1 Dynamical Systems

Given $A \in \mathbb{R}^{n\times n}$ and a function $f : [0,\infty) \to \mathbb{R}^n$, we are interested in the solution of the dynamical system

$$x'(t) = Ax(t) + f(t), \qquad t > 0, \tag{1a}$$
$$x(0) = x_0. \tag{1b}$$

The solution of the scalar differential equation

$$\xi'(t) = \lambda\xi(t) + \gamma(t), \qquad t > 0,$$
$$\xi(0) = \xi_0$$

is given by

$$\xi(t) = e^{\lambda t}\xi_0 + \int_0^t e^{\lambda(t-\tau)}\gamma(\tau)d\tau.$$

A similar expression holds for the solution of the dynamical system (1).

The scalar exponential function is given by

$$e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}.$$

We can define the matrix exponential of a square matrix $A \in \mathbb{R}^{n\times n}$ as follows.

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

Note that the matrix exponential is a matrix, $\exp(A) \in \mathbb{R}^{n\times n}$!

If we define the matrix valued function

$$t \mapsto E(t) \stackrel{\text{def}}{=} \exp(At) = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k,$$

then one can show that

$$E'(t) = AE(t) = E(t)A$$

and

$$E(0) = I.$$

In fact,

$$\begin{aligned}\frac{d}{dt}E(t) &= \sum_{k=0}^{\infty} \frac{d}{dt}\frac{1}{k!}A^k t^k = \sum_{k=1}^{\infty} \frac{1}{k!}A^k k t^{k-1} \\ &= \sum_{k=1}^{\infty} A \frac{1}{(k-1)!}A^{k-1}t^{k-1} = A\sum_{k=0}^{\infty}\frac{1}{k!}A^k t^k \\ &= AE(t).\end{aligned}$$

(Note that in the first equality above the order of summation and differentiation can be switched because the series converges uniformly.) Hence, the solution of the dynamical system (1) with $f = 0$ is given by

$$x(t) = E(t)x_0 = \exp(At)x_0.$$

One can show that the solution of the dynamical system (1) is given by

$$x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau))f(\tau)d\tau.$$

In MATLAB the matrix exponential can be computed using `expm`. Note that the MATLAB command `exp(A)` returns a matrix in which the scalar exponential is applied to each entry of A. This is *not* the matrix exponential of A.

```
>> A = [1 0; 0 2]

A =
     1     0
     0     2

>> expm(A)

ans =
    2.7183         0
         0    7.3891

>> exp(A)

ans =
    2.7183    1.0000
    1.0000    7.3891
```

If

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

is a diagonal matrix, then

$$A^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}$$

and

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_1^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_2^k \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}.$$

The matrix exponential of a diagonal matrix is a diagonal matrix. If one computes `exp(A)`, MATLAB just computes the (scalar) exponential of each entry, which explains the ones in the off-diagonals in the result of `exp(A)` above.

We will return to the solution of dynamical systems and the matrix exponential in the following sections, where we use the so-called diagonalization of the matrix A to gain more insight into the structure of the solution dynamical systems.

2 Overview of Eigenvectors and Eigenvalues

Let A be a square $n \times n$ matrix. We are interested in non-zero vectors v such that Av is a multiple of v, that is

$$Av = v\lambda, \tag{2}$$

for some scalar λ. A scalar λ and non-zero vector v that satisfy (2) are called an *eigenvalue (of A)* and *eigenvector (of A)*, respectively.

Why are we interested in eigenvalues and eigenvectors? If λ and v are an eigenvalue and corresponding eigenvector of A, then

$$A^2 v = A(Av) = Av\lambda = v\lambda^2.$$

More generally, for any positive integer k,

$$A^k v = v\lambda^k.$$

This implies, for example,

$$\exp(At)v = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k v = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k t^k v = e^{\lambda t} v.$$

Thus, if the initial data $x_0 = v$ in the dynamical system (1) with $f \equiv 0$ is an eigenvector of A with eigenvalue λ, then the solution is

$$x(t) = \exp(At)v = e^{\lambda t} v.$$

We can read off the behavior of the solution immediately. For example, if the eigenvalue is a negative real number, then $x(t) = e^{\lambda t} v \to 0$ as $t \to \infty$. We will return to dynamical systems later and also study other examples where the knowledge of eigenvalues and eigenvectors of A allow us to study properties of the solution of linear systems, dynamical systems, difference equations, and quadratic optimization problems. First we need to learn more about eigenvalues and eigenvectors of a square matrix A.

Does any square matrix have eigenvalues and eigenvectors? There exists a non-zero vector v such that (2) holds if and only if $\lambda I - A$ has a non-trivial nullspace, $\mathcal{N}(\lambda I - A) \neq \{0\}$, i.e., if and only if $\lambda I - A$ is not invertible ($\lambda I - A$ is singular).

Example 1 Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Applying Gaussian elimination to $\lambda I - A$ gives

$$\lambda I - A = \begin{pmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 4 \end{pmatrix} \to \begin{pmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 4 - \frac{4}{\lambda - 1} \end{pmatrix}.$$

The matrix $\lambda I - A$ is singular if and only if

$$(\lambda - 1)\left(\lambda - 4 - \frac{4}{\lambda - 1}\right) = (\lambda - 1)(\lambda - 4) - 4 = \lambda^2 - 5\lambda = 0.$$

We call the product of the diagonal entries of its row-reduced form of $\lambda I - A$ the *determinant* [1] *of* $\lambda I - A$ and we denote the determinant by $\det(\lambda I - A)$. In this example,

$$\det(\lambda I - A) = \lambda^2 - 5\lambda.$$

We have shown that λ is an eigenvalue of A if and only if $\det(\lambda I - A) = 0$. Note that for an $n \times n$ matrix A, the determinant $\det(\lambda I - A)$ is a polynomial of degree n in λ. This polynomial is called the *characteristic polynomial of* A and often denoted by $p_A(\lambda) = \det(\lambda I - A)$. In our example $\det(\lambda I - A) = \lambda^2 - 5\lambda = 0$ if and only if $\lambda = 0$ or $\lambda = 5$. Thus $\lambda = 0$ and $\lambda = 5$ are eigenvalues of A.

The MATLAB function `poly` can be used to compute the coefficients of the characteristic polynomial of A and `roots` can be used to compute the roots of the characteristic polynomial

```
>> A = [ 1 2; 2 4 ]

A =
     1     2
     2     4

>> p = poly(A)

p =
     1    -5     0

>> roots(p)

ans =
     0
     5
>>
```

The eigenvectors corresponding to $\lambda = 0$ and $\lambda = 5$ can be computed by finding a basis for $\mathcal{N}(\lambda I - A)$.

[1] Our definition of the determinant may differ from the definitions of the determinant you will find in books by a factor of -1. Since we are interested in scalars λ for which $\det(\lambda I - A) = 0$, this factor is not important in our context.

For $\lambda = 0$ we find that

$$\lambda I - A = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \to \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \Longrightarrow \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(\lambda I - A).$$

Hence, $v = (-2,1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda = 0$. (Note $v = \alpha(-2,1)^T$ for any nonzero scalar α is also an eigenvector.)

For $\lambda = 5$ we find that

$$\lambda I - A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \Longrightarrow \left\{ \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(\lambda I - A).$$

Hence, $v = (1/2,1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda = 5$. ◇

Example 2 Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Applying Gaussian elimination to $\lambda I - A$ gives

$$\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \to \begin{pmatrix} \lambda & -1 \\ 0 & \lambda + \frac{1}{\lambda} \end{pmatrix}.$$

In this example,

$$\det(\lambda I - A) = \lambda^2 + 1.$$

The eigenvalues of A are the roots of $p_A(\lambda) = \det(\lambda I - A) = \lambda^2 + 1$. There are two roots of this polynomial, but they are complex. For this matrix the eigenvalues are the complex numbers

$$\lambda = i \quad \text{and} \quad \lambda = -i.$$

The corresponding eigenvectors can be computed by finding a basis for $\mathcal{N}(\lambda I - A)$. This is done using Gaussian elimination, but we now have to work with complex valued matrix entries, rather than real numbers.

For $\lambda = i$ we find that

$$\lambda I - A = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \to \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix} \text{ (multiply row 1 by } i \text{ and add result to row 2)}$$

Hence

$$\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathcal{N}(iI - A)$. The vector $v = (-i,1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda = i$.

For $\lambda = -i$ we find that

$$\lambda I - A = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \to \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \text{(multiply row 1 by } i \text{ and subtract result from row 2)}$$

Hence

$$\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathcal{N}(-iI - A)$. The vector $v = (i, 1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda = -i$. ◇

Example 3 Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since

$$\lambda I - A \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{pmatrix}$$

we find that

$$\det(\lambda I - A) = (\lambda - 1)^2.$$

The eigenvalues of A are the roots of $p_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)^2$. The characteristic polynomial has a double root $\lambda = 1$. We say that $\lambda = 1$ is an eigenvalue with *algebraic multiplicity 2*. When we count (list) eigenvalues, we count (list) $\lambda = 1$ twice.

The corresponding eigenvectors for $\lambda = 1$ are computed by finding a basis for $\mathcal{N}(\lambda I - A)$. We find that

$$I - A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Longrightarrow \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(I - A).$$

Note that the dimension of $\mathcal{N}(I - A)$ is only one. The dimension of $\mathcal{N}(\lambda I - A)$ is called the *geometric multiplicity of* λ. In this example, $\lambda = 1$ has algebraic multiplicity 2, but geometric multiplicity 1. (We will show later that the geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.)

