Problem 1 (25 points) Let
\[
A = \begin{pmatrix}
1 & -1 & -1 \\
3 & -4 & -2 \\
2 & -3 & -2
\end{pmatrix}.
\]

(a) [10 pts.] Compute the LU-decomposition \( A = LU \) of \( A \).

(b) [7 pts.] Use the LU-deomposition computed in a) to solve \( Ax = b \) with \( b = (1, 1, 1)^T \).

(c) [8 pts.] Use the LU-deomposition computed in a) to solve \( A^T y = c \) with \( c = (1, 1, 1)^T \).

Show all of your work!

Solution

(a) [10 pts.] Compute the LU-decomposition \( A = LU \) of \( A \).

\[
\begin{pmatrix}
1 & -1 & -1 \\
3 & -4 & -2 \\
2 & -3 & -2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
3 & 1 & 0 \\
2 & 1 & 1
\end{pmatrix}\begin{pmatrix}
1 & -1 & -1 \\
0 & -1 & 1 \\
0 & 0 & -1
\end{pmatrix}
\]

\( =A \) \( =L \) \( =U \)

(b) [7 pts.] \( Ly = (1, 1, 1) \) has solution \( y = (1, -2, 1)^T \).
\( Ux = (1, -2, 1) \) has solution \( x = (1, 1, -1)^T \).

(c) [8 pts.] \( U^T z = (1, 1, 1) \) has solution \( z = (1, -2, -4)^T \).
\( L^T y = (1, -2, -4) \) has solution \( x = (3, 2, -4)^T \).
Problem 2 (20 points) Let \( x \in \mathbb{R}^n, y \in \mathbb{R}^m \) be nonzero vectors (i.e., not all \( x_i = 0 \) for \( i = 1, 2 \cdots, n \), and not all \( y_i = 0 \) for \( i = 1, 2 \cdots, m \)).

(a) [3 pts.] Let \( \| x \|_2 = \| y \|_2 = 1 \). What is the Frobenius norm of \( xy^T \), \( \| xy^T \|_F = \) ______? (Give a number.)

(b) [2 pts.] Let \( m = n \). Under what condition on the vectors \( x, y \in \mathbb{R}^n \) is \( (xy^T)^2 = xy^T \)?

(c) [5 pts.] Let \( m = n \geq 2 \). Do there exist \( x, y \in \mathbb{R}^n \) with \( x_1 \neq 0 \) and \( y_1 \neq 0 \) such that \( xy^T \) is invertible? (Give an example, or prove that no such vectors exist.)

(d) [10 pts.] Let \( I \) denote the \( n \times n \) identity matrix, and let \( x, y \in \mathbb{R}^n \) be vectors with \( y^T x \neq -1 \). Prove that \( I + xy^T \) is invertible and its inverse is \( (I + xy^T)^{-1} = I - \frac{1}{1 + y^T x} xy^T \).

Solution

(a) [3 pts.] Let \( \| x \|_2 = \| y \|_2 = 1 \). The entries of \( xy^T \in \mathbb{R}^{n \times m} \) are \( (xy^T)_{ij} = x_i y_j \). Hence,

\[
\| xy^T \|_F = \left( \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^2 y_j^2 \right)^{1/2} = \left( \sum_{i=1}^{n} x_i^2 \sum_{j=1}^{m} y_j^2 \right)^{1/2} = \left( \| x \|_2^2 \| y \|_2^2 \right)^{1/2} = 1.
\]

(b) [2 pts.] Let \( m = n \). \( (xy^T)^2 = xy^T xy^T = x(y^T x)y^T = (y^T x) xy^T = xy^T \) if \( y^T x = 1 \).

(c) [5 pts.] Let \( m = n \geq 2 \). There does NOT exist \( x, y \in \mathbb{R}^n \) with \( x_1 \neq 0 \) and \( y_1 \neq 0 \) such that \( xy^T \) is invertible.

Proof: There exists a nonzero vector \( z \) such that \( y^T z = 0 \). Hence \( xy^T z = x0 = 0 \) for a nonzero vector \( z \). Thus, \( xy^T \) is not invertible.

Alternative Proof: The \( i \)th row of the matrix \( xy^T \) is \( x_i y^T \):

\[
xy^T = \begin{pmatrix}
x_1 y^T \\
x_2 y^T \\
\vdots \\
x_n y^T 
\end{pmatrix}.
\]

Since the \((1,1)\) element of \( xy^T \) is \( x_1 y_1 \neq 0 \) we can subtract \( x_i/x_1 \) times the first row from row
\( i, \text{ for } i = 2, \ldots, n \) to get

\[
xy^T = \begin{pmatrix} x_1y^T \\ x_2y^T \\ \vdots \\ x_ny^T \end{pmatrix} \rightarrow \begin{pmatrix} x_1y^T \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]

Since the row reduced form of \( xy^T \) has zero diagonal elements, \( xy^T \) is singular.

