### CAAM 335: Matrix Analysis

#### Examination 3

**Wednesday, 6 December 2017.**

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**Instructions**

1. **Time limit:** 3 uninterrupted hours.

2. There are five questions worth a total of 100 points. Please do not look at the questions until you begin the exam.

3. You may not use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.

4. Please answer the questions thoroughly and justify all your answers. Show all your work to maximize partial credit.

5. Print your name on the line below:

   

6. Time started: ___________________________ Time completed: ___________________________

7. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

   

8. **Staple this page to the front of your exam.**
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Problem 1 (15 points)

Let

\[ A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \]

(a) (10 pts) Diagonalize \( A \), i.e., find a matrix \( V \in \mathbb{R}^{2 \times 2} \) and a diagonal matrix \( \Lambda \in \mathbb{R}^{2 \times 2} \) such that

\[ A = V \Lambda V^{-1}. \]

(b) (5 pts) Use part (a) to show that

\[ A^k = \frac{1}{2} \begin{pmatrix} 1 + b & -1 + b \\ -1 + b & 1 + b \end{pmatrix}. \]

What is \( b \)?
Problem 2 (25 points)

In this problem we will use the eigen-decomposition of $B$ to compute the solutions $x_k$ the so-called discrete time system

$$x_{k+1} = Bx_k + c, \quad k = 0, 1, \ldots$$

with given $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$.

(a) (5 pts) Given $\xi_0 \in \mathbb{R}$ and $\beta \neq 1$ and $\gamma \in \mathbb{R}$, consider the scalar discrete time equation

$$\xi_{k+1} = \beta \xi_k + \gamma, \quad k = 0, 1, \ldots$$

Show that the solution of (2) is

$$\xi_k = \beta^k \xi_0 + \frac{1 - \beta^k}{1 - \beta} \gamma \quad k = 0, 1, \ldots$$

(b) (5 pts) Let

$$B = Q\Lambda Q^T,$$

where $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n \times n}$.

Use (3) to transform (1) into a discrete time system

$$z_{k+1} = \Lambda z_k + d, \quad k = 0, 1, \ldots$$

with given $z_0 \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$.

How are the vectors $x_k$ and $z_k$ related? How are the vectors $c$ and $d$ related?

(c) (8 pts) Use parts (a) and (b) to compute the solutions $x_k \in \mathbb{R}^2$ of the discrete time system (1) with

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = Q\Lambda Q^T$$

and

$$x_0 = (1, -1)^T, \quad c = (1, 1)^T$$

(d) (7 pts) Let $B \in \mathbb{R}^{n \times n}$ be a symmetric matrix whose eigenvalues $\lambda_j$, $j = 1, \ldots, n$, satisfy $|\lambda_j| < 1$, $j = 1, \ldots, n$. Use parts (a) and (b) to show that the solution $x_k \in \mathbb{R}^n$, $k = 1, 2, \ldots$ of the discrete time system (1) satisfies

$$\lim_{k \to \infty} x_k = x^*,$$

where $x^* \in \mathbb{R}^n$ depends only on $c$, but not on $x_0$. 

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Problem 3 (30 points)

Let
\[
A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}.
\]

(a) (5 pts) Diagonalize \(A^T A\), i.e., find an orthogonal matrix \(V \in \mathbb{R}^{2 \times 2}\) and a diagonal matrix \(\Lambda \in \mathbb{R}^{2 \times 2}\) such that
\[
A^T A = V \Lambda V^T.
\]

Hint: the eigenvalues of \(A^T A\) are \(\lambda_1 = 25, \lambda_2 = 9\).

(b) (10 pts) Compute the SVD of \(A\), i.e., find an orthogonal matrix \(U \in \mathbb{R}^{3 \times 3}\), and a diagonal matrix \(\Sigma \in \mathbb{R}^{3 \times 2}\) such that
\[
A = U \Sigma V^T.
\]

(c) (5 pts) Let \(b = (1, 1, 1)^T\). Use the SVD of \(A\) to compute a solution of the least squares problem
\[
\min_{x \in \mathbb{R}^2} \|Ax - b\|_2
\]

(d) (5 pts) Let \(d = (1, 1)^T\). Use the SVD of \(A\) to compute a solution of the least squares problem
\[
\min_{y \in \mathbb{R}^3} \|A^T y - d\|_2
\]

(e) (5 pts) Provide bases for the four fundamental subspaces \(\mathcal{R}(A), \mathcal{N}(A), \mathcal{R}(A^T), \mathcal{N}(A^T)\).

Problem 4 (10 points)

The ministry of economy proposes to introduce two species, of which one is the predator and the other is the prey, to an island in order to attract wildlife tourism. The proposal is to initially bring in 100 in each population. Local resident fear that these population will ‘take over’ the island and oppose this. The government asks you for your help.

Biologists determined that populations of the predator \(x_1(t)\) and prey \(x_2(t)\) satisfy the dynamical system
\[
\begin{align*}
x_1'(t) &= x_1(t) + x_2(t), & t > 0, \\
x_2'(t) &= -x_1(t) + x_2(t), & t > 0, \\
x_1(0) &= 100, & x_2(0) = 100.
\end{align*}
\]

Is the solution to this system bounded, i.e., does there exists \(M > 0\) such that \(|x_1(t)| \leq M\) and \(|x_2(t)| \leq M\) for all \(t\)? Carefully justify your answer!

Recall: For a complex number \(\alpha + \beta i\) the exponential is \(e^{\alpha + \beta i} = e^\alpha (\cos(\beta) + i \sin(\beta))\).
Problem 5 (2 points each = 20 points)

State whether the following statements are true or false. No work or justification is needed. Please write "True" or "False" – if your choice is not clear, no credit will be given.

(a) All $m \times n$ matrices have eigenvalues.

(b) If $\lambda$ is an eigenvalue of $A$ and $\mu$ is an eigenvalue of $B$, then $\lambda + \mu$ is an eigenvalue of $A + B$.

(c) If $\lambda$ is an eigenvalue of $A$ and $\mu$ is an eigenvalue of $B$, then $\lambda \mu$ is an eigenvalue of $AB$.

(d) If $\lambda$ is an eigenvalue of $A$, then $t\lambda$ is an eigenvalue of $tA$ for all $t \in \mathbb{R}$.

(e) The sum of the algebraic multiplicities of the eigenvalues of $A \in \mathbb{R}^{n \times n}$ is always equal to $n$.

(f) The sum of the geometric multiplicities of the eigenvalues of $A \in \mathbb{R}^{n \times n}$ is always equal to $n$.

(g) An eigenvalue of $A \in \mathbb{R}^{n \times n}$ is also an eigenvalue of $A^T$.

(h) If $\lambda_i, i = 1, \ldots, n$, are eigenvalues of the symmetric matrix $A$, then the singular values of $A$ are $\sigma_i = |\lambda_i|, i = 1, \ldots, n$.

(i) If $A \in \mathbb{R}^{m \times n}$ has orthonormal columns, all singular values of $A$ are equal to one.

(j) The rank of the matrix $A \in \mathbb{R}^{m \times n}$ is equal to the number of its nonzero singular values.