the solution of (83) satisfies
\[ \mathcal{L}(x)(s) = (s-a)^{-1} \left( x_0 + \mathcal{L}(f)(s) \right). \] (92)

**Example 66**

(a) We compute the Laplace transform of $e^t$.

For $T > 0$ we have
\[ \int_0^T e^t e^{-st}dt = \int_0^T e^{(1-s)t}dt = \frac{1}{1-s}e^{(1-s)t} \bigg|_0^T = \frac{1}{1-s} \left( e^{(1-s)T} - 1 \right). \]

If $\text{Re}(1-s) < 0$, i.e., $\text{Re}(s) > 1$, then $\lim_{T \to \infty} e^{(1-s)T} = 0$. Hence, for $s$ with $\text{Re}(s) > 1$,
\[ \mathcal{L}(e^t)(s) = \lim_{T \to \infty} \int_0^\infty e^t e^{-st}dt = \lim_{T \to \infty} \frac{1}{1-s} \left( e^{(1-s)T} - 1 \right) \]
\[ = \frac{1}{s-1}. \]

(b) We compute the Laplace transform of $te^{-t}$.

For $T > 0$ we have $\int_0^T te^{-t}e^{-st}dt = \int_0^T te^{-(1+s)t}dt$. Applying the integration by parts formula (85) with $u'(t) = e^{-(1+s)t}$ and $v(t) = t$ gives
\[ \int_0^T te^{-(1+s)t}dt = \frac{-1}{1+s}te^{-(1+s)t} \bigg|_0^T - \int_0^T \frac{-1}{1+s}e^{-(1+s)t}dt \]
\[ = \frac{-1}{1+s}te^{-(1+s)t} \bigg|_0^T - \frac{1}{(1+s)^2}e^{-(1+s)t} \bigg|_0^T. \]
If \(\text{Re}(1 + s) > 0\), i.e., \(\text{Re}(s) > -1\), then \(\lim_{T \to \infty} e^{-(1+s)T} = 0\). Hence, for \(s\) with \(\text{Re}(s) > -1\),

\[
\mathcal{L}(te^{-t})(s) = \lim_{T \to \infty} \int_0^T te^{-st} \, dt
\]

\[
= \lim_{T \to \infty} \left( \frac{-1}{1 + s} t e^{-(1+s)t} \right|_0^T - \frac{1}{(1 + s)^2} e^{-(1+s)t} \right|_0^T = \left( 0 - 0 \right) - \left( 0 - \frac{1}{(1 + s)^2} \right)
\]

\[
= \frac{1}{(1 + s)^2}.
\]

(c) We compute the Laplace transform of \(\cos(t)\).

Let \(T > 0\). We apply the integration by parts formula (85) with \(u(t) = \cos(t)\) and \(v'(t) = \exp(-st)\) to obtain

\[
\int_0^T \cos(t)e^{-st} \, dt = -\frac{1}{s} \cos(t)e^{-st} \big|_0^T - \frac{1}{s} \int_0^T \sin(t)e^{-st} \, dt.
\]

We apply the integration by parts formula (85) again, but with \(u(t) = \sin(t)\) and \(v'(t) = \exp(-st)\) to obtain

\[
\int_0^T \sin(t)e^{-st} \, dt = -\frac{1}{s} \cos(t)e^{-st} \big|_0^T + \frac{1}{s^2} \sin(t)e^{-st} \big|_0^T - \frac{1}{s^2} \int_0^T \cos(t)e^{-st} \, dt.
\]

Hence,

\[
\left( 1 + \frac{1}{s^2} \right) \int_0^T \cos(t) e^{-st} \, dt = -\frac{1}{s} \cos(t) e^{-st} \big|_0^T + \frac{1}{s^2} \sin(t) e^{-st} \big|_0^T.
\]

If \(\text{Re}(s) > 0\), then \(\lim_{T \to \infty} \cos(t) e^{-st} = 0\), \(\lim_{T \to \infty} \sin(t) e^{-st} = 0\), and

\[
\lim_{T \to \infty} \left( 1 + \frac{1}{s^2} \right) \int_0^T \cos(t) e^{-st} \, dt = \frac{1}{s}.
\]

Hence, for \(s\) with \(\text{Re}(s) > 0\),

\[
\mathcal{L}(\cos)(s) = \lim_{T \to \infty} \int_0^T \cos(t) e^{-st} \, dt = \frac{s}{s^2 + 1}.
\]
The next example illustrates how we can use the Laplace transform to solve a dynamical system (83). The idea is to use (92) to compute the Laplace transform of the unknown solution \( x \) and then invert the Laplace transform. For now we find a (continuous) function so that its Laplace transform matches the right hand side in (92) and use Lerch’s Theorem below to argue that the unknown \( x \) is equal to that function.

**Theorem 67 (Lerch’s Theorem)** If \( f \) and \( g \) are continuous on \((0, \infty)\) and the Laplace transforms of \( f \) and \( g \) exist for all \( s \) with \( \text{Re}(s) > \alpha \) for some \( \alpha \in \mathbb{R} \), and are equal, i.e.,

\[
L(f)(s) = L(g)(s) \quad \text{for all} \quad \text{Re}(s) > \alpha,
\]

then \( f = g \).

**Example 68** Consider the dynamical system

\[
\begin{align*}
  x'(t) &= -x(t) + e^{-t}, \quad t > 0, \quad (93a) \\
  x(0) &= 2. \quad (93b)
\end{align*}
\]

The Laplace transform of the solution satisfies

\[
L(x)(s) = (s + 1)^{-1} \left( 2 + L(e^{-t})(s) \right) = (s + 1)^{-1} \left( 2 - \frac{1}{s + 1} \right) = \frac{2}{s + 1} + \frac{1}{(s + 1)^2}.
\]

(The Laplace transform of \( f(t) = \exp(-t) \) can be computed as in Example 66.) From Example 66 we recognize that \( 2/(s+1) \) is the Laplace transform of \( 2e^{-t} \) and \( 1/(s+1)^2 \) is the Laplace transform of \( te^{-t} \). Hence,

\[
L(x)(s) = \frac{2}{s + 1} + \frac{1}{(s + 1)^2} = L\left((2 + t)e^{-t}\right)(s),
\]

which implies that the solution of (93) is

\[
x(t) = (2 + t)e^{-t}. \quad (94)
\]

Plug (94) into (93) to double check that it is in fact the solution.

We have the following properties of the Laplace transform

\[
\begin{align*}
  L\left(\frac{d}{dt} x\right)(s) &= sL(x)(s) - x(0), \quad (95a) \\
  L(af + bg)(s) &= aL(f)(s) + bL(g)(s) \quad \text{for functions} \ f, g \ \text{and scalars} \ a, b. \quad (95b)
\end{align*}
\]
The Laplace transform of a vector of functions is obtained by applying the Laplace transform to each component. In particular, for a vector valued function $x : [0, \infty) \rightarrow \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n \times n}$ we have $\mathcal{L}(\frac{d}{dt}x)(s) = s\mathcal{L}(x)(s) - x(0)$ and 

$$\mathcal{L}(Ax)(s) = A\mathcal{L}(x)(s).$$

If we take the Laplace transform on either side of (82a) then 

$$s\mathcal{L}(x)(s) - x(0) = A\mathcal{L}(x)(s) + \mathcal{L}(f)(s),$$

or

$$(sI - A)\mathcal{L}(x)(s) = x_0 + \mathcal{L}(f)(s).$$

Thus by taking Laplace transforms on either side of (82a) and using the initial condition (82a) the Laplace transform of the unknown solution can be determined by solving a linear system:

$$\mathcal{L}(x)(s) = (sI - A)^{-1}(x_0 + \mathcal{L}(f)(s)).$$

(96)

**Example 69** Consider the dynamical system

$$x'(t) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x(t) + \begin{pmatrix} e^t \\ \cos(t) \end{pmatrix}, \quad t > 0;$$

(97a)

$$x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$  

(97b)

The Laplace transform of the solution satisfies

$$\mathcal{L}(x)(s) = \left( \begin{array}{cc} s - 2 & 1 \\ 1 & s - 2 \end{array} \right)^{-1} \left( \begin{array}{c} 2 \\ 1 \end{array} + \begin{pmatrix} \mathcal{L}(e^t)(s) \\ \mathcal{L}(\cos(t))(s) \end{pmatrix} \right)$$

$$= \left( \begin{array}{cc} s - 2 & 1 \\ 1 & s - 2 \end{array} \right)^{-1} \left( \begin{array}{c} 2 \\ 1 \end{array} + \begin{pmatrix} 1/(s - 1) \\ s/(s^2 + 1) \end{pmatrix} \right).$$

