CAAM 335: Matrix Analysis
Examination 1

Posted Wednesday, 27 September 2017.
Due 11am on Wednesday, 4 October 2017.

Instructions

1. Time limit: **3 uninterrupted hours**.

2. There are five questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.

3. You may not use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.

4. Please answer the questions thoroughly and justify all your answers. Show all your work to maximize partial credit.

5. Print your name on the line below:

   ____________________________________________________________

6. Time started: ___________________________ Time completed: _______________________

7. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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8. **Staple this page to the front of your exam.**
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Problem 1 (25 points) Let

\[
A = \begin{pmatrix}
1 & -1 & -1 \\
3 & -4 & -2 \\
2 & -3 & -2
\end{pmatrix}.
\]

(a) [10 pts.] Compute the LU-decomposition \( A = LU \) of \( A \).

(b) [7 pts.] Use the LU-decomposition computed in a) to solve \( Ax = b \) with \( b = (1, 1, 1)^T \).

(c) [8 pts.] Use the LU-deomposition computed in a) to solve \( A^T y = c \) with \( c = (1, 1, 1)^T \).

Show all of your work!

Problem 2 (20 points) Let \( x \in \mathbb{R}^n, y \in \mathbb{R}^m \) be nonzero vectors (i.e., not all \( x_i = 0 \) for \( i = 1, 2, \ldots, n \), and not all \( y_i = 0 \) for \( i = 1, 2, \ldots, m \)).

(a) [3 pts.] Let \( \|x\|_2 = \|y\|_2 = 1 \). What is the Frobenius norm of \( xy^T \), \( \|xy^T\|_F = \) ______? (Give a number.)

(b) [2 pts.] Let \( m = n \). Under what condition on the vectors \( x, y \in \mathbb{R}^n \) is \( (xy^T)^2 = xy^T \)?

(c) [5 pts.] Let \( m = n \geq 2 \). Do there exist \( x, y \in \mathbb{R}^n \) with \( x_1 \neq 0 \) and \( y_1 \neq 0 \) such that \( xy^T \) is invertible? (Give an example, or prove that no such vectors exist.)

(d) [10 pts.] Let \( I \) denote the \( n \times n \) identity matrix, and let \( x, y \in \mathbb{R}^n \) be vectors with \( y^T x \neq -1 \).

Prove that \( I + xy^T \) is invertible and its inverse is

\[
(I + xy^T)^{-1} = I - \frac{1}{1 + y^T x} xy^T.
\]

Problem 3 (18 points) Let \( A \in \mathbb{R}^{m \times n} \) and let \( B \in \mathbb{R}^{n \times n} \) be an invertible matrix.

Answer the following questions as thoroughly as possible (give a proof or counter example).

(a) [6 pts.] If \( \mathcal{R}(A) = \{0\} \) is \( A = 0 \)?

(b) [6 pts.] Is \( \mathcal{R}(A) = \mathcal{R}(AB) \)?

(c) [6 pts.] Is \( \mathcal{R}(A) = \mathcal{R}(tA) \) for a fixed nonzero scalar \( t \in \mathbb{R}, t \neq 0 \)?
Problem 4 (20 points) Consider the following truss (grey boxes represent walls)

![Diagram of a truss with labeled nodes 1, 2, 3, 4, 5]

The adjacency matrix in the elongation - displacement relation $e = Ax$ for this truss is

$$A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.$$

(a) [10 pts] Compute bases for $\mathcal{N}(A)$ and $\mathcal{R}(A)$.

(b) [5 pts] Are there stiffnesses $k_1, \ldots, k_5 \in \mathbb{R}$, $K = \text{diag}(k_1, \ldots, k_5)$ and forces $f \in \mathbb{R}^4$, so that the truss equation $A^TKAx = f$ has a unique solution? (Justify your answer!)

(c) [5 pts] Does the linear system $A^TAx = f$ have a solution for all right hand sides $f \in \mathbb{R}^4$? If not, what condition(s) does the right hand side $f$ need to satisfy for $A^TAx = f$ to have a solution? (Justify your answers!)
Problem 5 (17 points) Consider the following circuit.

(a) [10 pts]
- Write the voltage drops $e$ in terms of the potentials $x$ as $e = b - Ax$. What is $A$?
- Write the currents $y$ in terms of the voltage drops $e$ as $y = Ge$.
- Express Kirchhoff’s Current Law via $A^T y = 0$. (Write down Kirchhoff’s Current Law for each node to make sure that the matrix you obtain is the transpose of the matrix obtained in the first part.)
- Form $A^T GA$ and $A^T Gb$.

(b) [5 pts] Let $E = 15[V]$, and $R_1 = R_2 = R_3 = R_4 = R [\Omega]$. Solve $A^T GAx = A^T Gb$ for the potentials $x$.

(c) [2 pts] If the resistivity $R = R_1 = R_2 = R_3 = R_4 [\Omega]$ changes, do the potentials $x$ change? If so how? (Justify your answer.)