In this example, the vector $v = (1, 0)^T$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$. All other eigenvectors are nonzero multiples of $v = (1, 0)^T$, i.e., are linearly dependent from $v = (1, 0)^T$. ◇

To compute eigenvalues and corresponding eigenvectors for a square matrix $A \in \mathbb{R}^{n \times n}$ we have to perform the following steps

1. Compute the determinant $\det(\lambda I - A)$. That is compute the row reduced form of $\lambda I - A$. The determinant $\det(\lambda I - A)$ is equal to the product of the diagonals in the row reduced form.

2. The determinant $\det(\lambda I - A)$ is a polynomial of degree n in λ. The eigenvalues are the roots of this polynomial. Since any polynomial of degree n has exactly n roots (including multiplicities; some of the roots may be complex), any $n \times n$ matrix A has exactly n eigenvalues (including multiplicities). Even if the matrix is real, it may have complex eigenvalues.

3. For each of the (distinct) eigenvalues λ of A, compute a basis for $\mathcal{N}(\lambda I - A)$. Any basis vector of $\mathcal{N}(\lambda I - A)$ is an eigenvector corresponding to the eigenvalue λ.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A \in \mathbb{R}^{n\times n}$ and let $v_1, \ldots, v_n$ be corresponding eigenvectors. We have

$$Av_1 = v_1\lambda_1, \ldots, Av_n = v_n\lambda_n.$$

If we put these vectors as columns of a matrix, then

$$(Av_1, \ldots, Av_n) = (v_1\lambda_1, \ldots, v_n\lambda_n)\,.$$

If we define the matrix

$$V = (v_1, \ldots, v_n) \in \mathbb{C}^{n\times n}$$

and the diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathbb{C}^{n\times n},$$

then

$$AV = (Av_1, \ldots, Av_n) = (v_1\lambda_1, \ldots, v_n\lambda_n) = V\Lambda.$$

If the matrix V is invertible, that is if we can find n linearly independent eigenvectors $v_1, \ldots, v_n$, then

$$A = V\Lambda V^{-1}. \tag{3}$$

If there exists an invertible matrix V and a diagonal matrix Λ such that (3) holds, we say that *the matrix A is diagonalizable*.

Multiplication by the matrices V and V^{-1} transforms back and forth between regular Cartesian coordinates, and coordinates with respect to the basis of eigenvectors. In the eigenvector coordinates, the transformation A is represented by the diagonal matrix Λ. This is illustrated in figure 1.

The identity (3) is extremely useful. For example, we can compute

$$A^2 = V\Lambda V^{-1}V\Lambda V^{-1} = V\Lambda^2 V^{-1}$$

and, more generally

$$A^k = V\Lambda V^{-1}V\Lambda V^{-1}\ldots V\Lambda V^{-1} = V\Lambda^k V^{-1}.$$

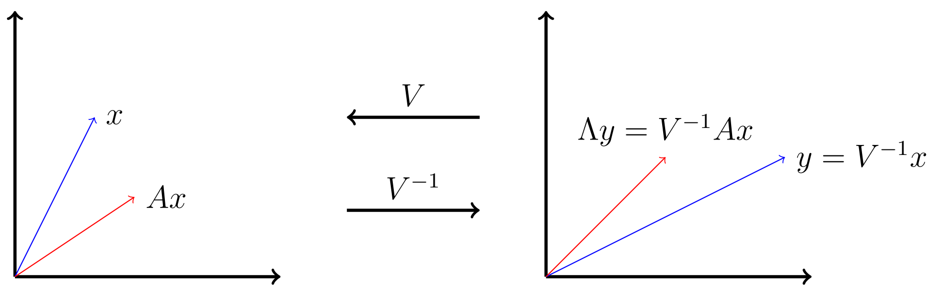


Figure 1: Consider the vectors x and Ax, shown at left. When x is expressed as a linear combination $x = \sum_{j=1}^{n} y_j v_j$ of the eigenvectors $v_1, \cdots, v_n$, i.e., in matrix form, $x = Vy$ for some vector y, the vector y is the coordinates of x with respect to the basis of eigenvectors. Since $A = V\Lambda V^{-1}$, the vector x transforms to $y = V^{-1}x$, while the vector Ax transforms to $V^{-1}Ax = \Lambda V^{-1}Vy = \Lambda y$.

Thus, essentially, we can take powers of a diagonalizable matrix by taking powers of the diagonal matrix, whose diagonal entries are the eigenvalues of the matrix. We will use (3) to the study linear systems, linear least squares problems, as well as properties of solutions to dynamical systems and differential equations. Unfortunately, as we have seen in Example 3 (see also Example 6 below), not every square matrix is diagonalizable and we will show that an identity like (3) can be derived where Λ is no longer a diagonal matrix, but has nonzero entries on the superdiagonal.

Example 4 Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

from Example 1. We have shown that

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} 0, \qquad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} 5.$$

Hence,

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} -2 & 1/2 \\ 1 & 1 \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} -2 & 1/2 \\ 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}}_{=\Lambda}.$$

Since V is invertible,

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} -2 & 1/2 \\ 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} -2/5 & 1/5 \\ 2/5 & 4/5 \end{pmatrix}}_{=V^{-1}}.$$

$\diamond$

Example 5 Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

from Example 2. We have shown that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} i, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} (-i).$$

Hence,

$$\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{=\Lambda}.$$

Since V is invertible,

$$\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{pmatrix}}_{=V^{-1}}.$$

$\diamond$

Example 6 Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

from Example 3. We have shown that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} 1, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} 1$$

Hence,

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=\Lambda}. \tag{4}$$

In this case all eigenvectors are multiples of $v = (1, 0)^T$. Hence we cannot find an invertible matrix V such that (4) holds. $\diamond$

In the following lectures we will pursue a more in-depth study eigenvalues and eigenvectors. In particular we will answer the following questions.

- What can be said about the properties of the eigenvalues and eigenvectors of a matrix. For example, what matrices have all real eigenvalues?

- For which classes of matrices can we find n linearly independent eigenvectors, i.e., which classes of matrices are diagonalizable?
- How can eigenvalues and eigenvectors be used in the study of linear systems, linear least squares problems, dynamical systems and differential equations?

3 Eigenvectors and Eigenvalues

Let A be a square $n \times n$ matrix. A (possibly complex) scalar λ is called an eigenvalue of A if there exists a non-zero vector v such that

$$Av = v\lambda. \tag{5}$$

In this section we prove several properties of eigenvalues and eigenvectors that we already observed in the examples studied in the previous section. For the first result, revisit Example 2.

Theorem 7 *If A is a real $n \times n$ matrix and $\lambda \in \mathbb{C}$ is an eigenvalue of A with corresponding eigenvector $v \in \mathbb{C}^n$, then $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{v} \in \mathbb{C}^n$.*

Proof: If $\lambda \in \mathbb{C}$ is an eigenvalue of A with corresponding eigenvector $v \in \mathbb{C}^n$, then

$$Av = v\lambda.$$

Taking the complex conjugate on either side gives

$$\overline{Av} = \overline{v\lambda}$$

For complex numbers z_1 and z_2 we have $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$. Therefore,

$$\overline{A}\overline{v} = \overline{Av} = \overline{v\lambda} = \overline{v}\overline{\lambda}.$$

Since A is real, $\overline{A} = A$ and we find that

$$A\overline{v} = \overline{v}\overline{\lambda}.$$

□

For the next result, revisit Example 1.

Theorem 8 *If A is a real $n \times n$ symmetric matrix then all eigenvalues and corresponding eigenvectors are real.*

Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with corresponding eigenvector $v \in \mathbb{C}^n$, i.e, let

$$Av = v\lambda.$$

Multiplying by $\overline{v}^T$ from the left gives

$$\overline{v}^T Av = \overline{v}^T v\lambda = \|v\|_2^2 \lambda,$$

where

$$\|v\|_2^2 = \sum_{j=1}^{n} |v_j|^2 \in \mathbb{R}.$$

If we consider the conjugates and transpose, we obtain

$$\overline{\overline{v}^T A v}^T = \overline{\|v\|_2^2 \lambda}^T.$$

Since $(\overline{\overline{v}^T A v})^T = (v^T A \overline{v})^T = \overline{v}^T A^T v = \overline{v}^T A v$ and $\overline{\|v\|_2^2 \lambda}^T = \|v\|_2^2 \overline{\lambda}$ we obtain

$$\overline{v}^T A v = \|v\|_2^2 \overline{\lambda}.$$

Hence

$$\|v\|_2^2 \lambda = \overline{v}^T A v = \|v\|_2^2 \overline{\lambda}.$$

Since v is a nonzero vector, $\|v\|_2^2 > 0$ and we obtain $\lambda = \overline{\lambda}$, which means that λ is a real number. □

The eigenvalues of A are those scalars λ for which $\lambda I - A$ is singular and these scalars can be determined by computing the roots of the characteristic polynomial

$$p_A(\lambda) = \det(\lambda I - A).$$

Since $p_A(\lambda)$ is a polynomial of degree n if A is an $n \times n$ matrix, and every polynomial of degree n has exactly n roots (counting multiplicities), every $n \times n$ matrix has n eigenvalues. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $A \in \mathbb{R}^{n \times n}$ and let $v_1, \dots, v_n$ be corresponding eigenvectors. We have

$$Av_1 = v_1 \lambda_1, \dots, Av_n = v_n \lambda_n.$$

If we put these vectors as columns of a matrix, then

$$AV = (Av_1, \dots, Av_n) = (v_1 \lambda_1, \dots, v_n \lambda_n) = V\Lambda.$$

where

$$V = (v_1, \dots, v_n) \in \mathbb{C}^{n \times n} \quad \text{and} \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

We are interested in characterizing matrices A for which we can find n linearly independent eigenvectors $v_1, \dots, v_n$, so that the matrix V is invertible.

