4 Complex Numbers

A complex number is given by $z = x + iy$, where $x$ and $y$ are real numbers and where $i$ is the imaginary unit, which satisfies $i^2 = -1$. Given $z = x + iy$, $x = \text{Re}(z)$ is the real part of $z$ and $y = \text{Im}(z)$ is the imaginary part of $z$. The set of all complex numbers is denoted by $\mathbb{C}$.

![Illustration of complex numbers](image)

The complex conjugate of a complex number $z = x + iy$ is given by $\overline{z} = x - iy$. The absolute value (or modulus or magnitude) of a complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}.$$ 

The addition and multiplication of complex numbers $z = a + ib$ and $w = c + id$ is defined as follows.

$$(a + ib) + (c + id) = a + c + i(b + d),$$
$$(a + ib)(c + id) = ac - bd + i(ad + bc).$$

Note that

$$|z|^2 = \overline{z}z = z\overline{z} = (a + ib)(a - ib) = a^2 + b^2.$$ 

The division of complex numbers $z = a + ib$ and $w = c + id$ is defined as follows.

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd + i(-ad + bc)}{c^2 + d^2}.$$
Polar Form

Every complex number $z = x + iy$ can be written as

$$z = r(cos \theta + i \sin \theta),$$

where $r = |z|$ is the length of $z$ and $\theta$ is such that $x = r \cos \theta$ and $y = r \sin \theta$. To make $\theta$ unique we require also that $\theta \in (-\pi, \pi].^1$ The representation (15) of a complex number is called the polar form.

![Figure 5: The polar form of a complex number](image)

To compute the polar form of $z = x + iy$, we first compute

$$r = |z| = \sqrt{x^2 + y^2}$$

The computation of the angle $\theta$ requires a bit more thought. The function $\tan(\theta) = \sin(\theta)/\cos(\theta)$ is not invertible. If we restrict $\tan$ to $(-\pi/2, \pi/2)$, then for every $y \in \mathbb{R}$ there exists a unique $\theta \in (-\pi/2, \pi/2)$ with $\tan(\theta) = y$. See Figure 6. The function $\arctan(y)$ returns this $\theta$.

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1Alternatively, we could have required that $\theta \in [0, 2\pi)$. 
Figure 6: The functions $\sin(\theta)$ and $\cos(\theta)$ (left plot), the function $\tan(\theta) = \sin(\theta)/\cos(\theta)$ (middle plot), and $\arctan(y)$ (right plot) is computed as the inverse of the branch of $\tan(\theta)$, $\theta \in (-\pi/2, \pi/2)$.

Since we require $\theta \in (-\pi, \pi]$, the angle in the polar form is given by

$$
\theta = \begin{cases} 
\pi/2 & \text{if } x = 0, y > 0 \\
-\pi/2 & \text{if } x = 0, y < 0, \\
\arctan(y/x) & \text{if } x > 0, \\
\arctan(y/x) + \pi & \text{if } x < 0, y \geq 0 \\
\arctan(y/x) - \pi & \text{if } x < 0, y < 0.
\end{cases}
$$

**Example 15** Consider the complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = -1 + \sqrt{3}i$. See Figure 7.

To compute the polar form of $z_1 = \sqrt{3} - i$, compute $r = 2$ and $\arctan(-1/\sqrt{3}) = -\pi/6$.

Since $x = \sqrt{3} > 0$, $z_1 = 2(\cos(-\pi/6) + i \sin(-\pi/6))$.

To compute the polar form of $z_2 = -1 + \sqrt{3}i$, compute $r = 2$ and $\arctan(\sqrt{3}/(-1)) = -\pi/3$.

Since $x = -1 < 0$ and $y = \sqrt{3} > 0$, $z_2 = 2(\cos(2\pi/3) + i \sin(2\pi/3))$.

If we require $\theta \in [0, 2\pi]$, then the angle in the polar form is given by

$$
\theta = \begin{cases} 
\pi/2 & \text{if } x = 0, y > 0 \\
3\pi/2 & \text{if } x = 0, y < 0, \\
\arctan(y/x) & \text{if } x > 0, y \geq 0 \\
\arctan(y/x) + \pi & \text{if } x < 0, y \geq 0 \\
\arctan(y/x) + 2\pi & \text{if } x > 0, y < 0 \\
\arctan(y/x) + \pi & \text{if } x < 0, y < 0.
\end{cases}
$$
Complex Vectors and Matrices

Given the vector
\[ z = (\zeta_1, \ldots, \zeta_n)^T \in \mathbb{C}^n. \]
its conjugate transpose is denoted by \( z^* \) and is the vector
\[ z^* = (\overline{\zeta}_1, \ldots, \overline{\zeta}_n). \]

The 2-norm of the vector \( z \) is given by
\[ \| z \|_2 = \sqrt{z^* z} = \sqrt{\sum_{j=1}^{n} |\zeta_j|^2}. \]

Given the complex \( m \times n \) matrix
\[ Z = (\zeta_{ij}) \in \mathbb{C}^{m \times n} \]
its conjugate transpose is denoted by \( Z^* \) and is given by
\[ Z^* = Z^T = (\overline{\zeta}_{ji}) \in \mathbb{C}^{n \times m}. \]