Since

$$\left( \begin{array}{cc} s - 2 & 1 \\ 1 & s - 2 \end{array} \right)^{-1} = \frac{1}{s^2 - 4s + 3} \left( \begin{array}{cc} s - 2 & -1 \\ -1 & s - 2 \end{array} \right) = \frac{1}{(s - 1)(s - 3)} \left( \begin{array}{cc} s - 2 & -1 \\ -1 & s - 2 \end{array} \right)$$

we can compute

$$\mathcal{L}(x)(s) = \frac{1}{(s - 1)(s - 3)} \left( \begin{array}{c} 2s - 5 \\ s - 4 \end{array} \right) + \frac{1}{(s - 1)^2(s^2 - 1)(s - 3)} \left( \begin{array}{c} s^3 - 3s^2 + 2s - 2 \\ -3s^2 + 2s - 1 \end{array} \right).$$

To compute the solution $x$ from the previous expression for $\mathcal{L}(x)(s)$ we need to build a few more tools. We will continue with the computation of the solution in Example 93. ⊗
16 Complex Functions

Every complex function \( f : \mathbb{C} \to \mathbb{C} \) with argument \( z = x + iy \) can be written as

\[
 f(x + iy) = u(x, y) + iv(x, y).
\]

The complex exponential function is given by \( \exp(z) = \sum_{k=0}^{\infty} z^k/k! \) and with \( z = x + iy \) leads to

\[
 e^{x+iy} = e^x \left( \cos(y) + i \sin(y) \right). \tag{98}
\]

As in the real case

\[
 \exp(z_1 + z_2) = \exp(z_1) \exp(z_2).
\]

The complex sine and cosine functions are

\[
 \sin(z) = \frac{1}{2i} \left( \exp(iz) - \exp(-iz) \right), \tag{99}
\]

\[
 \cos(z) = \frac{1}{2} \left( \exp(iz) + \exp(-iz) \right). \tag{100}
\]

**Definition 70** A complex function \( f : \mathbb{C} \to \mathbb{C} \) is differentiable at \( z_0 \) if the limit \( \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \) exists. In this case the derivative of \( f \) at \( z_0 \) is given by

\[
 f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.
\]

**Example 71**

1. Let \( f(z) = \text{Re}(z) = x \) and \( z_0 = x_0 + iy_0 \). We consider the difference quotient with \( z = z_0 + h \) and the difference quotient with \( z = z_0 + ih \). We find

\[
 \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{z + h - z_0} = \lim_{h \to 0} \frac{x_0 + h - x_0}{h} = 1
\]

and

\[
 \lim_{h \to 0} \frac{f(z_0 + ih) - f(z_0)}{z_0 + ih - z_0} = \lim_{h \to 0} \frac{x_0 - x_0}{ih} = 0.
\]

Since the two limits do not coincide, \( \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \) does not exist and, consequently, the function \( f(z) = \text{Re}(z) \) is not differentiable at any \( z \in \mathbb{C} \).

\[ \diamond \]

The previous example hints at conditions that need to be satisfied when \( f \) is differentiable at \( z_0 \). In this case for every sequence \( \{z_n\} \) of points converging to \( z_0 \) the limits \( \lim_{n \to \infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} \) must
coincide. If, in particular, we look at sequences of points converging to \( z_0 \) along the real axis and sequences of points converging to \( z_0 \) along the imaginary axis, their limits must coincide. That is, if the function \( f \) is differentiable at the point \( z_0 \), then

\[
\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{f(z_0 + i h) - f(z_0)}{i h} = f'(z_0)
\]

and

\[
\lim_{h \to 0} \frac{f(z_0 + i h) - f(z_0)}{i h} = \lim_{h \to 0} \frac{f(z_0 + i h) - f(z_0)}{h} = f'(z_0).
\]

If we insert \( f(x + iy) = u(x, y) + iv(x, y) \), \( z_0 = x_0 + iy_0 \), we obtain

\[
f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h}
\]

\[
= \frac{\partial}{\partial x} u(x_0, y_0) + i \frac{\partial}{\partial x} v(x_0, y_0)
\]

and

\[
f'(z_0) = \lim_{h \to 0} -i \frac{f(z_0 + i h) - f(z_0)}{h} = \lim_{h \to 0} -i \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h} + \frac{v(x_0, y_0 + h) - v(x_0, y_0)}{h}
\]

\[
= -i \frac{\partial}{\partial y} u(x_0, y_0) + \frac{\partial}{\partial y} v(x_0, y_0).
\]

Comparing the real and imaginary parts, we conclude that

\[
\frac{\partial}{\partial x} u(x, y) = \frac{\partial}{\partial y} v(x, y), \quad (101a)
\]

\[
\frac{\partial}{\partial y} u(x, y) = -\frac{\partial}{\partial x} v(x, y) \quad (101b)
\]

hold at \( z_0 = x_0 + iy_0 \). The equations (101) are the Cauchy-Riemann equations. Thus we have proven the following result.

**Theorem 72** If the function \( f(x + iy) = u(x, y) + iv(x, y) \) is differentiable at \( z_0 = x_0 + iy_0 \), then the Cauchy-Riemann equations (101) hold at \( z_0 = x_0 + iy_0 \).
Note that the Cauchy-Riemann equations are a necessary condition for differentiability. If a function is differentiable at a point, then the Cauchy-Riemann equations must be satisfied at that point. Conversely, if the Cauchy-Riemann equations are not satisfied at a point, then the function is not differentiable at that point. However, just because the Cauchy-Riemann equations hold at a point, does not allow one to conclude that the function is differentiable at that point. We illustrate this using some examples.

**Example 73**

1. In Example 71 we have already shown that the function \( f(z) = \text{Re}(z) = x \) is nowhere differentiable. Here we derive the same result using the Cauchy-Riemann equations. The partial derivatives of the real and imaginary part of \( f \) are

\[
\frac{\partial}{\partial x} u(x, y) = 1, \quad \frac{\partial}{\partial y} v(x, y) = 0,
\]
\[
\frac{\partial}{\partial y} u(x, y) = 0, \quad \frac{\partial}{\partial x} v(x, y) = 0.
\]

Hence the Cauchy-Riemann equations are nowhere satisfied and the function \( f(z) = \text{Re}(z) \) is nowhere differentiable.

2. Consider \( f(z) = (x^2 + y) + i(y^2 - x) \). We have

\[
\frac{\partial}{\partial x} u(x, y) = 2x, \quad \frac{\partial}{\partial y} v(x, y) = 2y,
\]
\[
\frac{\partial}{\partial y} u(x, y) = 1, \quad \frac{\partial}{\partial x} v(x, y) = -1.
\]

Hence the Cauchy-Riemann equations are only satisfied at \( z = x + ix \). The function \( f(z) = (x^2 + y) + i(y^2 - x) \) is not differentiable at any point \( z \notin \{x + ix : x \in \mathbb{R}\} \).

What about the differentiability at points \( z \in \{x + ix : x \in \mathbb{R}\} \)? The Cauchy-Riemann equations are only a necessary condition for differentiability. Thus, we cannot conclude that because the Cauchy-Riemann equations are satisfied the function is differentiable. In particular, Theorem 72 does not tell us anything about the differentiability at points \( z \in \{x + ix : x \in \mathbb{R}\} \).

