7 Eigenvectors and Eigenvalues

Let $A$ be a (possibly complex) square $n \times n$ matrix. A (possibly complex) scalar $\lambda$ is called an eigenvalue of $A$ if there exists a non-zero vector $v$ such that

$$Av = v\lambda.$$  \hspace{1cm} (27)

In this section we prove several properties of eigenvalues and eigenvectors that we already observed in the examples studied in the previous section. For the first result, revisit Example 17.

**Theorem 22** If $A$ is a real $n \times n$ matrix and $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ with corresponding eigenvector $v \in \mathbb{C}^n$, then $\overline{\lambda}$ is an eigenvalue of $A$ with corresponding eigenvector $\overline{v} \in \mathbb{C}^n$.

**Proof:** If $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ with corresponding eigenvector $v \in \mathbb{C}^n$, then

$$Av = v\lambda.$$ 

Taking the complex conjugate on either side gives

$$\overline{Av} = \overline{v\lambda}.$$ 

For complex numbers $z_1$ and $z_2$ we have $z_1\overline{z}_2 = \overline{z}_1z_2 = \overline{z}_1\overline{z}_2 = \overline{z}_2\overline{z}_1$. Therefore,

$$\overline{Av} = \overline{A}v = \overline{v\lambda}.$$ 

Since $A$ is real, $\overline{A} = A$ and we find that

$$A\overline{v} = \overline{v}\overline{\lambda}. \hspace{1cm} \square$$

For the next result, revisit Example 16.

A complex $n \times n$ matrix $A$ is called Hermitian if $A^* = A$. Real Hermitian matrices are symmetric matrices.

**Theorem 23** If $A$ is an $n \times n$ Hermitian matrix then all eigenvalues are real. If $A$ is a real $n \times n$ symmetric matrix then all eigenvalues and also the corresponding eigenvectors are real.

**Proof:** Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$ with corresponding eigenvector $v \in \mathbb{C}^n$, i.e, let

$$Av = v\lambda.$$ 

We have

$$\lambda\|v\|^2 = v^*v\lambda = v^*Av = v^*A^*v \quad \text{(since $A^* = A$)}$$

$$= (Av)^*v = (\lambda v)^*v = \overline{\lambda}v^*v = \overline{\lambda}\|v\|^2.$$
Since $v$ is a nonzero vector, $\|v\|^2 > 0$ and we obtain $\lambda = \overline{\lambda}$, i.e., that $\lambda$ is a real number.

If $A$ is a (real) symmetric matrix, then its eigenvalues are real (since $A$ is also Hermitian), and given an eigenvector $\lambda \in \mathbb{R}$, the null space of $\lambda I - A$ contains nonzero real vectors.

The eigenvalues of $A$ are those scalars $\lambda$ for which $\lambda I - A$ is singular and these scalars can be determined by computing the roots of the characteristic polynomial

$$p_A(\lambda) = \det(\lambda I - A).$$

Since $p_A(\lambda)$ is a polynomial of degree $n$ if $A$ is an $n \times n$ matrix, and every polynomial of degree $n$ has exactly $n$ roots (counting multiplicities), every $n \times n$ matrix has $n$ eigenvalues. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A \in \mathbb{R}^{n \times n}$ and let $v_1, \ldots, v_n$ be corresponding eigenvectors. We have

$$Av_1 = v_1 \lambda_1, \ldots, Av_n = v_n \lambda_n.$$

If we put these vectors as columns of a matrix, then

$$AV = (Av_1, \ldots, Av_n) = (v_1 \lambda_1, \ldots, v_n \lambda_n) = V \Lambda.$$

where

$$V = (v_1, \ldots, v_n) \in \mathbb{C}^{n \times n} \quad \text{and} \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

We are interested in characterizing matrices $A$ for which we can find $n$ linearly independent eigenvectors $v_1, \ldots, v_n$, so that the matrix $V$ is invertible.

**Theorem 24 (Eigenvectors corresponding to distinct eigenvalues)** Eigenvectors $v_1, v_2, \ldots, v_m$ of $A \in \mathbb{C}^{n \times n}$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ (i.e., if $j \neq k$, then $\lambda_j \neq \lambda_k$) are linearly independent.