Theorem 9 (Eigenvectors corresponding to distinct eigenvalues) *Eigenvectors $v_1, v_2, \cdots, v_m$ of $A \in \mathbb{C}^{n \times n}$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ (i.e., if $j \neq k$, then $\lambda_j \neq \lambda_k$). are linearly independent.*

Proof: Suppose that the eigenvectors $v_1, v_2, \cdots, v_k$ are linearly independent for some $k < m$, but that $v_1, v_2, \cdots, v_k, v_{k+1}$ are linearly dependent. Then there exist constants $c_1, c_2, \cdots, c_k$ not all zero such that

$$v_{k+1} = \sum_{j=1}^{k} c_j v_j. \tag{6}$$

Multiplication of the left-hand side of (6) by A yields

$$Av_{k+1} = \lambda_{k+1} v_{k+1} = \lambda_{k+1} \sum_{j=1}^{k} c_j v_j,$$

while multiplication of the right-hand side of (6) by A gives

$$A \sum_{j=1}^{k} c_j v_j = \sum_{j=1}^{k} c_j A v_j = \sum_{j=1}^{k} c_j \lambda_j v_j.$$

Hence $\lambda_{k+1} \sum_{j=1}^{k} c_j v_j = \sum_{j=1}^{k} c_j \lambda_j v_j$, which implies

$$\sum_{j=1}^{k} c_j (\lambda_j - \lambda_{k+1}) v_j = 0.$$

Since $\lambda_j \neq \lambda_{k+1}$ for any $j \in \{1, 2, \cdots, k\}$, and the coefficents c_j are not all zero, we conclude that $v_1, v_2, \cdots, v_k$ are linearly dependent, in contradiction to our assumption. Therefore, by induction, the whole set $v_1, v_2, \cdots, v_m$ must be linearly independent. □

Example 10 Consider

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

We compute the row reduced form of $\lambda I - A$,

$$\begin{pmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & -2 & \lambda \end{pmatrix} \to \begin{pmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 2/(\lambda - 1) \end{pmatrix}$$

to determine the characteristic polynomial of A,

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda - 2)((\lambda - 1)\lambda - 2) = (\lambda - 2)(\lambda^2 - \lambda - 2).$$

Hence the eigenvalues are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = -1$.

To compute eigenvectors we compute bases for $\mathcal{N}(2I - A)$ and $\mathcal{N}(-I - A)$:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(2I - A)$$

and

$$\left\{ \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(-I - A).$$

The eigenvectors $(1, 0, 0)^T$ and $(0, -1/2, 1)^T$ belonging to the eigenvalues 2 and 0 are linearly independent and the eigenvectors $(0, 1, 1)^T$ and $(0, -1/2, 1)^T$ belonging to the eigenvalues 2 and 0 are linearly independent.

In this case 2 is an eigenvalue with multiplicity two and the dimension of $\mathcal{N}(2I - A)$ is two. We have

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 1 & 1 \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{=\Lambda}$$

and V is invertible. Hence,

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2/3 & 1/3 \\ 0 & -2/3 & 2/3 \end{pmatrix}}_{=V^{-1}}$$

$\diamond$

Example 11 Consider

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We compute the row reduced form of $\lambda I - A$,

$$\begin{pmatrix} \lambda - 2 & -1 & -1 \\ 0 & \lambda - 1 & -1 \\ 0 & -1 & \lambda - 1 \end{pmatrix} \to \begin{pmatrix} \lambda - 2 & -1 & -1 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 1 - 1/(\lambda - 1) \end{pmatrix}$$

to determine the characteristic polynomial of A,

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda - 2)((\lambda - 1)^2 - 1) = (\lambda - 2)^2 \lambda.$$

Hence the eigenvalues are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 0$.

To compute eigenvectors we compute bases for $\mathcal{N}(2I - A)$ and $\mathcal{N}(-A)$:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(2I - A)$$

and

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(-A).$$

Clearly, the eigenvectors $(1, 0, 0)^T$ and $(0, -1, 1)^T$ are linearly independent. Note that in this example, $\lambda_1 = \lambda_2 = 2$ is a double eigenvalue of A but that the dimension of $\mathcal{N}(2I - A)$ is one. In this case there does *not* exist an invertible matrix V with $AV = A\Lambda$, where $\Lambda = \text{diag}(2, 2, 0)$. $\diamond$

A simple consequence of Theorem 9 is that if all n eigenvalues of $A \in \mathbb{R}^{n \times n}$ are distinct (i.e., are all simple roots of the characteristic polynomial $p_A(\lambda)$), then the matrix A has a complete set of n linearly independent eigenvectors. This is the case in Example 1 (see also Example 4) and in the following example.

Example 12 Consider the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The row reduced form of $\lambda I - A$ is given by

$$(\lambda I - A) = \begin{pmatrix} \lambda + 1 & 0 & 0 \\ -1 & \lambda - 1 & 0 \\ 0 & -1 & \lambda \end{pmatrix} \to \begin{pmatrix} \lambda + 1 & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ 0 & -1 & \lambda \end{pmatrix} \to \begin{pmatrix} \lambda + 1 & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Hence

$$p_A(\lambda) = det(\lambda I - A) = \lambda(\lambda + 1)(\lambda - 1)$$

The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$.

To compute eigenvectors corresponding to an eigenvalue λ_j of A, we have to compute the null-space $\mathcal{N}(\lambda_j I - A)$. We obtain

$$0I - A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \Rightarrow \mathcal{N}(0I - A) = \text{span}\Big\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Big\},$$

$$1I - A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow \mathcal{N}(1I - A) = \text{span}\Big\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Big\},$$

$$(-1)I - A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & -1 & -1 \end{pmatrix} \Rightarrow \mathcal{N}((-1)I - A) = \text{span}\Big\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Big\}.$$

Thus, $v_1 = (0,0,1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 0$, $v_2 = (0,1,1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$, and $v_3 = (2,-1,1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = -1$.

Note that if v is an eigenvector corresponding to the eigenvalue λ, then for any scalar $t \neq 0$ the vector tv is also an eigenvector corresponding to the eigenvalue λ. Therefore we often normalize eigenvectors v such that they satisfy $\overline{v}^T v = 1$, i.e., given a *nonzero* vector $\tilde{v} \in \mathcal{N}(\lambda I - A)$ we set $v = \tilde{v}/\|\tilde{v}\|_2$, where $\|\tilde{v}\|_2 = \sqrt{\sum_{j=1}^n \overline{\tilde{v}}_j \tilde{v}_j}$. Note that in general eigenvalues are complex and eigenvectors are vectors in $\mathbb{C}^n$, even if the matrix A is real.

If we normalize the previously computed eigenvectors we find that $v_1 = (0,0,1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 0$, $v_2 = (0,1,1)^T/\sqrt{2}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$, and $v_3 = (2,-1,1)^T/\sqrt{6}$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = -1$.

We can compute eigenvalues and corresponding eigenvectors using MATLAB function `eig`.

```
>> A = [-1 0 0; 1 1 0; 0 1 0]
A =
    -1     0     0
     1     1     0
     0     1     0

>> [V,Lambda]= eig(A)

V =
         0         0    0.8165
         0    0.7071   -0.4082
    1.0000    0.7071    0.4082

Lambda =
```

```
0         0         0
0         1         0
0         0        -1
```

In this example, `[V,Lambda]= eig(A)` returns a 3×3 matrix V and 3×3 diagonal matrix Λ such that the diagonal entries in Λ are the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$ of A and the columns of V are the corresponding normalized eigenvectors $v_1 = (0, 0, 1)^T$, $v_2 = (0, 1, 1)^T/\sqrt{2}$, $v_3 = (2, -1, 1)^T/\sqrt{6}$.

The eigenvalues - eigenvector relationships give

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{=\Lambda}.$$

(We use the normalized eigenvectors, but the same relation hold if do not normalize.) The matrix V is invertible. If we multiple both sides by V^{-1} from the left, we obtain

$$\begin{aligned} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{=\Lambda} \begin{pmatrix} 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}^{-1} \\ &= \underbrace{\begin{pmatrix} 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{=\Lambda} \cdot \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ \frac{1}{\sqrt{2}} & \sqrt{2} & 0 \\ \frac{\sqrt{6}}{2} & 0 & 0 \end{pmatrix}}_{=V^{-1}}. \end{aligned}$$

$\diamond$

In many physical applications of matrix theory, the matrices that arise are symmetric. This typically arises due to some kind of physical invariance or conservation law. We have already shown in Theorem 8 that eigenvalues and eigenvectors of symmetric matrices are real. We next show a few more important properties of symmetric matrices. First, we show that eigenvectors corresponding to distinct eigenvalues of symmetric matrices are not only linearly independent but even orthogonal. We have observed this already in Example 1. Then we will show that for symmetric matrices we can always find n linearly independent eigenvectors. In fact, we will show that for symmetric matrices we can always find n orthogonal eigenvectors.