To check the differentiability of \( f \) at \( z_0 = x_0 + ix_0 \) we have to consider difference quotients.
with \( z = z_0 + h + ik \), i.e., we have to consider

\[
\lim_{(h,k) \to (0,0)} \frac{f(z_0 + h + ik) - f(z_0)}{z_0 + h + ik - z_0}
= \lim_{(h,k) \to (0,0)} \frac{((x_0 + h)^2 + (x_0 + k)) + i((x_0 + k)^2 - (x_0 + h)) - [(x_0^2 + x_0) + i(x_0^2 - x_0)]}{h + ik}
= \lim_{(h,k) \to (0,0)} \frac{2x_0 h + h^2 + k + i(2x_0k + k^2 - h)}{h + ik}
= \lim_{(h,k) \to (0,0)} \frac{2x_0(h + ik) + (h^2 + k^2) + (k - ih)}{h + ik}
= \lim_{(h,k) \to (0,0)} \frac{2x_0(h + ik) + (h^2 + k^2) - i(h + ik)}{h + ik}
= 2x_0 - i + \lim_{(h,k) \to (0,0)} \frac{h^2 + k^2}{h + ik}
= 2x_0 - i + \lim_{(h,k) \to (0,0)} \frac{(h^2 + k^2)(h - ik)}{h^2 + k^2}
= 2x_0 - i + 0.
\]

Thus, the function \( f(z) = (x^2 + y) + iy^2 - x \) is differentiable at \( z_0 = x_0 + ix_0 \) and the derivative is

\[
f'(z_0) = 2x_0 - i, \quad \text{for} \quad z_0 = x_0 + ix_0
\]

3. Consider

\[
f(z) = \begin{cases} \frac{x^{4/3} y^{5/3} + i x^{5/3} y^{4/3}}{x^2 + y^2}, & \text{if} \ z \neq 0, \\ 0, & \text{if} \ z = 0. \end{cases}
\]

At \( z = 0 \), we have

\[
\frac{\partial}{\partial x} u(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h - 0} = \lim_{h \to 0} \frac{1}{h} \frac{h^{4/3} 0^{5/3}}{h^2} = 0
\]

and, similarly,

\[
\frac{\partial}{\partial y} u(0,0) = \frac{\partial}{\partial x} v(0,0) = \frac{\partial}{\partial y} v(0,0) = 0.
\]

The Cauchy-Riemann equations are satisfied at \( z_0 = 0 \), however, the function is not differentiable at \( z_0 = 0 \). To see this consider the difference quotient with \( z = z_0 + h \) and the difference quotient with \( z = z_0 + h + ih \). We find that

\[
\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h - 0} = \lim_{h \to 0} \frac{1}{h} \frac{h^{4/3} 0^{5/3} + i h^{5/3} 0^{4/3}}{h^2 + 0^2} = 0
\]
and

\[
\lim_{h \to 0} \frac{f(z_0 + h + ih) - f(z_0)}{h + ih} = \lim_{h \to 0} \frac{1}{(1 + i)h} \frac{h^{4/3}h^{5/3} + ih^{5/3}h^{4/3}}{h^2 + h^2} = \lim_{h \to 0} \frac{1}{1 + i} \frac{h^3 + ih^3}{2h^3} = \frac{1}{2}.
\]

Since the two limits do not coincide, \( f \) is not differentiable at \( z_0 = 0 \).

A sufficient condition for the differentiability of a function is given in the following theorem.

**Theorem 74** Let \( f(x + iy) = u(x, y) + iv(x, y) \) be defined on some open set \( G \) including the point \( z_0 = x_0 + iy_0 \). If the partial derivatives of \( u \) and \( v \) exist in \( G \), are continuous at \( (x_0, y_0) \), and satisfy the Cauchy-Riemann equations (101) at \( z_0 \), then \( f \) is differentiable at \( z_0 \).

**Example 75**

1. Consider \( f(z) = (x^2 + y) + i(y^2 - x) \). In Example 73 we have established that \( f \) is differentiable at any \( z = x + ix \). We will argue that \( f \) is differentiable at any \( z = x + ix \) using Theorem 74. We have

\[
\frac{\partial}{\partial x} u(x, y) = 2x, \quad \frac{\partial}{\partial y} v(x, y) = 2y, \quad \frac{\partial}{\partial y} u(x, y) = 1, \quad \frac{\partial}{\partial x} v(x, y) = -1.
\]

The partial derivatives of \( u \) and \( v \) exist in \( \mathbb{C} \), are continuous at any point in \( \mathbb{C} \). Since the Cauchy-Riemann equations are satisfied at \( z = x + ix \), Theorem 74 implies that \( f \) is differentiable at any \( z = x + ix \).

2. Consider

\[
f(z) = \begin{cases} 
\frac{x^{4/3}y^{5/3} + ix^{5/3}y^{4/3}}{x^2 + y^2}, & \text{if } z \neq 0, \\
0 & \text{if } z = 0.
\end{cases}
\]

In Example 73 we have shown that \( f \) is not differentiable at \( z_0 = 0 \). Compute the partial derivatives of \( u \) and \( v \) and show that they are not continuous at \( (x_0, y_0) = (0, 0) \).
The following well-known identities for the differentiation of real functions also hold for the differentiation of complex functions.

\[
(f + g)'(z_0) = f'(z_0) + g'(z_0),
\]
\[
(af)'(z_0) = af'(z_0) \quad \text{for a constant } a \in \mathbb{C},
\]
\[
(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0),
\]
\[
\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.
\]
17 Partial Fraction Expansions

The solution of dynamical systems

\[ x'(t) = Ax(t) + f(t), \quad t > 0, \]
\[ x(0) = x_0. \]

using the Laplace transform leads to

\[ (\mathcal{L}x)(s) = (sI - A)^{-1}\left(x_0 + (\mathcal{L}f)(s)\right). \]

The inverse \((sI - A)^{-1}\) can be computed using Gaussian elimination and executing Gaussian elimination shows that \((sI - A)^{-1}\) is a matrix with entries given by rational functions of the type

\[ \frac{p_{ij}(s)}{\det(sI - A)}, \]

where \(\det(sI - A)\) is the characteristic polynomial of degree \(n\) of \(A\) and \(p_{ij}(s)\) is a polynomial of degree less than \(n\).

**Example 76**

- Consider

  \[ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]

  \[ (sI - A)^{-1} = \frac{1}{s^2 - 4s + 3} \begin{pmatrix} s - 2 & -1 \\ -1 & s - 2 \end{pmatrix}. \]

  The characteristic polynomial of \(A\) is \(\det(sI - A) = s^2 - 4s + 3 = (s - 1)(s - 3)\). Consequently, the eigenvalues are given by \(\lambda_1 = 3\) and \(\lambda_2 = 1\).

- Consider

  \[ A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

  This is matrix in the reaction kinetics ODE (see Example 55) with \(k_1 = k_2 = 1\).

  \[ (sI - A)^{-1} = \frac{1}{s(1+s)^2} \begin{pmatrix} s(s + 1) & 0 & 0 \\ s & s(s + 1) & 0 \\ 1 & (s + 1) & (s + 1)^2 \end{pmatrix}. \]

  The characteristic polynomial of \(A\) is \(\det(sI - A) = (s + 1)^2s\) and the eigenvalues of \(A\) are given by \(\lambda_1 = -1\) (with algebraic multiplicity 2) and \(\lambda_2 = 0\).
To compute the inverse Laplace transform we will have to compute complex integrals with integrands of the type \( p_{ij}(s) / \det(sI - A) \) or \( p_{ij}(s) f_j(s) / \det(sI - A) \), where \( h_j \) is a component of the right hand side function \( h \) in the dynamical system. To compute these complex integrals, it will be beneficial to expand the rational functions \( p_{ij}(s) / \det(sI - A) \) in a partial fraction expansion.

We first consider the case in which all roots of the denominator are simple.

In this section we will use both \( s \) and \( z \) to denote a complex number.

Let \( g : \mathbb{C} \to \mathbb{C} \) be a complex polynomial of order \( m \) with roots \( \lambda_1, \ldots, \lambda_n \) of multiplicity one, i.e.,

\[
g(z) = c_n (z - \lambda_1) \cdots (z - \lambda_n)
\]

Furthermore, let \( h : \mathbb{C} \to \mathbb{C} \) be a complex polynomial of order at most \( n - 1 \). The rational function \( h/g \) has a partial fraction expansion of the form

\[
\frac{h(z)}{g(z)} = \frac{r_1}{z - \lambda_1} + \frac{r_2}{z - \lambda_2} + \cdots + \frac{r_n}{z - \lambda_n}.
\]

The roots \( \lambda_1, \ldots, \lambda_n \) of the denominator polynomial \( g \) that are not also roots of the numerator polynomial \( h \) are called poles of \( h/g \). We assume that the roots of \( h \) are different than \( \lambda_1, \ldots, \lambda_n \), i.e., that \( \lambda_1, \ldots, \lambda_n \) are the poles of \( h/g \).

To compute the coefficients \( r_1, \ldots, r_n \) we proceed as follows. Multiply \( h/g \) by \( z - \lambda_j \). This gives

\[
(z - \lambda_j) \frac{h(z)}{g(z)} = (z - \lambda_j) \left( \frac{r_1}{z - \lambda_1} + \cdots + \frac{r_n}{z - \lambda_n} \right).
\]

This identity holds for all \( z \) at which \( h(z)/g(z) \) is defined, i.e., for all \( z \notin \{\lambda_1, \ldots, \lambda_n\} \). If we take the limit

\[
\lim_{z \to \lambda_j} (z - \lambda_j) \frac{h(z)}{g(z)} = \lim_{z \to \lambda_j} (z - \lambda_j) \frac{r_1}{z - \lambda_1} + \cdots + \frac{r_n}{z - \lambda_n} = r_j,
\]

we obtain the \( j \)th coefficient.