**Proof:** Suppose that the eigenvectors $v_1, v_2, \ldots, v_k$ are linearly independent for some $k < m$, but that $v_1, v_2, \ldots, v_k, v_{k+1}$ are linearly dependent. Then there exist constants $c_1, c_2, \ldots, c_k$ not all zero such that

$$v_{k+1} = \sum_{j=1}^k c_j v_j. \quad (28)$$

Multiplication of the left-hand side of (28) by $A$ yields

$$Av_{k+1} = \lambda_{k+1} v_{k+1} = \lambda_{k+1} \sum_{j=1}^k c_j v_j.$$
while multiplication of the right-hand side of (28) by $A$ gives
\[ A \sum_{j=1}^{k} c_j v_j = \sum_{j=1}^{k} c_j A v_j = \sum_{j=1}^{k} c_j \lambda_j v_j. \]

Hence $\lambda_{k+1} \sum_{j=1}^{k} c_j v_j = \sum_{j=1}^{k} c_j \lambda_j v_j$, which implies
\[ \sum_{j=1}^{k} c_j (\lambda_j - \lambda_{k+1}) v_j = 0. \]

Since $\lambda_j \neq \lambda_{k+1}$ for any $j \in \{1, 2, \cdots, k\}$, and the coefficients $c_j$ are not all zero, we conclude that $v_1, v_2, \cdots, v_k$ are linearly dependent, in contradiction to our assumption. Therefore, the vectors $v_1, v_2, \cdots, v_k, v_{k+1}$ are linearly independent.

Using induction, the vectors $v_1, v_2, \cdots, v_m$ are linearly independent. \qed

**Example 25** Consider
\[ A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}. \]

We compute the row reduced form of $\lambda I - A$,
\[ \begin{pmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & -2 & \lambda \end{pmatrix} \rightarrow \begin{pmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 2/(\lambda - 1) \end{pmatrix} \]

to determine the characteristic polynomial of $A$,
\[ p_A(\lambda) = \det(\lambda I - A) = (\lambda - 2)((\lambda - 1)\lambda - 2) = (\lambda - 2)(\lambda^2 - \lambda - 2). \]

Hence the eigenvalues are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = -1$.

To compute eigenvectors we compute bases for $\mathcal{N}(2I - A)$ and $\mathcal{N}(-I - A)$:
\[ \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(2I - A) \]
and
\[ \left\{ \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(-I - A). \]
Each of the basis vectors \((1, 0, 0)^T\) and \((0, 1, 1)^T\) is linearly independent from \((0, -1/2, 1)^T\), as guaranteed by Theorem 24.

In this case 2 is an eigenvalue with algebraic multiplicity two and the dimension of \(\mathcal{N}(2I - A)\) is two. We have:

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 1 \\
0 & 2 & 0 \\
\end{bmatrix}
= A
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -1/2 \\
0 & 1 & 1 \\
\end{bmatrix}
= V
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -1/2 \\
0 & 1 & 1 \\
\end{bmatrix}
= V
\begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}
= \Lambda
\]

and \(V\) is invertible. Hence,

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 1 \\
0 & 2 & 0 \\
\end{bmatrix}
= A
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -1/2 \\
0 & 1 & 1 \\
\end{bmatrix}
= V
\begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}
= V^{-1}
\]

\[\diamond\]

**Example 26** Consider

\[
A = \begin{pmatrix}
2 & 1 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{pmatrix}
\]

We compute the row reduced form of \(\lambda I - A\),

\[
\begin{pmatrix}
\lambda - 2 & -1 & -1 \\
0 & \lambda - 1 & -1 \\
0 & -1 & \lambda - 1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\lambda - 2 & -1 & -1 \\
0 & \lambda - 1 & -1 \\
0 & 0 & \lambda - 1 - 1/(\lambda - 1) \\
\end{pmatrix}
\]

to determine the characteristic polynomial of \(A\),

\[
p_A(\lambda) = \det(\lambda I - A) = (\lambda - 2)((\lambda - 1)^2 - 1) = (\lambda - 2)^2 \lambda.
\]

Hence the eigenvalues are \(\lambda_1 = \lambda_2 = 2\) and \(\lambda_3 = 0\).

To compute eigenvectors we compute bases for \(\mathcal{N}(2I - A)\) and \(\mathcal{N}(-A)\):

\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(2I - A)
\]

and

\[
\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(-A).
\]
Clearly, the eigenvectors $(1, 0, 0)^T$ and $(0, -1, 1)^T$ are linearly independent. Note that in this example, $\lambda_1 = \lambda_2 = 2$ is an eigenvalue with algebraic multiplicity two but the dimension of $N(2I - A)$ is one (the geometric multiplicity of the eigenvalue 2 is only one. In this case there does not exist an invertible matrix $V$ with $AV = A\Lambda$, where $\Lambda = \text{diag}(2, 2, 0)$.