Theorem 13 *Let λ and μ be two distinct eigenvalues of a symmetric matrix. If x and y are eigenvectors corresponding to λ and μ, respectively, then x and y are orthogonal, $x^T y = 0$.*

Proof: Since λ and μ are eigenvalues of A with eigenvectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$,

$$Ax = \lambda x, \quad Ay = \mu y.$$

We multiply the first equation by y^T and the second by x^T.

$$y^T Ax = \lambda y^T x, \quad x^T Ay = \mu x^T y.$$

Since A is symmetric, we have

$$y^T Ax = (y^T Ax)^T = x^T A^T y = x^T Ay.$$

Hence, $\lambda y^T x = \mu x^T y$, which implies

$$(\lambda - \mu) y^T x = 0.$$

Since $\lambda \neq \mu$ this proves $y^T x = 0$. $\square$

Example 14 Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The row reduced form of $\lambda I - A$ is given by

$$(\lambda I - A) = \begin{pmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{pmatrix} \to \begin{pmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - 1 - \frac{1}{\lambda - 1} & -1 - \frac{1}{\lambda - 1} \\ 0 & -1 - \frac{1}{\lambda - 1} & \lambda - 1 - \frac{1}{\lambda - 1} \end{pmatrix}$$
$$\to \begin{pmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda(\lambda - 2)/(\lambda - 1) & -1 - \frac{1}{\lambda - 1} \\ 0 & 0 & \lambda(\lambda - 3)/(\lambda - 2) \end{pmatrix}.$$

Hence,

$$p_A(\lambda) = det(\lambda I - A) = \lambda^2 (\lambda - 3).$$

The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 3$.

To compute eigenvectors corresponding to an eigenvalue λ_j of A, we have to compute the null-space $\mathcal{N}(\lambda_j I - A)$. We obtain

$$0I - A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \Rightarrow \mathcal{N}(0I - A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\},$$
$$3I - A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \Rightarrow \mathcal{N}(3I - A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Thus, $v_1 = (-1,1,0)^T$ and $v_2 = (-1,0,1)^T$ are eigenvectors corresponding to the eigenvalue $\lambda_1 = \lambda_2 = 0$ and $v_3 = (1,1,1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = 3$. It is easily verified that $v_1^T v_3 = 0$ and $v_2^T v_3 = 0$.

The eigenvalues - eigenvector relationships give

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{=\Lambda}.$$

The matrix V is invertible. If we multiple both sides by V^{-1} from the left, we obtain

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} &= \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{=\Lambda} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \\ &= \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}}_{=V^{-1}}. \end{aligned}$$

$\diamond$

We now state the result on diagonalizability of symmetric matrices $A \in \mathbb{R}^{n \times n}$. The proof of this result requires a few more preparations and will be given later.

Theorem 15 (Diagonalizability of Symmetric Matrices) *For any symmetric matrix $A \in \mathbb{R}^{n \times n}$, $A^T = A$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ ($Q^T = Q^{-1}$) and a diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that*

$$A = Q \Lambda Q^T$$

Note that in Example 14 we have computed three linearly independent eigenvectors v_1, v_2, v_3. The eigenvectors v_1, v_2 corresponding to the eigenvalue 0 are both orthogonal to the eigenvector v_3 corresponding to the eigenvalue 3. However, v_1 and v_2 are not orthogonal. Thus we need to find a way to extract from the eigenvectors v_1, v_2 two orthogonal eigenvectors. This can be accomplished using Gram-Schmidt orthogonalization, which will be introduced in the following section.

4 Gram-Schmidt Orthogonalization

Given a basis $\{v_1, \ldots, v_m\}$ of a subspace $S \subset \mathbb{R}^n$, the Gram-Schmidt process constructs an orthonormal basis $\{q_1, \ldots, q_m\}$, i.e., basis where the basis vectors satisfy

$$q_i^T q_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases},$$

of the same subspace S. More precisely, for $k = 1, \ldots, m$ the Gram-Schmidt process successively computes an orthonormal basis $\{q_1, \ldots, q_k\}$ from $\{v_1, \ldots, v_k\}$ such that both bases span the same subspace. The idea is to use orthogonal projection to remove components along the existing basis vectors, leaving an orthogonal set. The geometric idea is illustrated in Figure 2.

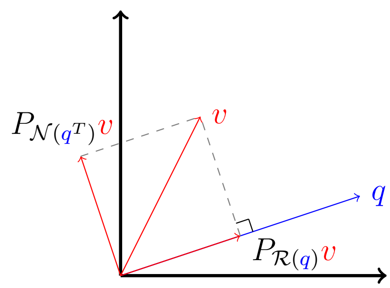


Figure 2: The projectors $P_{\mathcal{R}(q)} = qq^T/(q^Tq)$ and $P_{\mathcal{N}(q^T)} = I - P_{\mathcal{R}(q)}$ decompose the vector v into the component $P_{\mathcal{R}(q)}v$ along q, and the orthogonal component $P_{\mathcal{N}(q^T)}v$.

The steps of the Gram-Schmidt process are described next.

$k = 1$. If $k = 1$, then we need to find a vector q_1 with $q_1^T q_1 = 1$ such that

$$\text{span}\{q_1\} = \text{span}\{v_1\}.$$

The vector q_1 is obtained by normalizing v_1:

$$q_1 = v_1/\|v_1\|_2.$$

$k = 2$. Given q_1 we want to compute q_2 such that $q_1^T q_2 = 0$, $q_2^T q_2 = 1$, and

$$\text{span}\{q_1, q_2\} = \text{span}\{v_1, v_2\}.$$

First note that since $\text{span}\{q_1\} = \text{span}\{v_1\}$ we have $\text{span}\{v_1, v_2\} = \text{span}\{q_1, v_2\}$. Moreover, by the Fundamental Theorem of Linear Algebra we can write v_2 as the sum of vector in $\text{span}\{q_1\} = \mathcal{R}(Q_1)$, where Q_1 is the matrix $Q_1 = (q_1) \in \mathbb{R}^{n\times 1}$, and a vector in the orthogonal complement of $\mathcal{R}(Q_1)$. We can use projections to express these vectors. The projection onto $\mathcal{R}(Q_1)$ is given by $P_{\mathcal{R}(Q_1)} = Q_1(Q_1^T Q_1)^{-1} Q_1^T$. Since $q_1^T q_1 = 1$, $Q_1^T Q_1 = 1$ and the projection is given by

$$P_{\mathcal{R}(Q_1)} = Q_1 Q_1^T.$$

Hence,

$$v_2 = Q_1 Q_1^T v_2 + (I - Q_1 Q_1^T) v_2.$$

Since $Q_1 Q_1^T v_2 \in \mathcal{R}(Q_1) = \text{span}\{q_1\}$ we have

$$\text{span}\{q_1, v_2\} = \text{span}\{q_1, (I - Q_1 Q_1^T) v_2\}.$$

The vector

$$\widetilde{q}_2 = (I - Q_1 Q_1^T) v_2 = v_2 - (q_1^T v_2) q_1$$

is orthogonal to q_1. We just need to normalize it to obtain

$$q_2 = \widetilde{q}_2 / \|\widetilde{q}_2\|_2.$$

We have

$$\text{span}\{v_1, v_2\} = \text{span}\{q_1, v_2\} = \text{span}\{q_1, (I - Q_1 Q_1^T) v_2\} = \text{span}\{q_1, q_2\}$$

$k > 2$. Given $q_1, \ldots, q_{k-1}$ with

$$q_i^T q_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} . \end{cases}$$

and $\text{span}\{q_1, \ldots, q_{k-1}\} = \text{span}\{v_1, \ldots, v_{k-1}\}$, we want to compute q_k such that $q_j^T q_k = 0$, $j = 1, \ldots, k-1$, $q_k^T q_k = 1$, and

$$\text{span}\{q_1, \ldots, q_{k-1}, q_k\} = \text{span}\{v_1, \ldots, v_{k-1}, v_k\}.$$

Since $\text{span}\{q_1, \ldots, q_{k-1}\} = \text{span}\{v_1, \ldots, v_{k-1}\}$ we have

$$\text{span}\{v_1, \ldots, v_{k-1}, v_k\} = \text{span}\{q_1, \ldots, q_{k-1}, v_k\}.$$

By the Fundamental Theorem of Linear Algebra we can write v_k as the sum of vector in $\text{span}\{q_1, \ldots, q_{k-1}\} = \mathcal{R}(Q_{k-1})$, where Q_{k-1} is the matrix $Q_{k-1} = (q_1, \ldots, q_{k-1}) \in \mathbb{R}^{n \times k-1}$, and a vector in the orthogonal complement of $\mathcal{R}(Q_{k-1})$. We can use projections to express these vectors. The projection onto $\mathcal{R}(Q_{k-1})$ is given by $P_{\mathcal{R}(Q_{k-1})} = Q_{k-1}(Q_{k-1}^T Q_{k-1})^{-1} Q_{k-1}^T$. Since the columns $q_1, \ldots, q_{k-1}$ of Q_{k-1} are orthonormal, $Q_{k-1}^T Q_{k-1} = I$ and the projection is given by

$$P_{\mathcal{R}(Q_{k-1})} = Q_{k-1} Q_{k-1}^T.$$

Hence,

$$v_k = Q_{k-1} Q_{k-1}^T v_k + (I - Q_{k-1} Q_{k-1}^T) v_k.$$

Since $Q_1 Q_1^T v_2 \in \mathcal{R}(Q_1) = \text{span}\{q_1\}$ we have

$$\text{span}\{q_1, \ldots, q_{k-1}, v_2\} = \text{span}\{q_1, \ldots, q_{k-1}, (I - Q_{k-1}Q_{k-1}^T)v_k\}.$$

The vector

$$\widetilde{q}_k = (I - Q_{k-1}Q_{k-1}^T)v_k = v_k - \sum_{j=1}^{k-1}(q_j^T v_k)q_j$$

is orthogonal to $q_1, \ldots, q_{k-1}$. We just need to normalize it to obtain

$$q_k = \widetilde{q}_k / \|\widetilde{q}_k\|_2.$$

We have

$$\begin{aligned}\text{span}\{v_1, \ldots, v_{k-1}, v_k\} &= \text{span}\{q_1, \ldots, q_{k-1}, v_k\} = \text{span}\{q_1, \ldots, q_{k-1}, (I - Q_{k-1}Q_{k-1}^T)v_k\} \\ &= \text{span}\{q_1, \ldots, q_{k-1}, q_k\}.\end{aligned}$$

We use the Gram-Schmidt method to compute orthonormal eigenvectors, i.e., to compute orthonormal bases for the eigenspaces $\mathcal{N}(\lambda_j I - A)$.