**Example 77** Consider

\[
\frac{1}{(z - i)(z - 1)} = \frac{r_1}{z - i} + \frac{r_2}{z - 1}.
\]

Multiply by \( z - 1 \):

\[
\frac{1}{z - 1} = \frac{z - i}{(z - i)(z - 1)} = (z - i) \frac{r_1}{z - i} + (z - i) \frac{r_2}{z - 1} = r_1 + (z - i) \frac{r_2}{z - 1}.
\]
Evaluate at $z = i$:

\[ \frac{-i - 1}{2} = \frac{(-i - 1)}{(i - 1)(-i - 1)} = \frac{1}{i - 1} = r_1. \]

Multiply by $z - 1$:

\[ \frac{1}{z - i} = \frac{z - 1}{(z - i)(z - 1)} = \frac{r_1}{z - i} + \frac{r_2}{z - 1} = \frac{(z - 1)r_1}{z - i} + r_2. \]

Evaluate at $z = 1$:

\[ \frac{1 + i}{2} = \frac{1 + i}{(1 - i)(1 + i)} = \frac{1}{1 - i} = r_2. \]

Thus, the partial fraction expansion of $1/((z - i)(z - 1))$ is

\[ \frac{1}{(z - i)(z - 1)} = -\frac{1}{2} - \frac{1}{2}i + \frac{1}{2} + \frac{1}{2}i. \]

\[ \Box \]

**Example 78** Consider

\[ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]

The inverse $(sI - A)^{-1}$ can be computed using Gaussian elimination or using Matlab.

\[ \gg \text{syms } s \]
\[ \gg A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}; \]
\[ \gg \text{inv}( s\text{*eye}(2)-A) \]

\[ \text{ans} = \begin{bmatrix} \frac{s - 2}{s^2 - 4s + 3} & \frac{1}{s^2 - 4s + 3} \\ \frac{-1}{s^2 - 4s + 3} & \frac{s - 2}{s^2 - 4s + 3} \end{bmatrix} \]

\[ (sI - A)^{-1} = \frac{1}{s^2 - 4s + 3} \begin{pmatrix} s - 2 & -1 \\ -1 & s - 2 \end{pmatrix}. \]

The characteristic polynomial of $A$ is $p_A(s) = \det(sI - A) = s^2 - 4s + 3 = (s - 1)(s - 3)$ and the eigenvalues are given by $\lambda_1 = 3$ and $\lambda_2 = 1$.

We now compute partial fraction expansions of the entries of $(sI - A)^{-1}$.

\[ \bullet \]

\[ \frac{s - 2}{(s - 3)(s - 1)} = \frac{r_1}{s - 3} + \frac{r_2}{s - 1}. \]

Multiply by $s - 3$ to obtain $(s - 2)/(s - 1) = r_1 + (s - 3)r_2/(s - 1)$. Evaluate at $s = 3$ to get $r_1 = 1/2$.

Multiply by $s - 1$ to obtain $(s - 2)/(s - 3) = (s - 1)r_1/(s - 3) + r_2$. Evaluate at $s = 1$ to get $r_2 = 1/2$. 

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\[
\frac{-1}{(s - 3)(s - 1)} = \frac{r_1}{s - 3} + \frac{r_2}{s - 1}.
\]

Multiply by \(s - 3\) to obtain \(-1/(s - 1) = r_1 + (s - 3)r_2/(s - 1)\). Evaluate at \(s = 3\) to get \(r_1 = -1/2\).

Multiply by \(s - 1\) to obtain \(-1/(s - 3) = (s - 1)r_1/(s - 3) + r_2\). Evaluate at \(s = 1\) to get \(r_2 = 1/2\).

Insert partial fraction expansion into \((sI - A)^{-1}\)

\[
\frac{1}{s^2 - 4s + 3} \begin{pmatrix} s - 2 & -1 \\ -1 & s - 2 \end{pmatrix} = \frac{1}{s - 3} \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} + \frac{1}{s - 1} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = P_1 + P_2.
\]

Note that \(P_1^2 = P_1\), \(P_2^2 = P_2\), and \(P_1P_2 = 0\). In fact, we have seen these matrices \(P_1\) and \(P_2\) before. See Example 38. It is no accident that the matrices \(P_1\) and \(P_2\) from the eigen-decomposition appear here again.

We can compute this partial fraction expansion also using Matlab. For example the partial fraction expansion of \(s^2 - 4s + 3\) can be computed using

\[
\begin{align*}
&\text{>> [r,p,k]=residue([0,1,-2],[1,-4,3])} \\
r &= [0.5000, 0.5000] \\
p &= [3, 1] \\
k &= []
\end{align*}
\]

\(p\) contains the roots of \(s^2 - 4s + 3\) and \(r\) contains the scalars \(r_k\) corresponding to the roots \(\lambda_k\). Thus

\[
\frac{s - 2}{s^2 - 4s + 3} = \frac{1/2}{s - 3} + \frac{1/2}{s - 1},
\]

which is of course identical to what we have computed by hand. \(\diamond\)
Next we consider the case in which the denominator polynomial has multiple roots. Let $g : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree $n$ with roots $\lambda_1, \ldots, \lambda_k$ of multiplicities $m_1, \ldots, m_k$, i.e.,

$$g(z) = c_n(z - \lambda_1)^{m_1} \cdots (z - \lambda_k)^{m_k}$$

and $\sum_{j=1}^{k} m_j = n$. Furthermore, let $h : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree at most $n - 1$. The rational function $r(z) = h(z)/g(z)$ can be written in the form

$$r(z) = \sum_{j=1}^{k} \sum_{\ell=1}^{m_j} \frac{r_{j,\ell}}{(z - \lambda_j)^\ell}$$

for all $z \notin \{\lambda_1, \ldots, \lambda_n\}$.

We first consider an example on how to compute the coefficients $r_{j,\ell}$. Let

$$s^2 = \frac{r_{1,1}}{s+1} + \frac{r_{1,2}}{(s+1)^2} + \frac{r_{1,3}}{(s+1)^3}.\quad \text{We multiply by } (s+1)^3 \text{ and take the limit } s \to -1. \text{ This gives}$$

$$1 = \lim_{s \to -1} s^2 = \lim_{s \to -1} r_{1,1} + r_{1,2} + r_{1,3} = r_{1,3}.$$  

To compute the other coefficients, we note that $s^2 = r_{1,1} + r_{1,2} + r_{1,3}$ holds for all $s \neq -1$, and therefore

$$\frac{d}{ds}s^2 = \frac{d}{ds} (r_{1,1} + r_{1,2} + r_{1,3})$$

and

$$\frac{d^2}{ds^2}s^2 = \frac{d^2}{ds^2} (r_{1,1} + r_{1,2} + r_{1,3})$$

for all $s \neq -1$. Thus,

$$-2 = \lim_{s \to -1} 2s = \lim_{s \to -1} \frac{d}{ds}s^2$$

$$= \lim_{s \to -1} \frac{d}{ds} (r_{1,1} + r_{1,2} + r_{1,3}) = \lim_{s \to -1} 2r_{1,1} + r_{1,2} = r_{1,2}$$

and

$$2 = \lim_{s \to -1} 2s = \lim_{s \to -1} \frac{d^2}{ds^2}s^2 = \lim_{s \to -1} \frac{d^2}{ds^2} (r_{1,1} + r_{1,2} + r_{1,3}) = \lim_{s \to -1} 2r_{1,1} = 2r_{1,1}.$$
The partial fraction expansion is
\[
\frac{s^2}{(s + 1)^3} = \frac{1}{s + 1} + \frac{-2}{(s + 1)^2} + \frac{1}{(s + 1)^3}.
\]

More generally, the rational function \( r(z) = h(z)/g(z) \) can be written in the form
\[
r(z) = \sum_{j=1}^{k} \sum_{\ell=1}^{m_j} r_{j,\ell} \frac{1}{(z - \lambda_j)^\ell}
\]
where
\[
\lambda \notin \{\lambda_1, \ldots, \lambda_k\}, \\
r_{j,\ell} = \lim_{z \to \lambda_j} \frac{1}{(m_j - \ell)!} \frac{d^{m_j-\ell}}{dz^{m_j-\ell}} [(z - \lambda_j)^{m_j} r(z)].
\]

**Example 79** Consider
\[
A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

This is matrix in the reaction kinetics ODE (see Example 55) with \( k_1 = k_2 = 1 \). We have
\[
(sI - A)^{-1} = \frac{1}{s(1 + s)^2} \begin{pmatrix} s(s + 1) & 0 & 0 \\ s & s(s + 1) & 0 \\ 1 & (s + 1) & (s + 1)^2 \end{pmatrix}.
\]

The characteristic polynomial of \( A \) is \( p_A(s) = \det(sI - A) = s + 1)^2 s \) and the eigenvalues of \( A \) are given by \( \lambda_1 = -1 \) (with algebraic multiplicity 2) and \( \lambda_2 = 0 \).