(d) [10 pts.] To prove that \( I + xy^T \) is invertible and its inverse is

\[
(I + xy^T)^{-1} = I - \frac{1}{1 + y^Tx} xy^T
\]

we need to show that

\[
(I + xy^T) \left( I - \frac{1}{1 + y^Tx} xy^T \right) = I.
\]

We get

\[
(I + xy^T) \left( I - \frac{1}{1 + y^Tx} xy^T \right) \\
= I - \frac{1}{1 + y^Tx} xy^T + xy^T \left( I - \frac{1}{1 + y^Tx} xy^T \right) \\
= I - \frac{1}{1 + y^Tx} xy^T + xy^T \left( \frac{1}{1 + y^Tx} xy^T \right) \\
= I - \frac{1}{1 + y^Tx} xy^T + \frac{y^Tx}{1 + y^Tx} xy^T \\
\begin{align*}
= I + \left( -\frac{1}{1 + y^Tx} + 1 - \frac{y^Tx}{1 + y^Tx} \right) xy^T \\
= I.
\end{align*}
\]
Problem 3 (18 points) Let $A \in \mathbb{R}^{m \times n}$ and let $B \in \mathbb{R}^{n \times n}$ be an invertible matrix.

Answer the following questions as thoroughly as possible (give a proof or counter example).

(a) [6 pts.] If $\mathcal{R}(A) = \{0\}$ is $A = 0$?

(b) [6 pts.] Is $\mathcal{R}(A) = \mathcal{R}(AB)$?

(c) [6 pts.] Is $\mathcal{R}(A) = \mathcal{R}(tA)$ for a fixed nonzero scalar $t \in \mathbb{R}$, $t \neq 0$?

Solution

(a) [6 pts.] If $\mathcal{R}(A) = \{0\}$ is $A = 0$? Yes.

Proof: Let $e_j$ be the $j$-th unit vector. Then $a_j = Ae_j$ is the $j$-th columns of $A$. By definition of $\mathcal{R}(A)$, $a_j = Ae_j \in \mathcal{R}(A) = \{0\}$. This means $a_j = Ae_j = 0$. Since this holds for all $j$’s, all columns of $A$ are the zero vector, i.e., $A = 0$.

(b) [6 pts.] Is $\mathcal{R}(A) = \mathcal{R}(AB)$? Yes.

Proof: $\mathcal{R}(AB) \subset \mathcal{R}(A)$: Let $y \in \mathcal{R}(AB)$. By definition of the range space, there exists $x \in \mathbb{R}^n$ such that $y = ABx = A(Bx) = Az$ where $z = Bx$. Hence, $y \in \mathcal{R}(A)$.

$\mathcal{R}(A) \subset \mathcal{R}(AB)$: Let $y \in \mathcal{R}(A)$. By definition of the range space, there exists $x \in \mathbb{R}^n$ such that $y = Ax = A(t^{-1}x) = tAz$ where $z = t^{-1}x$. Hence, $y \in \mathcal{R}(AB)$.

(c) [6 pts.] Is $\mathcal{R}(A) = \mathcal{R}(tA)$ for a fixed nonzero scalar $t \in \mathbb{R}$, $t \neq 0$? Yes.

Proof: $\mathcal{R}(A) \subset \mathcal{R}(tA)$: Let $y \in \mathcal{R}(A)$. By definition of the range space, there exists $x \in \mathbb{R}^n$ such that $y = Ax = tA(t^{-1}x) = tAz$ where $z = t^{-1}x$. Hence, $y \in \mathcal{R}(tA)$.

$\mathcal{R}(tA) \subset \mathcal{R}(A)$: Let $y \in \mathcal{R}(tA)$. By definition of the range space, there exists $x \in \mathbb{R}^n$ such that $y = tAx = A(tx) = Az$ where $z = tx$. Hence, $y \in \mathcal{R}(A)$.
Problem 4 (20 points) Consider the following truss (grey boxes represent walls)

The adjacency matrix in the elongation - displacement relation $e = Ax$ for this truss is

$$A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\end{pmatrix}.$$

(a) [10 pts] Compute bases for $\mathcal{N}(A)$ and $\mathcal{R}(A)$.

(b) [5 pts] Are there stiffnesses $k_1, \ldots, k_5 \in \mathbb{R}$, $K = \text{diag}(k_1, \ldots, k_5)$ and forces $f \in \mathbb{R}^4$, so that the truss equation $A^T K A x = f$ has a unique solution? (Justify your answer!)