Example 16 In Example 14 we have shown that the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 3$. The corresponding eigenspaces, i.e, the nullspaces $\mathcal{N}(0I - A)$ and $\mathcal{N}(3I - A)$ are

$$\begin{aligned}\mathcal{N}(0I - A) &= \text{span}\Big\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Big\}, \\ \mathcal{N}(3I - A) &= \text{span}\Big\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Big\}.\end{aligned}$$

Furthermore, we have shown that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}}_{=V^{-1}}.$$

The matrix V of eigenvectors is invertible, but not orthonormal. This is due to the fact that we have computed bases of the eigenspaces $\mathcal{N}(0I - A)$ and $\mathcal{N}(3I - A)$, but not orthonormal bases.

We can do this now using Gram-Schmidt. To compute an orthonormal basis for $\mathcal{N}(0I - A)$ we proceed as follows. Let $v_1 = (-1, 1, 0)^T$ and $v_2 = (-1, 0, 1)^T$. An orthonormal basis is obtained by computing

$$\begin{aligned} q_1 &= v_1/\|v_1\|_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} /\sqrt{2}, \\ \widetilde{q}_2 &= v_2 - (v_2^T q_1)\, q_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}, \\ q_2 &= \widetilde{q}_2/\|\widetilde{q}_2\| = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}. \end{aligned}$$

To compute an orthonormal basis for $\mathcal{N}(3I - A)$ we just need to normalize the original basis vector. The normalized basis vector is

$$q_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

It is easy to verify that the vectors q_1, q_2, q_3 are orthogonal.

The vectors q_1, q_2 are eigenvectors corresponding to the eigenvalue 0 and the vector q_3 is an eigenvectors corresponding to the eigenvalue 3. Hence

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1/\sqrt{2} & -1//\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1//\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{=Q} = \underbrace{\begin{pmatrix} -1/\sqrt{2} & -1//\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1//\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{=\Lambda}.$$

Since the matrix Q is orthogonal, $Q^{-1} = Q^T$ and we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}}_{=Q^T}.$$

If we use Matlab to compute the eigendecomposition of A we obtain

```
>> A = ones(3,3);
>> [Q,Lambda]=eig(A);
>> Q

Q =

    0.4082    0.7071    0.5774
    0.4082   -0.7071    0.5774
   -0.8165         0    0.5774

>> Lambda

Lambda =

   -0.0000         0         0
         0         0         0
         0         0    3.0000
```

Note that the first column in the matrix `Q` computed by Matlab is $-q_2$ and the second column in the matrix `Q` computed by Matlab is $-q_1$. Thus, the first two columns in the matrix `Q` computed by Matlab are just another orthonormal basis for $\mathcal{N}(0I - A)$. $\diamond$

5 Diagonalization of Symmetric Matrices

5.1 Diagonalization of Symmetric Matrices

For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, $A^T = A$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ ($Q^T = Q^{-1}$) and a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that

$$A = Q\Lambda Q^T.$$

One example of the diagnalization of a symmetric matrix was presented in Example 16. (Example 4 is another example, but in Example 4 we have not normalized the eigenvectors, and therefore the matrix V of eigenvectors in Example 4 is not orthonormal.)

If the columns of Q are denoted $q_1, \dots, q_n$, and the diagonal entries of Λ, (the eigenvalues of A) are denoted by $\lambda_1, \dots, \lambda_n$, then the rules of matrix-matrix multiplication imply

$$A = Q\Lambda Q^T = \sum_{j=1}^{n} \lambda_j q_j^T q_j.$$

Example 17 In Example 16 we have shown that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}}_{=Q^T}$$

$$= 0 \left[\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2}\,, & 1/\sqrt{2}\,, & 0 \end{pmatrix} + \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} -1/\sqrt{6}\,, & -1/\sqrt{6}\,, & 2/\sqrt{6} \end{pmatrix} \right]$$

$$+ 3 \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3}\,, & 1/\sqrt{3}\,, & 1/\sqrt{3} \end{pmatrix}$$

$$= 0 \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} + 3 \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

The matrices

$$\begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

are the orthogonal projections onto the eigenspaces $\mathcal{N}(0I - A)$ and $\mathcal{N}(3I - A)$, respectively. $\diamond$

Example 18 In Example 4 we have shown that

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} -2 & 1/2 \\ 1 & 1 \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} -2 & 1/2 \\ 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}}_{=\Lambda}.$$

Since the eigenspaces $\mathcal{N}(0I - A)$ and $\mathcal{N}(5I - A)$ are both one dimensional, we only need to normalize the eigenvectors v_1, v_2 to obtain an orthonormal basis:

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}}_{=Q^T}.$$

◇

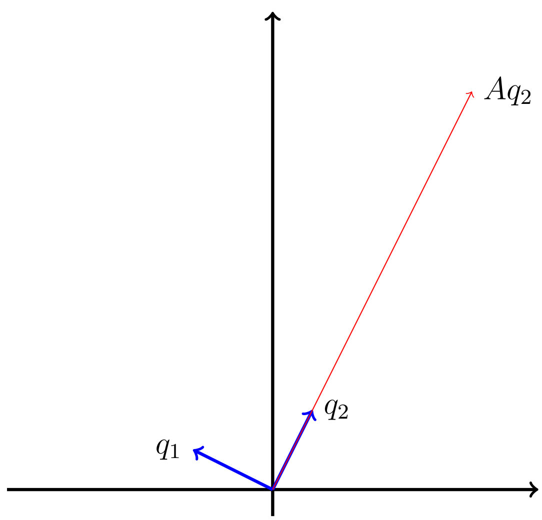


Figure 3: xxx

5.2 Proof of Theorem 15

For completeness, we re-state Theorem 15.

Theorem 15 (Diagonalizability of Symmetric Matrices) *For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, $A^T = A$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ ($Q^T = Q^{-1}$) and a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that*

$$A = Q\Lambda Q^T$$

Proof: The proof uses induction over the size of the matrix.

The result is trivially true for 1×1 matrices.

Assume that for every symmetric matrix $A_n \in \mathbb{R}^{n\times n}$ there exists an orthogonal matrix $Q_n \in \mathbb{R}^{n\times n}$ ($Q_n^T = Q_n^{-1}$) and a diagonal matrix $\Lambda_n \in \mathbb{R}^{n\times n}$ such that

$$A_n = Q_n \Lambda_n Q_n^T.$$

We will show that for every symmetric matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ ($Q^T = Q^{-1}$) and a diagonal matrix $\Lambda \in \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$A = Q \Lambda Q^T.$$

Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be any matrix. Let λ be an eigenvalue of A with to eigenvector $q \in \mathbb{R}^{n+1}$, such that $q^T q = 1$. Consider the subspace

$$S = \left\{ x \in \mathbb{R}^n \ : \ q^T x = 0 \right\}.$$

Note that $S = \mathcal{N}(q^T)$. Since $q^T \in \mathbb{R}^{1\times(n+1)}$, $\dim(S) = n$.

Let $\{p_1, \ldots, p_n\}$ be an orthonormal basis for S (it can be computed using Gram-Schmidt). Then $\{q, p_1, \ldots, p_n\}$ is an orthonormal basis for $\mathbb{R}^{n+1}$. We define $P = (p_1, \ldots, p_n) \in \mathbb{R}^{(n+1)\times n}$. The matrix

$$\begin{pmatrix} q & p_1 & \ldots & p_n \end{pmatrix} = \begin{pmatrix} q & P \end{pmatrix}$$

is an orthogonal matrix.

$$\begin{pmatrix} q & P \end{pmatrix}^T A \begin{pmatrix} q & P \end{pmatrix} = \begin{pmatrix} q^T A q & q^T A P \\ P^T A q & P^T A P \end{pmatrix}.$$

Since $Aq = \lambda q$ and ${}^T q = 1$, we have $q^T A q = \lambda$. Furthermore $q^T A P = (Aq)^T P = \lambda q^T P = 0$, since the columns of P are basis vectors of S and therefore satisfy $q^T p_j$. Thus, we obtain

$$\begin{pmatrix} q & P \end{pmatrix}^T A \begin{pmatrix} q & P \end{pmatrix} = \begin{pmatrix} q^T A q & q^T A P \\ P^T A q & P^T A P \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & P^T A P \end{pmatrix}.$$

The matrix $P^T A P$ is of size $n \times n$. Therefore by induction hypothesis, there there exists an orthogonal matrix $\widetilde{Q}_n \in \mathbb{R}^{n\times n}$ ($\widetilde{Q}_n^T = \widetilde{Q}_n^{-1}$) and a diagonal matrix $\Lambda_n \in \mathbb{R}^{n\times n}$ such that

$$P^T A P = \widetilde{Q}_n \Lambda_n \widetilde{Q}_n^T.$$

Moreover,

$$\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix}^T \begin{pmatrix} \lambda & 0 \\ 0 & P^T A P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \widetilde{Q}_n^T P^T A P \widetilde{Q}_n \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \Lambda_n \end{pmatrix}.$$

and

$$\left(\begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix} \right)^T A \left(\begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix}^T \begin{pmatrix} \lambda & 0 \\ 0 & P^T A P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \Lambda_n \end{pmatrix}}_{=\Lambda}$$

The matrix

$$Q = \begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix}$$

is orthogonal, since

$$\left(\begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix} \right)^T \left(\begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix}^T \begin{pmatrix} q^T q & q^T P \\ P^T q & P^T P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}_n^T \widetilde{Q}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}.$$