We compute the partial fractions of the entries of \((sI - A)^{-1}\). For
\[
\frac{1}{s(s + 1)^2} = \frac{r_1}{s} + \frac{r_{2,1}}{s + 1} + \frac{r_{2,2}}{(s + 1)^2},
\]
we find
\[
r_1 = \frac{1}{(s + 1)^2} \big|_{s=0} = 1,
\]
\[
r_{2,1} = \left( \frac{1}{s} \right)' \big|_{s=-1} = \frac{-1}{s^2} \big|_{s=-1} = -1,
\]
\[
r_{2,2} = \frac{1}{s} \big|_{s=-1} = -1.
\]
Check:
\[
\frac{1}{s} + \frac{-1}{s+1} + \frac{-1}{(s+1)^2} = \frac{(s+1)^2}{s(s+1)^2} - \frac{s(s+1)}{s(s+1)^2} - \frac{s}{s(s+1)^2}
\]
\[
= \frac{(s^2 + 2s + 1) - (s^2 + s) - s}{s(s+1)^2} = \frac{1}{s(s+1)^2}.
\]

We can compute this partial fraction expansion also using Matlab

\[
>> \text{[}r, p, k\text{]} = \text{residue([0, 0, 0, 1], [1, 2, 1, 0])}
\]
\[
r =
\begin{cases}
-1 \\
-1 \\
1
\end{cases}
\]
\[
p =
\begin{cases}
-1 \\
-1 \\
0
\end{cases}
\]
\[
k =
\begin{cases}
[]
\end{cases}
\]

For
\[
\frac{1}{(s+1)^2} = \frac{r_{1,1}}{s+1} + \frac{r_{1,2}}{(s+1)^2},
\]

we find
\[
\begin{align*}
\left. r_{1,1} \right|_{s=-1} &= 0, \\
\left. r_{1,2} \right|_{s=-1} &= 1.
\end{align*}
\]

Of course, we could have read off these coefficients directly from (102).

The partial fraction expansions for the other entries of \((sI - A)^{-1}\) can be computed analogously. Inserting the partial fraction expansion into \((sI - A)^{-1}\) gives
\[
(sI - A)^{-1} = \frac{1}{s} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{s+1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix} + \frac{1}{(s+1)^2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Note that \(P_1^2 = P_1, P_2^2 = P_2, P_1 P_2 = 0, P_1 D_2 = D_2 P_1 = 0, P_2 D_2 = D_2 P_2 = D_2.\)

\[
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\]
18 Complex Integration

Let

\[ C = \{ z(t) : t \in [t_1, t_2] \} \]

where \( z \) is continuously differentiable and one-to-one. Such a set \( C \) is called a directed smooth curve. The integral of a continuous function \( f : C \to \mathbb{C} \) on the directed smooth curve is defined as

\[ \int_C f(z) \, dz \overset{\text{def}}{=} \int_{t_1}^{t_2} f(z(t))z'(t) \, dt. \]

Example 80 Let \( C = \{ t + it^2 : t \in [0, 1] \} \) and \( f(z) = z^2 \). Then

\[ \int_C f(z) \, dz = \int_0^1 (t + it^2)^2(1 + 2ti) \, dt \]
\[ = \int_0^1 (t^2 - t^4 + 2t^3i)(1 + 2ti) \, dt \]
\[ = \int_0^1 (t^2 - 5t^4) + i(4t^3 - 2t^5) \, dt \]
\[ = (t^3/3 - t^5) + i(t^4 - t^6/3) \bigg|_0^1 = -2/3 + 2/3 i. \]

The orientation of the curve matters. Changing the orientation switches the sign of the integral. Let \( C_- = \{(1 - t) + i(1 - t)^2 : t \in [0, 1] \} \) and \( f(z) = z^2 \). Then

\[ \int_{C_-} f(z) \, dz = \int_0^1 ((1 - t) + i(1 - t)^2)^2 (-1 - 2(1 - t)i) \, dt \]
\[ = \int_0^1 -((1 - t)^2 - 5(1 - t)^4) - i(4(1 - t)^3 - 2(1 - t)^5) \, dt \]
\[ = ((1 - t)^3/3 - (1 - t)^4) + i((1 - t)^4 - (1 - t)^6/3) \bigg|_0^1 = 2/3 - 2/3 i. \]
Example 81 Let $C(a, r) = \{ a + re^{it} : t \in [0, 2\pi] \}$ be the circle around $a$ with radius $r$.

$$\int_{C(a, r)} (z - a)^m dz = \int_0^{2\pi} (a + re^{it} - a)^m rie^{it} dt$$

$$= r^{m+1}i \int_0^{2\pi} e^{i(m+1)t} dt$$

$$= r^{m+1}i \left( \int_0^{2\pi} \cos((m + 1)t) dt + i \int_0^{2\pi} \sin((m + 1)t) dt \right)$$

$$= \left\{ \begin{array}{ll}
2\pi i & \text{if } m = -1 \\
0 & \text{else.}
\end{array} \right.$$

\[ \diamond \]

Theorem 82 (Cauchy’s Theorem) If the function $f$ is differentiable on and inside the closed curve $C$, then

$$\int_C f(z) dz = 0.$$

Theorem 83 (Curve Replacement Lemma) Let $C_1$ and $C_2$ be two closed curves with the same orientation such that $C_1$ lies in the inside of $C_2$. If the function $f$ is differentiable on the curves $C_1$ and $C_2$ and in the region between the curves $C_1$ and $C_2$, then

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz.$$

Notation: In the following

$$C(a, r) \overset{\text{def}}{=} \{ a + re^{it} : t \in [0, 2\pi] \}$$

denotes the circle in the complex plane with center $a$ and radius $r$.

Example 84 In Example 79 we have computed the partial fraction expansion

$$\frac{1}{z(z + 1)^2} = \frac{1}{z} + \frac{-1}{z + 1} + \frac{-1}{(z + 1)^2}.$$

If $C$ is a closed curve encircling the points 0 and $-1$, then, by the Curve Replacement Lemma

$$\int_C \frac{1}{z(z + 1)^2} dz = \int_{C(0,1/3)} \frac{1}{z(z + 1)^2} dz + \int_{C(-1,1/3)} \frac{1}{z(z + 1)^2} dz,$$

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where \( C(0, 1/3) \) and \( C(-1, 1/3) \) are the circles with center 0 and \(-1\), respectively, and radii 1/3. The integrals can be computed using the partial fraction expansion, Cauchy’s Theorem, and Example 81. We have

\[
\int_C \frac{1}{z(z+1)^2} \, dz = \int_{C(0,1/3)} \frac{1}{z(z+1)^2} \, dz + \int_{C(-1,1/3)} \frac{1}{z(z+1)^2} \, dz
\]

\[
= \int_{C(0,1/3)} \frac{1}{z} \, dz + \int_{C(0,1/3)} \frac{-1}{z+1} \, dz + \int_{C(-1,1/3)} \frac{-1}{z+1} \, dz
\]

\[
= 2\pi i \text{ by Ex. 81} \quad = 0 \text{ by Cauchy’s Thm.} \quad = 0 \text{ by Cauchy’s Thm.}
\]

\[
+ \int_{C(-1,1/3)} \frac{1}{z} \, dz + \int_{C(-1,1/3)} \frac{-1}{z+1} \, dz + \int_{C(-1,1/3)} \frac{-1}{z+1} \, dz
\]

\[
= 0 \text{ by Cauchy’s Thm.} \quad = -2\pi i \text{ by Ex. 81} \quad = 0 \text{ by Ex. 81}
\]

\[
= 2\pi i - 2\pi i = 0.
\]

The previous example indicates how to integrate rational functions in partial fraction expansion.