The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$. The geometric multiplicity of an eigenvalue is the dimension of the null space $N(\lambda I - A)$. In Example 26 the eigenvalue $\lambda = 2$ has algebraic multiplicity two and geometric multiplicity one; the eigenvalue $\lambda = 0$ has algebraic multiplicity one and geometric multiplicity one. One can show that the algebraic multiplicity of an eigenvalue is always greater than or equal to its geometric multiplicity. We can find $n$ linearly independent eigenvectors if and only if for all eigenvalues the algebraic multiplicity is equal to the geometric multiplicity.

A simple consequence of Theorem 24 is that if all $n$ eigenvalues of $A \in \mathbb{R}^{n \times n}$ are distinct (i.e., are all simple roots of the characteristic polynomial $p_A(\lambda)$), then the matrix $A$ has a complete set of $n$ linearly independent eigenvectors. This is the case in Example 16 (see also Example 19) and in the following example.

**Example 27** Consider the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The matrix $(\lambda I - A)$ is a lower triangular matrix and

$$p_A(\lambda) = \det(\lambda I - A) = \lambda(\lambda + 1)(\lambda - 1).$$

The eigenvalues of $A$ are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$.

To compute eigenvectors corresponding to an eigenvalue $\lambda_j$ of $A$, we have to compute the null-space $N(\lambda_j I - A)$. We obtain

$$0I - A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \Rightarrow N(0I - A) = \text{span}\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\},$$

$$1I - A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow N(1I - A) = \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\},$$

$$(-1)I - A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & -1 & -1 \end{pmatrix} \Rightarrow N((-1)I - A) = \text{span}\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\}.$$
Thus, \( v_1 = (0, 0, 1)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda_1 = 0 \), \( v_2 = (0, 1, 1)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda_2 = 1 \), and \( v_3 = (2, -1, 1)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda_3 = -1 \).

Note that if \( v \) is an eigenvector corresponding to the eigenvalue \( \lambda \), then for any scalar \( t \neq 0 \) the vector \( tv \) is also an eigenvector corresponding to the eigenvalue \( \lambda \). Therefore we often normalize eigenvectors \( v \) such that they satisfy \( v^*v = 1 \), i.e., given a nonzero vector \( \tilde{v} \in \mathcal{N}(\lambda I - A) \) we set \( v = \tilde{v}/\|\tilde{v}\|_2 = \sqrt{\sum_{j=1}^{n} \tilde{v}_j \tilde{v}_j} \). Note that in general eigenvalues are complex and eigenvectors are vectors in \( \mathbb{C}^n \), even if the matrix \( A \) is real.

If we normalize the previously computed eigenvectors we find that \( v_1 = (0, 0, 1)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda_1 = 0 \), \( v_2 = (0, 1, 1)^T / \sqrt{2} \) is an eigenvector corresponding to the eigenvalue \( \lambda_2 = 1 \), and \( v_3 = (2, -1, 1)^T / \sqrt{6} \) is an eigenvector corresponding to the eigenvalue \( \lambda_3 = -1 \).

We can compute eigenvalues and corresponding eigenvectors using MATLAB function \texttt{eig}.

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0.8165 \\
0 & 0.7071 & -0.4082 \\
1.0000 & 0.7071 & 0.4082
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

In this example, \( [V, \text{Lambda}] = \text{eig}(A) \) returns a \( 3 \times 3 \) matrix \( V \) and \( 3 \times 3 \) diagonal matrix \( \Lambda \) such that the diagonal entries in \( \Lambda \) are the eigenvalues \( \lambda_1 = 0 \), \( \lambda_2 = 1 \), \( \lambda_3 = -1 \) of \( A \) and the columns of \( V \) are the corresponding normalized eigenvectors \( v_1 = (0, 0, 1)^T \), \( v_2 = (0, 1, 1)^T / \sqrt{2} \), \( v_3 = (2, -1, 1)^T / \sqrt{6} \).
The eigenvalues - eigenvector relationships give

\[
\begin{bmatrix}
-1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & \frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & \frac{2}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\(= V \)

\(= V \)

\(= \Lambda \)

(We use the normalized eigenvectors, but the same relation hold if do not normalize.) The matrix \(V\) is invertible. If we multiple both sides by \(V^{-1}\) from the left, we obtain

\[
\begin{bmatrix}
-1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & \frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & \frac{2}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
0 & 1 & 0
\end{bmatrix}
\]

\(= \Lambda \)

\(= \Lambda \)

\(= V^{-1} \)

\(= V^{-1} \)