Thus we have shown that $A = Q \Lambda Q^T$ with an orthogonal matrix $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{(n+1)\times(n+1)}$. $\square$

6 Diagonalization of Matrices and the Solution of Linear Systems

6.1 Symmetric Matrices

Let A be a real symmetric matrix, $A \in \mathbb{R}^{n\times n}$, $A = A^T$. We are interested in the solution of the linear system

$$Ax = b.$$

For any symmetric matrix $A \in \mathbb{R}^{n\times n}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ ($Q^{-1} = Q^T$) and a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n\times n}$ such that

$$A = Q\Lambda Q^T.$$

Let $q_1, \dots, q_n$ be the columns of Q and let $\lambda_1, \dots, \lambda_n$ be the diagonal entries of Λ (the eigenvalues of A). Then

$$A = Q\Lambda Q^T = \sum_{j=1}^{n} \lambda_j q_j q_j^T.$$

implies that the range space of A is the span of all orthonormal eigenvectors corresponding to non-zero eigenvalues,

$$\mathcal{R}(A) = \text{span}\left\{q_j \ : \ \lambda_j \neq 0\right\},$$

and the null space of A is the span of all orthonormal eigenvectors corresponding to zero eigenvalues,

$$\mathcal{N}(A) = \text{span}\left\{q_j \ : \ \lambda_j = 0\right\}.$$

Note that this representation of the range and the null space shows that $\mathcal{R}(A) \perp \mathcal{N}(A)$, which we of course already know from the Fundamental Theorem of Linear Algebra.

Using the diagonalization $A = Q\Lambda Q^T$, the linear system $Ax = b$ becomes

$$Q\Lambda Q^T x = b.$$

If we multiply both sides by Q^T from the left and define the new unknown $z = Q^T x$ we obtain

$$\Lambda z = Q^T b.$$

While in $Ax = b$ every equation generally depends on every unknown $x_1, \dots, x_n$, the equations $\Lambda z = Q^T b$ are

$$\begin{aligned} \lambda_1 z_1 &= q_1^T b, \\ &\vdots \\ \lambda_n z_n &= q_n^T b. \end{aligned}$$

If an eigenvalue λ_j is zero, the corresponding equation is solvable if and only if $q_j^T b = 0$.

The matrix A is invertible if and only if all eigenvalues are nonzero. In this case the inverse is

$$A^{-1} = Q\Lambda^{-1}Q^T.$$

If an eigenvalue of A is zero, we can define the *pseudo inverse* of A. For a real symmetric matrix A the pseudo inverse is given by

$$A^\dagger = Q\Lambda^\dagger Q^T,$$

where for a diagonal matrix

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$$

the pseudo inverse is defined as

$$\Lambda^\dagger = \text{diag}(\lambda_1^\dagger, \dots, \lambda_n^\dagger)$$

with

$$\lambda_j^\dagger = \begin{cases} 1/\lambda_j & \text{if } \lambda_j \neq 0, \\ 0 & \text{if } \lambda_j = 0, \end{cases} \qquad j = 1, \dots, n.$$

That is,

$$A^\dagger = Q\Lambda^\dagger Q^T = \sum_{\lambda_j \neq 0} \frac{1}{\lambda_j} q_j q_j^T.$$

Example 19 In Example 17 we have shown that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 0 \left[\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \left(-1/\sqrt{2}\,,\ 1/\sqrt{2}\,,\ 0\right) + \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \left(-1/\sqrt{6}\,,\ -1/\sqrt{6}\,,\ 2/\sqrt{6}\right) \right]$$
$$+ 3 \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \left(1/\sqrt{3}\,,\ 1/\sqrt{3}\,,\ 1/\sqrt{3}\right).$$

The pseudoinverse of the matrix above is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^\dagger = \frac{1}{3} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \left(1/\sqrt{3}\,,\ 1/\sqrt{3}\,,\ 1/\sqrt{3}\right) = \begin{pmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix}.$$

In MATLAB the pseudoinverse of a square matrix A can be computed using `pinv(A)`. ◇

6.2 Nonsymmetric Matrices

Let A be a real matrix, $A \in \mathbb{R}^{n\times n}$ and assume that A is diagonalizable, i.e., assume there exists a nonsignular matrix $V \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^{n\times n}$ such that

$$A = V\Lambda V^{-1}.$$

Note that not every non-symmetric matrix is diagonalizable (see, e.g., Examples 3 and 6).

We can use the diagonalization of A to solve the linear system $Ax = b$. Using the diagonalization $A = V\Lambda V^{-1}$, the linear system $Ax = b$ becomes

$$V\Lambda V^{-1}x = b.$$

If we multiply both sides by V^{-1} from the left and define the new unknown $z = V^{-1}x$ we obtain

$$\Lambda z = V^{-1}b.$$

Define $d = V^{-1}b$. While in $Ax = b$ every equation generally depends on every unknown $x_1, \dots, x_n$, the equations $\Lambda z = V^{-1}b$ are

$$\lambda_1 z_1 = d_1,$$
$$\vdots$$
$$\lambda_n z_n = d_n.$$

If an eigenvalue λ_j is zero, the corresponding equation is solvable if and only if the j component of $d = V^{-1}b$ satisfies $d_j = 0$.

The matrix A is invertible if and only if all eigenvalues are nonzero. In this case the inverse is

$$A^{-1} = V\Lambda^{-1}V^{-1}.$$

If an eigenvalue of A is zero, we can define the pseudo inverse of A. We will define the pseudo inverse of non-symmetric (and even non-square) matrices later.

Example 20 The eigenvalues of

$$A = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

are $\lambda_1 = 1$, $\lambda_2 = 2i$, $\lambda_3 = -2i$ with corresponding eigenvectors $v_1 = (0, 0, 1)^T$, $v_3 = (-2, -2i, -1)^T$, and $v_3 = (-2, 2i, -1)^T$. The eigenvectors are linearly independent. Hence A is diagonalizable

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 \\ 0 & -2i & 2i \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix} \begin{pmatrix} 0 & -2 & -2 \\ 0 & -2i & 2i \\ 1 & -1 & -1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 0 & -2 & -2 \\ 0 & -2i & 2i \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix} \begin{pmatrix} -1/2 & 0 & 1 \\ -3/4 & 3i/4 & 0 \\ -3/4 & -3i/4 & 0 \end{pmatrix}.$$

We can also write

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1/2 , & 0 , & 1 \end{pmatrix} + 2i \begin{pmatrix} -2 \\ -2i \\ -1 \end{pmatrix} \begin{pmatrix} -3/4 , & 3i/4 , & 0 \end{pmatrix}$$
$$- 2i \begin{pmatrix} -2 \\ 2i \\ -1 \end{pmatrix} \begin{pmatrix} -3/4 , & -3i/4 , & 0 \end{pmatrix}.$$

The inverse is given by

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -2 & -2 \\ 0 & -2i & 2i \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2i & 0 \\ 0 & 0 & -1/2i \end{pmatrix} \begin{pmatrix} -1/2 & 0 & 1 \\ -3/4 & 3i/4 & 0 \\ -3/4 & -3i/4 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1/2 & 0 \\ -1/2 & 0 & 0 \\ -1/2 & 1/4 & 1 \end{pmatrix}.$$

◇

7 Diagonalization of Matrices and the Solution of Dynamical Systems

Given $A \in \mathbb{R}^{n \times n}$, we are interested in the solution of the dynamical system

$$x'(t) = Ax(t), \qquad t > 0, \tag{7a}$$
$$x(0) = x_0. \tag{7b}$$

The solution of the scalar differential equation

$$\xi'(t) = \lambda \xi(t), \qquad t > 0, \tag{8a}$$
$$\xi(0) = \xi_0 \tag{8b}$$

is given by

$$\xi(t) = e^{\lambda t} \xi_0. \tag{9}$$

We will show that the diagonalization of A, i.e., the spectral decomposition of A can be used to transform the dynamical system (7) into a set of n independent scalar differential equations of the form (8).

Suppose there exists a nonsingular matrix $V \in \mathbb{C}^{n \times n}$ of eigenvectors and a diagonal matrix $\Lambda \in \mathbb{C}^{n \times n}$ of eigenvalues. (Note that we are guaranteed to find n linearly independent eigenvectors if A is symmetric or if A has n distinct eigenvalues. Note also that there exist matrices A that cannot be diagonalized and we will will discuss later what to do in that case.) Since V is invertible, we have

$$A = V \Lambda V^{-1}. \tag{10}$$

If we insert (10) into the dynamical system we obtain

$$x'(t) = Ax(t) = V \Lambda \underbrace{V^{-1} x(t)}_{\stackrel{\text{def}}{=} z(t)}.$$

If we multiply this equation by V^{-1} and use $\frac{d}{dt}(V^{-1}x(t)) = V^{-1}x'(t)$, we obtain

$$z'(t) = \frac{d}{dt}(V^{-1}x(t)) = V^{-1}x'(t) = V^{-1}V\Lambda z(t) = \Lambda z(t).$$

At $t = 0$ we have

$$z_0 \stackrel{\text{def}}{=} z(0) = V^{-1}x(0) = V^{-1}x_0.$$

Using the diagonalization of A we have transformed (7) into the system

$$z'(t) = \Lambda z(t), \qquad t > 0, \tag{11a}$$
$$z(0) = z_0. \tag{11b}$$

Since $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix, (11) is equivalent to n independent scalar differential equations.

$$\begin{aligned} z_1'(t) &= \lambda_1 z_1(t), \qquad t > 0, \\ z_1(0) &= z_{1,0}, \\ &\vdots \\ z_n'(t) &= \lambda_n z_n(t), \qquad t > 0, \\ z_n(0) &= z_{n,0}. \end{aligned}$$