Let

\[
r(z) = \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{r_{j,k}}{(z-\lambda_j)^k}
\]

and let \( C \) be a closed curve encircling all poles \( \lambda_1, \ldots, \lambda_h \) of the rational function \( q \). If \( C_j \) is the circle around \( \lambda_j \) with radius small enough so that all other poles are outside \( C_j \) and the circles \( C_1, \ldots, C_h \) do not intersect, see Figure 27, then

\[
\int_{C} r(z) \, dz = \int_{C} \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{r_{j,k}}{(z-\lambda_j)^k} \, dz
\]

\[
= \sum_{j=1}^{h} \int_{C_j} \sum_{k=1}^{m_j} \frac{r_{j,k}}{(z-\lambda_j)^k} \, dz
\]

\[
= \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{r_{j,k}}{(z-\lambda_j)^k} \int_{C_j} \frac{1}{(z-\lambda_j)^k} \, dz = 2\pi i \sum_{j=1}^{h} \frac{r_{j,1}}{j-1}
\]

\[
= 2\pi i \text{ if } \ell = j \text{ and } k = 1; \quad = 0 \text{ else}
\]

**Theorem 85 (Cauchy’s Integral Formula)** If the function \( f \) is differentiable on and inside the closed curve \( C \), then

\[
f(a) = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z-a} \, dz
\]

for each \( a \) lying inside \( C \).
Figure 27: The poles $\lambda_1, \ldots, \lambda_h$ of the rational function $r$, the curve $C$ and the circles used to compute $\int_C r(z) \, dz$.

**Proof:** Let $a$ be a point inside $C$ and let $r > 0$ such that $C(a, r)$ is inside $C$. By the Curve Replacement Lemma

$$\int_C \frac{f(z)}{z-a} \, dz = \lim_{r \to 0} \int_{C(a, r)} \frac{f(z)}{z-a} \, dz$$

$$= \lim_{r \to 0} \int_{C(a, r)} \frac{f(a)}{z-a} \, dz + \int_{C(a, r)} \frac{f(z)-f(a)}{z-a} \, dz$$

$$= f(a)2\pi i + \lim_{r \to 0} \int_{C(a, r)} \frac{f(z)-f(a)}{z-a} \, dz.$$  

Since $f$ is differentiable at $a$, $\lim_{z \to a} \frac{f(z)-f(a)}{z-a} = f'(a)$. In particular, $\frac{f(z)-f(a)}{z-a}$ is bounded for all $z$ sufficiently close to $a$, i.e., there exists $M > 0$ and $\epsilon > 0$ such that

$$\left| \frac{f(z)-f(a)}{z-a} \right| \leq M$$

for all $z$ with $|z-a| \leq \epsilon$. This implies

$$\left| \int_{C(a, r)} \frac{f(z)-f(a)}{z-a} \, dz \right| \leq M2\pi r$$

for all $r \leq \epsilon$. Hence,

$$\int_C \frac{f(z)}{z-a} \, dz = \lim_{r \to 0} \int_{C(a, r)} \frac{f(z)}{z-a} \, dz = f(a)2\pi i + \lim_{r \to 0} \int_{C(a, r)} \frac{f(z)-f(a)}{z-a} \, dz = f(a)2\pi i.$$
The Cauchy Integral Formula provides an alternative representation of $f$ inside the circle. For any $a$ lying inside $C$ we have

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \, dz. \quad (103)$$

The right hand side in (103) can be differentiated with respect to $a$:

$$\frac{d}{da} \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \, dz = \frac{1}{2\pi i} \int_C \frac{d}{da} \frac{f(z)}{z-a} \, dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} \, dz.$$

Therefore, because of the identity (103), $f$ is differentiable and

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} \, dz. \quad (104)$$

The right hand side in (104) can be differentiated with respect to $a$:

$$\frac{d}{da} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} \, dz = \frac{1}{2\pi i} \int_C \frac{d}{da} \frac{f(z)}{(z-a)^2} \, dz = \frac{2}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} \, dz.$$

Therefore, because of the identity (104), $f$ is twice differentiable and

$$f''(a) = \frac{2}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} \, dz. \quad (105)$$

We can repeat this process to obtain

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \, dz$$

for any integer $n$ and any $a$ inside the curve $C$. In particular, we have shown that if the function $f$ is differentiable on and inside the closed curve $C$, if is infinitely often differentiable inside the closed curve $C$!

**Theorem 86 (Cauchy’s Integral Formula)** If the function $f$ is differentiable on and inside the closed curve $C$, then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \, dz \quad (106)$$

for each $a$ lying inside $C$.

In the next example we will use curve Replacement Lemma and Cauchy’s Integral Formula, Theorem 85, to compute complex integrals.
Example 87 1. Let $C(0, 3) = \{3e^{it} : t \in [0, 2\pi]\}$. Compute the integral
\[
\int_{C(0,3)} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)}\,dz.
\]
Let $C(1, 1/2)$ be the circle around $a = 1$ with radius $r = 1/2$ and let $C(2, 1/2)$ be the circle around $a = 2$ with radius $r = 1/2$. By the Curve Replacement Lemma
\[
\int_{C(0,3)} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)}\,dz = \int_{C(1,1/2)} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)}\,dz + \int_{C(2,1/2)} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)}\,dz.
\]
The function $f(z) = \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)}$ is differentiable on $C(1, 1/2)$ and inside $C(1, 1/2)$. Hence, by the Cauchy Integral Formula (103),
\[
\int_{C(1,1/2)} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)}\,dz = 2\pi i \frac{\sin(\pi 1^2) + \cos(\pi 1^2)}{(1 - 2)} = -2\pi i(\sin(\pi) + \cos(\pi)) = 2\pi i.
\]
The function $f(z) = \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)}$ is differentiable on $C(2, 1/2)$ and inside $C(2, 1/2)$. Hence, by the Cauchy Integral Formula (103),
\[
\int_{C(2,1/2)} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)}\,dz = 2\pi i \frac{\sin(\pi 2^2) + \cos(\pi 2^2)}{(2 - 1)} = 2\pi i(\sin(4\pi) + \cos(4\pi)) = 2\pi i.
\]
Consequently,
\[
\int_{C(0,3)} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)}\,dz = 2\pi i + 2\pi i = 4\pi i.
\]

2. Let $C(0, 3) = \{3e^{it} : t \in [0, 2\pi]\}$. Compute the integral
\[
\int_{C(0,3)} \frac{e^{2z}}{(z + 1)^4}\,dz.
\]
We use equation (106) with $a = -1$, $n = 3$, $f(z) = e^{2z}$. We can apply this formula, since $a = -1$ is inside $C(0, 3)$ and $f(z) = e^{2z}$ is differentiable on $C(0, 3)$ and inside $C(0, 3)$.
\[
f(z) = e^{2z}, \quad f'(z) = 2e^{2z}, \quad f''(z) = 4e^{2z}, \quad f'''(z) = 8e^{2z}.
\]
\[
\int_{C(0,3)} \frac{e^{2z}}{(z + 1)^4}\,dz = \frac{2\pi i}{3!} 8e^{2(-1)} = \frac{16\pi i}{6} e^{-2} = \frac{8\pi i}{3} e^{-2}.
\]
Our approach to compute integrals shown in the previous examples can be generalized. This will lead to the so-called Residue Theorem, which is an immediate consequence of the Curve Replacement Lemma and Cauchy’s Integral Formula, Theorem 85. Next, we derive the Residue Theorem.

Let \( g \) be a polynomial with roots \( \lambda_1, \ldots, \lambda_k \) of degree \( m_1, \ldots, m_h \), respectively, i.e., let

\[
g(z) = c(\lambda_1 - z)^{m_1} \cdots (\lambda_h - z)^{m_h}
\]

with some constant \( c \). Furthermore let \( C \) be a closed curve encircling each of the roots \( \lambda_1, \ldots, \lambda_k \) and let the function \( f \) be differentiable on and inside \( C \). We want to compute

\[
\int_C \frac{f(z)}{g(z)} \, dz.
\]

We proceed as in the computation of the integral of the partial fraction expansion of a rational function. Let \( C_j \) be a circle around \( \lambda_j \) with radius small enough so that all other roots are outside \( C_j \) and the circles \( C_1, \ldots, C_h \) do not intersect. By the curve replacement lemma

\[
\int_C \frac{f(z)}{g(z)} \, dz = \sum_{j=1}^h \int_{C_j} \frac{f(z)}{g(z)} \, dz = \sum_{j=1}^h \int_{C_j} \frac{\tilde{f}_j(z)}{(\lambda_j - z)^{m_j}} \, dz.
\]