(c) [5 pts] Does the linear system $A^T A x = f$ have a solution for all right hand sides $f \in \mathbb{R}^4$? If not, what condition(s) does the right hand side $f$ need to satisfy for $A^T A x = f$ to have a solution? (Justify your answers!)
Solution

(a) [10 pts]

\[ A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[ x_4 \text{ is the free variable} \]

(b) [5 pts] No. Let \( v = (0, 1, 0, 1)^T \in \mathcal{N}(A) \).

If \( A^T K A x = f \), then \( A^T K A (x + \alpha v) = A^T K Ax + \alpha A^T K Av = f + \alpha A^T K 0 = f \) for all \( \alpha \in \mathbb{R} \).

(c) [5 pts]

Approach 1: If \( v \in \mathcal{N}(A) \), then \( A^T Ax = f \) implies \( 0 = v^T A^T Ax = v^T f \). Thus, \( A^T Ax = f \) only has a solution for right hand sides \( f \) that satisfy \( 0 = v^T f = f_2 + f_4 \).

Approach 2:

\[ A^T A = \begin{pmatrix}
0 & 0 & 0 & 1 & -1 \\
0 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 2 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 0 & 2 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix}
\]

Use Gaussian elimination:

\[ \begin{pmatrix}
2 & 0 & 0 & 0 & f_1 \\
0 & 1 & 0 & -1 & f_2 \\
0 & 0 & 2 & 0 & f_3 \\
0 & -1 & 0 & 1 & f_4
\end{pmatrix} \rightarrow \begin{pmatrix}
2 & 0 & 0 & 0 & f_1 \\
0 & 1 & 0 & -1 & f_2 \\
0 & 0 & 2 & 0 & f_3 \\
0 & 0 & 0 & 0 & f_2 + f_4
\end{pmatrix}
\]

Has solution only if \( f_2 + f_4 = 0 \). Solution is \( x_4 \) free variable

\[ x_3 = f_3/2, \quad x_2 = f_2 + x_4, \quad x_1 = f_1/2. \]
Problem 5 (17 points) Consider the following circuit.

(a) [10 pts]
- Write the voltage drops \( e \) in terms of the potentials \( x \) as \( e = b - Ax \). What is \( A \)?
- Write the currents \( y \) in terms of the voltage drops \( e \) as \( y = Ge \).
- Express Kirchhoff’s Current Law via \( A^T y = 0 \). (Write down Kirchhoff’s Current Law for each node to make sure that the matrix you obtain is the transpose of the matrix obtained in the first part.)
- Form \( A^T GA \) and \( A^T Gb \).

(b) [5 pts] Let \( E = 15[V] \), and \( R_1 = R_2 = R_3 = R_4 = R [\Omega] \). Solve \( A^T GAx = A^T Gb \) for the potentials \( x \).

(c) [2 pts] If the resistivity \( R = R_1 = R_2 = R_3 = R_4 [\Omega] \) changes, do the potentials \( x \) change? If so how?

Solution

(a) [10 pts] The equations for the circuit are \( e = b - Ax \) and \( y = Ge \).

\[
b = \begin{pmatrix} E \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}
\]

\[
G = \text{diag}(1/R_1, 1/R_2, 1/R_3, 1/R_4)
\]
We have

\[ A^T G = \begin{pmatrix} 1/R_1 & -1/R_2 & -1/R_3 & 0 \\ 0 & 1/R_3 & -1/R_4 & 0 \end{pmatrix}, \]

\[ A^T G A = \begin{pmatrix} 1/R_1 & -1/R_2 & -1/R_3 & 0 \\ 0 & 1/R_3 & -1/R_4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \]

\[ = \begin{pmatrix} 1/R_1 + 1/R_2 + 1/R_3 & -1/R_3 \\ -1/R_3 & 1/R_3 + 1/R_4 \end{pmatrix} \]

and

\[ A^T G b = \begin{pmatrix} 1/R_1 & -1/R_2 & 0 \\ 0 & 1/R_3 & -1/R_4 \end{pmatrix} \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E/R_1 \\ 0 \\ 0 \end{pmatrix}. \]

(b) [5 pts] If \( R_1 = R_2 = R_3 = R_4 = R \), then \( A^T G A x = A^T G b \) is equivalent to

\[ \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} \begin{pmatrix} 3 & -1 \end{pmatrix} \begin{pmatrix} 15 \\ 0 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} 3 & -1 \end{pmatrix} \begin{pmatrix} 15 \\ 5/3 \end{pmatrix} \end{pmatrix} \text{ row 2 + (1/3)*row 1} \]

Back-substitution gives

\( x_2 = 3, \quad x_1 = 18/3 = 6. \)

Check solution:

\[ \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \end{pmatrix}. \]

(c) [2 pts] Since \( G = R^{-1} I \), \( A^T G A x = A^T G b \) is equivalent to \( R^{-1} A^T A x = R^{-1} A^T b \) is equivalent to \( A^T A x = A^T b \). Thus changing resistivity \( R \) does not change the potentials.