Each of these scalar differential equations is of the form (8) and the solutions of these scalar equations are

$$\begin{aligned} z_1(t) &= e^{\lambda_1 t} z_{1,0}, \\ &\vdots \\ z_n(t) &= e^{\lambda_n t} z_{n,0}. \end{aligned}$$

In vector notation,

$$z(t) = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} z_0 = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} V^{-1} x_0.$$

Since $z(t) = V^{-1}x(t)$, the solution of the system (7) is given by

$$x(t) = V z(t) = V \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} V^{-1} x_0$$

Note that for a diagonalizable matrix A,

$$\begin{aligned} \exp(A) &= \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_{k=0}^{\infty} \frac{1}{k!} (V \Lambda V^{-1})^k = \sum_{k=0}^{\infty} \frac{1}{k!} V \Lambda^k V^{-1} \\ &= V \left(\sum_{k=0}^{\infty} \frac{1}{k!} \Lambda^k \right) V^{-1} = V \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} V^{-1}. \end{aligned} \tag{12}$$

Hence,

$$x(t) = V \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} V^{-1} x_0 = \exp(At) x_0. \tag{13}$$

Note the similarity between the solution formulas (9) for scalar equations and (13) for systems. In MATLAB the *matrix exponential* defined in (12) can be computed using `expm(A)`. Note that this is very different from the output of the MATLAB command `exp(A)` which simply applies the scalar exponential function to every matrix entry of A.

We can extend our technique to dynamical systems

$$x'(t) = Ax(t) + f(t), \qquad t > 0, \tag{14a}$$

$$x(0) = x_0. \tag{14b}$$

with inhomegeneous right hand sides $f : [0, \infty) \to \mathbb{R}^n$. The solution of the scalar differential equation

$$\xi'(t) = \lambda \xi(t) + g(t), \qquad t > 0, \tag{15a}$$

$$\xi(0) = \xi_0 \tag{15b}$$

is given by

$$\xi(t) = e^{\lambda t}\xi_0 + \int_0^t e^{\lambda(t-\tau)} g(\tau) d\tau. \tag{16}$$

If we insert (10) into the dynamical system with inhomogeneous right hand side, then

$$x'(t) = Ax(t) + f(t) = V\Lambda \underbrace{V^{-1}x(t)}_{\stackrel{\text{def}}{=} z(t)} + f(t)$$

If we multiply this equation by V^{-1} and use $\frac{d}{dt}(V^{-1}x(t)) = V^{-1}x'(t)$, we obtain

$$z'(t) = \frac{d}{dt}(V^{-1}x(t)) = V^{-1}x'(t) = V^{-1}V\Lambda z(t) + \underbrace{V^{-1}f(t)}_{\stackrel{\text{def}}{=} g(t)} = \Lambda z(t) + g(t).$$

At $t = 0$ we have

$$z_0 \stackrel{\text{def}}{=} z(0) = V^{-1}x(0) = V^{-1}x_0.$$

$$z'(t) = \Lambda z(t) + g(t), \qquad t > 0, \tag{17a}$$

$$z(0) = z_0. \tag{17b}$$

Since $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix, (17) is equivalent to n independent scalar differential equations.

$$\begin{aligned} z_1'(t) &= \lambda_1 z_1(t) + g_1(t), \qquad t > 0, \\ z_1(0) &= z_{1,0}, \\ &\vdots \\ z_n'(t) &= \lambda_n z_n(t) + g_n(t), \qquad t > 0, \\ z_n(0) &= z_{n,0}, \end{aligned}$$

and the solutions are

$$z_j(t) = e^{\lambda_j t} z_{j,0} + \int_0^t e^{\lambda_j (t-\tau)} g_j(\tau) d\tau.$$

If we insert this into $x(t) = Vz(t)$, we arrive at

$$\begin{aligned}
x(t) = Vz(t) = V &\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} V^{-1} x_0 + V \int_0^t \begin{pmatrix} e^{\lambda_1 (t-\tau)} & & \\ & \ddots & \\ & & e^{\lambda_n (t-\tau)} \end{pmatrix} g(\tau) d\tau \\
&= V \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} V^{-1} x_0 + \int_0^t V \begin{pmatrix} e^{\lambda_1 (t-\tau)} & & \\ & \ddots & \\ & & e^{\lambda_n (t-\tau)} \end{pmatrix} V^{-1} f(\tau) d\tau \\
&= \exp(At) x_0 + \int_0^t \exp(A(t-\tau)) f(\tau) d\tau.
\end{aligned}$$

Again, note the similarity between the solution formulas (16) for scalar equations and

$$x(t) = \exp(At) x_0 + \int_0^t \exp(A(t-\tau)) f(\tau) d\tau \tag{18}$$

for systems.

Before we return to circuit and truss examples, we consider a few simple dynamical systems.

Example 21 The symmetric matrix

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

has eigenvalues $\lambda_1 = -2, \lambda_2 = -4$ and corresponding orthonormal eigenvectors $v_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T / \sqrt{2}$, $v_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T / \sqrt{2}$. In this case

$$\underbrace{\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}}_{= A} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{= V} \underbrace{\begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}}_{= \Lambda} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{= V^T = V^{-1}}$$

We have

$$\exp(At) = V \exp(\Lambda t) V^T = \tfrac{1}{2} \begin{pmatrix} \exp(-2t) + \exp(-4t) & \exp(-2t) - \exp(-4t) \\ \exp(-2t)) - \exp(-4t) & \exp(-2t) + \exp(-4t) \end{pmatrix}.$$

Let $x_0 = (2,3)^T$. The solutions z of $z'(t) = \text{diag}(-2,-4)z(t)$ and x of $x'(t) = Ax(t)$ decay exponentially. See Figure 4. Since V is orthogonal,

$$\|x(t)\|_2^2 = \|Vz(t)\|_2^2 = z(t)^T V^T V z(t) = z(t)^T z(t) = \|z(t)\|_2^2.$$

Since $t \mapsto \|z(t)\|_2$ decays exponentially, $t \mapsto \|x(t)\|_2$ also decays exponentially. ⋄

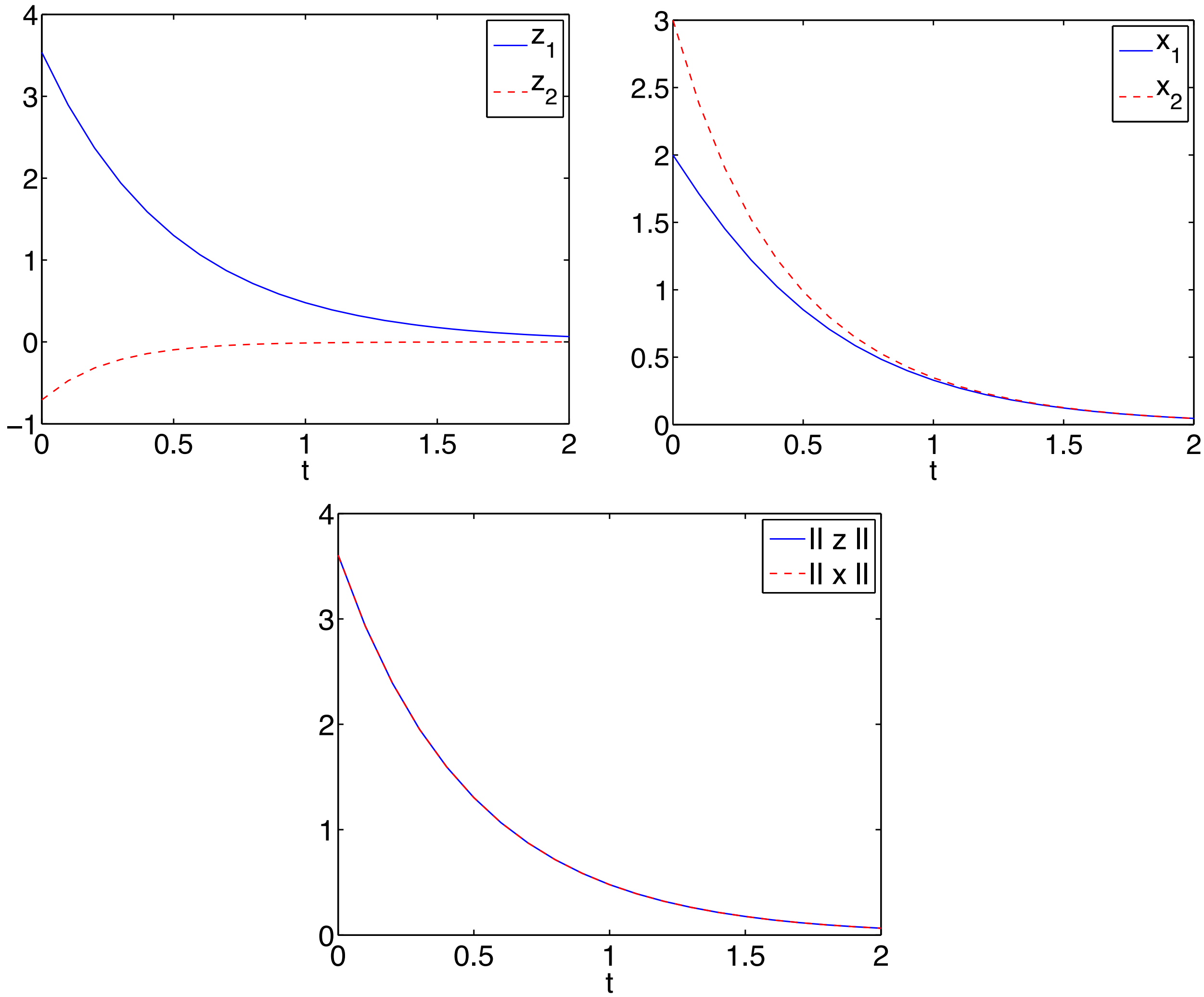


Figure 4: The solutions z of z of $z'(t) = \Lambda z(t)$ (top left plot) and x of $x'(t) = Ax(t)$ (top right plot) for the matrix in Example 21 decay exponentially since all eigenvalues of A are negative. Since the matrix V of eigenvectors is orthonormal $\|x(t)\|_2 = \|z(t)\|_2$ (bottom plot).