If we define

\[
\tilde{f}_j(z) \overset{\text{def}}{=} \frac{f(z)}{c(\lambda_1 - z)^{m_1} \cdots (\lambda_j-1 - z)^{m_{j-1}}(\lambda_{j+1} - z)^{m_j+1} \cdots (\lambda_h - z)^{m_h}} = (\lambda_j - z)^{m_j} \frac{f(z)}{g(z)},
\]

then

\[
\int_C \frac{f(z)}{g(z)} \, dz = \sum_{j=1}^h \int_{C_j} \frac{f(z)}{g(z)} \, dz = \sum_{j=1}^h \int_{C_j} \frac{\tilde{f}_j(z)}{(\lambda_j - z)^{m_j}} \, dz.
\]

Furthermore,

\[
\int_{C_j} \frac{\tilde{f}_j(z)}{(\lambda_j - z)^{m_j}} \, dz = \frac{2\pi i}{(m_j - 1)!} \frac{d^{m_j-1}}{dz^{m_j-1}} \tilde{f}_j(\lambda_j) = \frac{2\pi i}{(m_j - 1)!} \lim_{z \to \lambda_j} \frac{d^{m_j-1}}{dz^{m_j-1}} \left( (z - \lambda_j)^{m_j} \frac{f(z)}{g(z)} \right).
\]

Thus we have proven the Residue Theorem.
Theorem 88 (The Residue Theorem) If $g$ is a polynomial with roots $\lambda_1, \ldots, \lambda_k$ of degree $m_1, \ldots, m_h$, respectively, and $C$ is a closed curve encircling each of the roots $\lambda_1, \ldots, \lambda_k$ and $h$ is a differentiable on an inside $C$, then

$$\int_C \frac{f(z)}{g(z)} dz = 2\pi i \sum_{j=1}^{h} \text{res}(\lambda_j; f/g),$$

where

$$\text{res}(\lambda_j; f/g) = \lim_{z \to \lambda_j} \frac{1}{(m_j - 1)!} \frac{d^{m_j-1}}{dz^{m_j-1}} \left( (z - \lambda_j)^{m_j} \frac{f(z)}{g(z)} \right)$$

is the residue of $f/g$ at $\lambda_j$.

If it is clear what functions $h$ and $g$ we take the residue of, then we simply write $\text{res}(\lambda_j)$ instead of $\text{res}(\lambda_j; f/g)$.

Example 89 1. Let $C = \{3e^{it} : t \in [0, 2\pi]\}$. In Example 87 we have shown that

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)} dz = 4\pi i.$$

We apply the Residue Theorem directly to recompute the integral. The residues are

$$\text{res}(1) = \lim_{z \to 1} (z - 1) \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)} = \frac{\sin(\pi) + \cos(\pi)}{1} = 1,$$

$$\text{res}(2) = \lim_{z \to 2} (z - 2) \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)} = \frac{\sin(4\pi) + \cos(4\pi)}{1} = 1.$$

Hence

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z - 1)(z - 2)} dz = 2\pi i (1 + 1) = 4\pi i.$$

2. Let $C = \{3e^{it} : t \in [0, 2\pi]\}$. In Example 87 we have shown that

$$\int_C \frac{e^{2z}}{(z + 1)^4} dz = \frac{8\pi i}{3} e^{-2}.$$

We apply the Residue Theorem directly to recompute the integral. The residue is

$$\text{res}(-1) = \lim_{z \to -1} \frac{1}{(4 - 1)!} \frac{d^3}{dz^3} e^{2z} = \lim_{z \to -1} \frac{1}{6} 2^3 e^{2z} = \frac{4}{3} e^{-2}.$$
Hence
\[ \int_{C} \frac{e^{2z}}{(z+1)^4} \, dz = 2\pi i \text{res}(-1) = \frac{8\pi i}{3} e^{-2}. \]

\[ \diamond \]

**Example 90**

1. Let \( C(1, 3) = \{ 1 + 3e^{it} : t \in [0, 2\pi] \} \). Compute
\[ \int_{C(1,3)} \frac{z^2}{(z+1)(z-1)^2}. \]

We apply the residue theorem. The roots of \((z+1)(z-1)^2\) are \( \lambda_1 = -1 \) with multiplicity \( m_1 = 1 \) and \( \lambda_2 = 1 \) with multiplicity \( m_2 = 2 \). Both roots are inside \( C(1,3) \). We apply the residue theorem with
\[
 f(z) = z^2, \quad g(z) = (z+1)(z-1)^2.
\]

The residues are
\[
 \text{res}(-1) = \lim_{z \to -1} \frac{1}{(1-1)!} \left( (z+1) \frac{z^2}{(z+1)(z-1)^2} \right) = \frac{1}{(z-1)^2} = 1/4,
\]
\[
 \text{res}(1) = \lim_{z \to 1} \frac{1}{(2-1)!} \frac{d}{dz} \left( (z-1)^2 \frac{z^2}{(z+1)(z-1)^2} \right) = \lim_{z \to 1} \frac{d}{dz} \left( \frac{z^2}{(z+1)} \right) = \lim_{z \to 1} \frac{2z(z+1) - z^2}{(z+1)^2} = 3/4.
\]

Hence
\[ \int_{C(1,3)} \frac{z^2}{(z+1)(z-1)^2} = 2\pi i \left( \frac{1}{4} + \frac{3}{4} \right) = 2\pi i. \]

2. Now, let \( C(1, 1) = \{ 1 + e^{it} : t \in [0, 2\pi] \} \). Compute
\[ \int_{C(1,1)} \frac{z^2}{(z+1)(z-1)^2}. \]
Again, we apply the residue theorem. The roots of \((z + 1)(z - 1)^2\) are \(\lambda_1 = -1\) with multiplicity \(m_1 = 1\) and \(\lambda_2 = 1\) with multiplicity \(m_2 = 2\). ONLY the root \(\lambda_2 = 1\) is inside \(C(1, 1)\). Hence we apply the residue theorem with

\[
f(z) = \frac{z^2}{(z + 1)}, \quad g(z) = (z - 1)^2.
\]

\[
\text{res}(1) = \lim_{z \to 1} \frac{1}{(2 - 1)!} \frac{d}{dz} \left( (z - 1)^2 \frac{z^2}{(z + 1)(z - 1)^2} \right)
= \lim_{z \to 1} \frac{d}{dz} \left( \frac{z^2}{(z + 1)} \right)
= \lim_{z \to 1} \left( \frac{2z(z + 1) - z^2}{(z + 1)^2} \right) = \frac{3}{4}.
\]

(The calculation is identical to the computation of \(\text{res}(1)\) in the previous part.) Hence

\[
\int_{C(1, 1)} \frac{z^2}{(z + 1)(z - 1)^2} = 2\pi i \frac{3}{4} = \frac{3\pi i}{2}.
\]
19  The Inverse Laplace Transform

An important application of the Residue Theorem is the computation of the Inverse Laplace Transform. The inverse Laplace transform of a rational function \( r \) is given by

\[
(\mathcal{L}^{-1}r)(t) = \frac{1}{2\pi i} \int_C r(z)e^{zt}dz \quad \text{for } t > 0,
\]

where \( C \) is a closed curve that encloses all poles \( \lambda_1, \ldots, \lambda_k \) of \( r \).

The integral in (107) can be computed using the Residue Theorem and we obtain

\[
(\mathcal{L}^{-1}r)(t) = \sum_{j=1}^{h} \text{res}(\lambda_j; r(z)e^{zt}) \quad \text{for } t > 0.
\]

**Example 91** In Example 66 we have computed the Laplace transforms of several functions. Now we apply the inverse Laplace transform to check whether we get the original functions back.

