CAAM 335: Matrix Analysis  
(updated August 22, 2016)

Leontief Input-Output Model


The following example is adapted from 1 http://media.pearsoncmg.com/aw/aw_lay_linearalg_3/cs_apps/leontief.pdf.

The paper Wassily Leontief, "Input-Output Economics," Scientific American, October 1951, pp. 15–21 used a 42 sector model of the 1947 US economy. In his 1951 paper Leontief reports that the solution of a 42 sector model for the 1939 US economy, which as we will see shortly is the solution of a system of linear equations of size $42 \times 42$, took 56 hours on the Harvard Mark II computer.

- We consider an economy with four sectors. The first three sectors, ‘Agriculture’, ‘Manufacturing’, ‘Services’, produce goods and services. The last sector in any input-output model is the so-called open sector, which only consumes goods and services.

- Input-output table for the economy:

<table>
<thead>
<tr>
<th></th>
<th>Agriculture</th>
<th>Manufacturing</th>
<th>Services</th>
<th>Open Sector</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agriculture</td>
<td>34.69</td>
<td>4.92</td>
<td>5.62</td>
<td>39.24</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>5.28</td>
<td>61.82</td>
<td>22.99</td>
<td>60.02</td>
</tr>
<tr>
<td>Services</td>
<td>10.45</td>
<td>25.95</td>
<td>42.03</td>
<td>130.65</td>
</tr>
<tr>
<td>Total Gross Output</td>
<td>84.56</td>
<td>163.43</td>
<td>219.03</td>
<td></td>
</tr>
</tbody>
</table>

Reading the table is straightforward; for example, the agriculture sector produced output worth 84.56 billion dollars. Of the agricultural production parts were used as inputs in other sectors as follows: 34.69 billion dollars of agricultural output was consumed by the agriculture sector itself, 5.28 billion dollars of manufacturing output was consumed by the agriculture sector, etc.

These data are used to generate an equilibrium model of the economy.

1 Accessed August 22, 2016
• To create the $3 \times 3$ consumption matrix $C$ from the input-output table, we divide each column of the $3 \times 3$ subtable by the corresponding Total Gross Output (TGO). This gives

$$C = \begin{pmatrix}
0.4102 & 0.0301 & 0.0257 \\
0.0624 & 0.3783 & 0.1050 \\
0.1236 & 0.1588 & 0.1919
\end{pmatrix}.$$ 

The entries of $C$ contain the inputs consumed per unit of sector output. $C_{ij}$ contains the inputs consumed from sector $i$ per unit of sector output in sector $j$.

• From the consumption matrix $C$ and the open sector demand vector 

$$d = (39.24, 60.02, 130.65)^T,$$
we can compute equilibrium levels of production for each sector. These equilibrium levels are the production levels which will just meet the intermediate demands of the sectors of the economy plus the final demands of each sector.

If $x_i$ is the equilibrium level of production for sector $i$, then

$$x_i = c_{i,1}x_1 + c_{i,2}x_2 + c_{i,3}x_3 + d_i,$$

or in matrix-vector notation,

$$x = Cx + d.$$ 

• The computation of the equilibrium level production vector $x$ requires the solution of a system of linear equations

$$(I - C)x = d.$$ 

• For the $3 \times 3$ example this is done in leontief.m. The resulting vector displayed in a pie-chart is shown below.
• The matrix \((I - C)^{-1}\) has several important economic interpretations: The \((i, j)\)th entry in the matrix \((I - C)^{-1}\) is the amount by which sector \(i\) must change its production level to satisfy an increase of 1 unit in the final demand from sector \(j\).

• The input-output model also leads to several mathematical questions:

  – The entries of \((I - C)^{-1}\) should be non-negative. (Why?)
    Note that since \(C\) has non-negative entries, the off-diagonal entries of the \(A = I - C\) are non-positive. One can show that if the real parts of all of the so-called eigenvalues of \(A\) are positive, then all entries of \(A^{-1} = (I - C)^{-1}\) are non-negative.

  – The eigenvalues of the matrix \(I - C\) corresponding to the \(3 \times 3\) example are shown in the following plot.