(a) We compute the inverse Laplace transform of \( r(s) = (s - 1)^{-1} \). We have \( \mathcal{L}^{-1}((s - 1)^{-1})(t) = \text{res}(1; r(s)e^{st}) \) and \( \text{res}(1; r(s)e^{st}) = e^t \). Hence,

\[
\mathcal{L}^{-1}((s - 1)^{-1})(t) = e^t \quad \text{for } t > 0.
\]

(b) We compute the inverse Laplace transform of \( r(s) = (1 + s)^{-2} \). We have \( \mathcal{L}^{-1}((s + 1)^{-2})(t) = \text{res}(-1; r(s)e^{st}) \) and

\[
\text{res}(-1; r(s)e^{st}) = \frac{d}{ds} e^{st}\bigg|_{s=-1} = te^{-t}.
\]

Hence,

\[
\mathcal{L}^{-1}((1 + s)^{-2})(t) = te^{-t} \quad \text{for } t > 0.
\]

(c) We compute the inverse Laplace transform of \( r(s) = s/(s^2 + 1) \). The roots of \( s^2 + 1 \) are \( s = i, -i \). We have \( \mathcal{L}^{-1}\left(s/(s^2 + 1)\right)(t) = \text{res}(i; e^{st}r(s)) + \text{res}(-i; e^{st}r(s)) \) with

\[
\text{res}(i; r(s)e^{st}) = \lim_{s \to i} (s - i)e^{st}s/(s^2 + 1) = e^{st}s/(s + i)\bigg|_{s=i} = \frac{1}{2}e^{it},
\]

\[
\text{res}(-i; r(s)e^{st}) = \lim_{s \to -i} (s + i)e^{st}s/(s^2 + 1) = e^{st}s/(s - i)\bigg|_{s=-i} = \frac{1}{2}e^{-it}.
\]

Hence,

\[
\mathcal{L}^{-1}\left(s/(s^2 + 1)\right)(t) = \frac{1}{2}(e^{it} + e^{-it}) = \cos(t) \quad \text{for } t > 0.
\]
Example 92 We want to solve the ordinary differential equation

\[ x'(t) + x(t) = e^{-t} \sin(t), \quad t > 0, \quad (109a) \]
\[ x(0) = 0. \quad (109b) \]

We define

\[ X(s) \stackrel{\text{def}}{=} L(x)(s). \]

Applying the Laplace transform to both sides of (109a) and using the initial condition gives

\[ L(x'(t) + x(t)) = sX(s) - x(0) + X(s) = (s + 1)X(s), \]
\[ L(\exp(-t) \sin(t)) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s + 1 + i)(s + 1 - i)}. \]

Hence the Laplace transform of the solution satisfies

\[ X(s) = \frac{1}{(s + 1)(s + 1 + i)(s + 1 - i)}. \]

It has 3 poles at \( \lambda_1 = -1, \lambda_2 = -1 - i, \) and \( \lambda_3 = -1 + i. \)

To get the solution, we compute the inverse Laplace transform via the Residue theorem:

\[ \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1} \left( \frac{1}{(s + 1)(s + 1 + i)(s + 1 - i)} \right) \]
\[ = \sum_{j=1}^{3} \text{res}(\lambda_j; X(s)e^{st}) \]
\[ = \frac{e^{st}}{(s + 1 - i)(s + 1 + i)} \bigg|_{s=1} + \frac{e^{st}}{(s + 1)(s + 1 + i)} \bigg|_{s=-1+i} + \frac{e^{st}}{(s + 1)(s + 1 - i)} \bigg|_{s=-1-i} \]
\[ = \exp(-t) - \frac{1}{2} (\exp(-t + it) + \exp(-t - it)) \]
\[ = \exp(-t) \left[ 1 - \frac{\exp(it) + \exp(-it)}{2} \right] \]
\[ = \exp(-t)(1 - \cos(t)). \]

Thus the solution of (109) is

\[ x(t) = \exp(-t)(1 - \cos(t)). \]

Insert \( x(t) = \exp(-t)(1 - \cos(t)) \) into (109) to verify that it is in fact the solution. \( \diamond \)
Example 93 In Example 69 we have begun to solve the dynamical system

\[ x'(t) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x(t) + \begin{pmatrix} e^t \\ \cos(t) \end{pmatrix}, \quad t > 0, \quad (110a) \]

\[ x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad (110b) \]

and we have shown that the Laplace transform of the solution satisfies

\[ X(s) \overset{\text{def}}{=} \mathcal{L}(x)(s) = \frac{1}{(s-1)(s-3)} \left( \begin{array}{cc} s-2 & -1 \\ -1 & s-2 \end{array} \right) \left( \begin{array}{c} 2 \\ 1 \end{array} \right) + \left( \frac{1}{s-1} \right) \left( \frac{1}{s/(s^2+1)} \right) \]

\[ = \frac{1}{(s-1)(s-3)} \left( \begin{array}{c} 2s-5 \\ s-4 \end{array} \right) + \frac{1}{(s-1)^2(s-3)} \left( \begin{array}{c} s-2 \\ -1 \end{array} \right) + \frac{1}{(s-1)(s-3)(s-i)(s+i)} \left( \begin{array}{c} -s \\ s/(s-2) \end{array} \right). \quad (111) \]

Note that to compute the inverse Laplace transform it may be better not to combine these three terms. Computing the inverse Laplace transform via (108) and the Residue Theorem gives

\[ \mathcal{L}^{-1} \left( \frac{1}{(s-1)(s-3)} \left( \begin{array}{c} 2s-5 \\ s-4 \end{array} \right) \right) \]

\[ = \frac{1}{s-3} \left( \frac{(2s-5)e^{st}}{(s-4)e^{st}} \right) \bigg|_{s=1} + \frac{1}{s-1} \left( \frac{(2s-5)e^{st}}{(s-4)e^{st}} \right) \bigg|_{s=3} \]

\[ = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} e^t + \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} e^{3t}, \quad (112a) \]

\[ \mathcal{L}^{-1} \left( \frac{1}{(s-1)^2(s-3)} \left( \begin{array}{c} s-2 \\ -1 \end{array} \right) \right) \]

\[ = \frac{1}{(s-3)^2} \left( \frac{[1 + t(s-2)](s-3)e^{st} - (s-2)e^{st}}{-t(s-3)e^{st} + e^{st}} \right) \bigg|_{s=1} + \frac{1}{(s-1)^2} \left( \frac{(s-2)e^{st}}{-e^{st}} \right) \bigg|_{s=3} \]

\[ = \begin{pmatrix} t/2 - 1/4 \\ t/2 + 1/4 \end{pmatrix} e^t + \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix} e^{3t}, \quad (112b) \]
and
\[
\mathcal{L}^{-1}\left(\frac{1}{(s - 1)(s - 3)(s - i)(s + i)} \left(\begin{array}{c}
-s \\
s(s - 2)
\end{array}\right)\right) = \frac{1}{(s - 3)(s - i)(s + i)} \left(\begin{array}{c}
-s_{est} \\
s(s - 2)_{est}
\end{array}\right)_{s = 1} + \frac{1}{(s - 1)(s - i)(s + i)} \left(\begin{array}{c}
-s_{est} \\
s(s - 2)_{est}
\end{array}\right)_{s = 3} + \frac{1}{(s - 1)(s - 3)(s + i)} \left(\begin{array}{c}
-s_{est} \\
s(s - 2)_{est}
\end{array}\right)_{s = i} + \frac{1}{(s - 1)(s - 3)(s - i)} \left(\begin{array}{c}
-s_{est} \\
s(s - 2)_{est}
\end{array}\right)_{s = -i}
\]
\[
= \left(\begin{array}{c}
1/4 \\
1/4
\end{array}\right) e^t + \left(\begin{array}{c}
-3/20 \\
3/20
\end{array}\right) e^{3t} + \frac{1}{8 + 4i} \left(\begin{array}{c}
-i \\
-1 - 2i
\end{array}\right) e^{it} + \frac{1}{8 - 4i} \left(\begin{array}{c}
i \\
-1 + 2i
\end{array}\right) e^{-it}
\]
\[
= \left(\begin{array}{c}
1/4 \\
1/4
\end{array}\right) e^t + \left(\begin{array}{c}
-3/20 \\
3/20
\end{array}\right) e^{3t} + \frac{1}{20} \left(\begin{array}{c}
-1 - 2i \\
-4 - 3i
\end{array}\right) e^{it} + \frac{1}{20} \left(\begin{array}{c}
-1 + 2i \\
-4 + 3i
\end{array}\right) e^{-it}
\]
\[
= \left(\begin{array}{c}
1/4 \\
1/4
\end{array}\right) e^t + \left(\begin{array}{c}
-3/20 \\
3/20
\end{array}\right) e^{3t} - \left(\begin{array}{c}
1/10 \\
4/10
\end{array}\right) \cos(t) + \left(\begin{array}{c}
1/5 \\
3/10
\end{array}\right) \sin(t).
\]

Adding (112a-c) gives
\[
x(t) = \mathcal{L}^{-1}(X(s)) = \left(\begin{array}{c}
3/2 \\
2
\end{array}\right) e^t + \left(\begin{array}{c}
3/5 \\
-3/5
\end{array}\right) e^{3t} + \left(\begin{array}{c}
1/2 \\
1/2
\end{array}\right) t e^t - \left(\begin{array}{c}
1/10 \\
4/10
\end{array}\right) \cos(t) + \left(\begin{array}{c}
1/5 \\
3/10
\end{array}\right) \sin(t).
\]