Example 22 The matrix

$$A = \begin{pmatrix} 8 & -10 \\ 12 & -14 \end{pmatrix}.$$

has eigenvalues $\lambda_1 = -2, \lambda_2 = -4$ with corresponding to eigenvectors $v_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ and $v_2 = \begin{pmatrix} 5 & 6 \end{pmatrix}^T$. Since the eigenvectors are linearly indepdent,

$$\underbrace{\begin{pmatrix} 8 & -10 \\ 12 & -14 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} 6 & -5 \\ -1 & 1 \end{pmatrix}}_{=V^{-1}}$$

We have

$$\exp(At) = V \exp(\Lambda t) V^{-1} = \begin{pmatrix} 6\exp(-2t) - 5\exp(-4t) & -5\exp(-2t) + 5\exp(-4t) \\ 6\exp(-2t) - 6\exp(-4t) & -5\exp(-2t) + 6\exp(-4t) \end{pmatrix}.$$

Let $x_0 = (2,3)^T$. The solution z of $z'(t) = \text{diag}(-2,-4)z(t)$ decays exponentially. Since the eigenvectors v_1 and v_2 are not orthogonal,

$$\|x(t)\|_2^2 \neq \|z(t)\|_2^2.$$

The solution x of $x'(t) = Ax(t)$ no longer decreases monotonically for small t, For large t the solution x decays exponentially. See Figure 5. $\diamond$

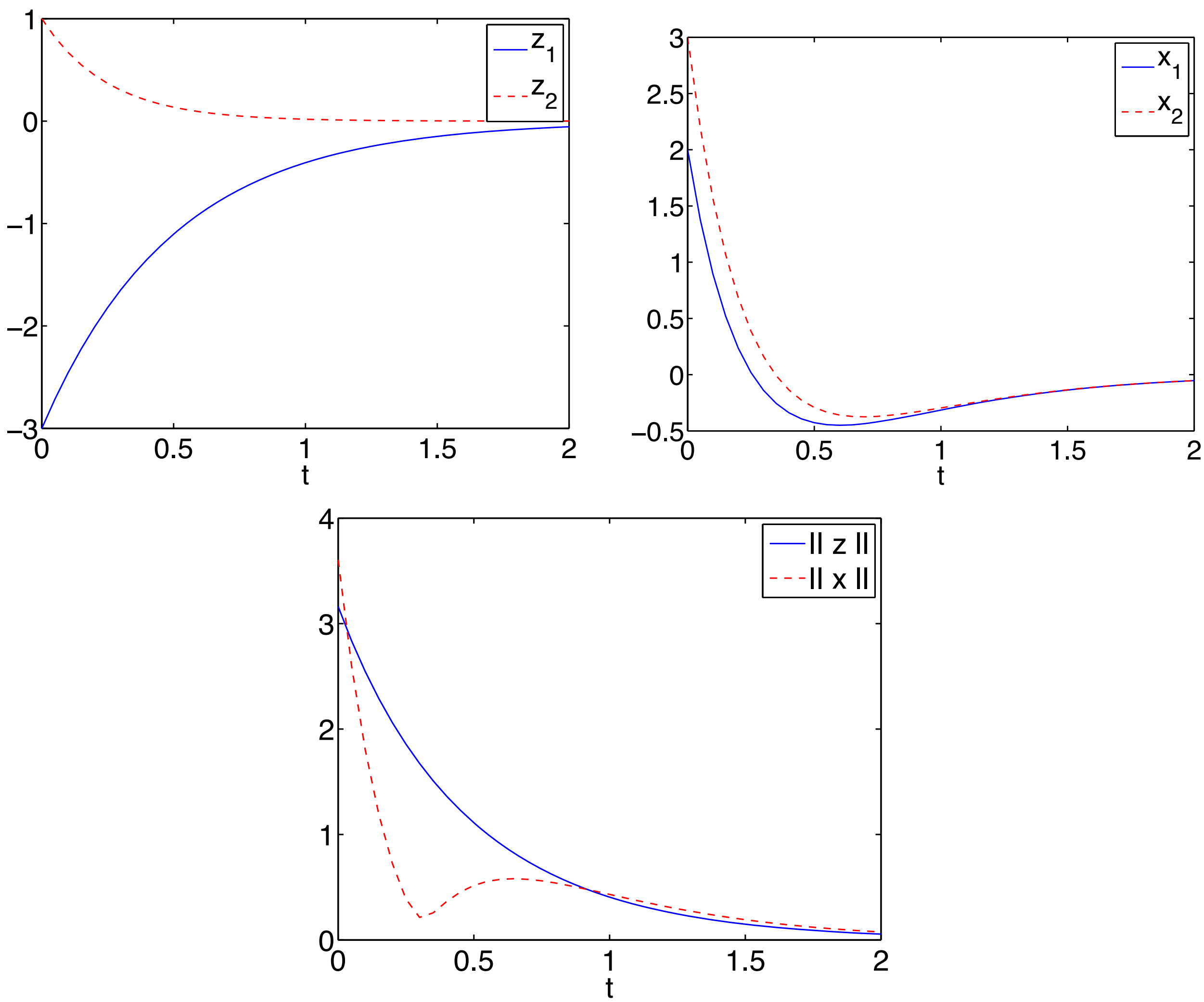


Figure 5: The solution z of $z'(t) = \Lambda z(t)$ (top left plot) for the matrix in Example 22 decay exponentially since all eigenvalues of A are negative. Since the eigenvectors of A are not orthogonal, the solution x of $x'(t) = Ax(t)$ (top right plot) is not monotonically decreasing. For large t the solution x decays exponentially, at the same rate as the solution z. The norms of the solutions z and x are shown in the plot in the center of the second row.

Example 23 The matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has complex eigenvalues $\lambda_1 = i, \lambda_2 = -i$ and corresponding eigenvectors $v_1 = \begin{pmatrix} 1 & -i \end{pmatrix}^T, v_2 = \begin{pmatrix} 1 & i \end{pmatrix}^T$. See Example 5. Since V is invertible,

$$\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{pmatrix}}_{=V^{-1}}.$$

We have

$$\exp(At) = V \exp(\Lambda t) V^{-1} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Let $x_0 = (2, 3)^T$. Since the eigenvalues of A are purely imaginary (the real parts of the eigenvalues are zero), the solution to the dynamical system $x'(t) = Ax(t)$ is oscillatory. See Figure 6. $\diamond$

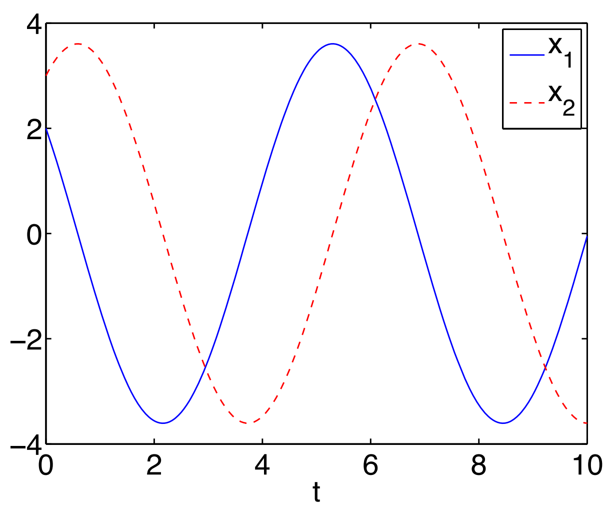


Figure 6: The solution x of $x'(t) = Ax(t)$ for the matrix in Example 23 oscillates since all eigenvalues of A are complex and have zero real parts.

Example 24 The matrix

$$A = \begin{pmatrix} -3/2 & 1 \\ -5/4 & -5/2 \end{pmatrix}.$$

has complex eigenvalues $\lambda_1 = -2+i, \lambda_2 = -2-i$ and corresponding eigenvectors $v_1 = \begin{pmatrix} -2+i & 5 \end{pmatrix}^T, v_2 = \begin{pmatrix} -2-i & 5 \end{pmatrix}^T$. Since V is invertible,

$$\underbrace{\begin{pmatrix} 3/2 & 1 \\ 5/4 & -5/2 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} -2+i & -2-i \\ 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} -2+i & 0 \\ 0 & -2-i \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} i/8 & 1/10+5i/100 \\ -i/8 & 1/10-5i/100 \end{pmatrix}}_{=V^{-1}}.$$

We have

$$\exp(At) = V\exp(\Lambda t)V^{-1} = \begin{pmatrix} \big(\cos(t)+\frac{1}{2}\sin(t)\big)e^{-2t} & \sin(t)e^{-2t} \\ -(5/4)\sin(t)e^{-2t} & \big(\cos(t)-\frac{1}{2}\sin(t)\big)e^{-2t} \end{pmatrix}.$$

Let $x_0 = (2,3)^T$. Since the eigenvalues of A are complex, the the solution to the dynamical system $x'(t) = Ax(t)$ is oscillatory. Since the real parts of the eigenvalues of A are negative, the solution decays. See Figure 7. ◇

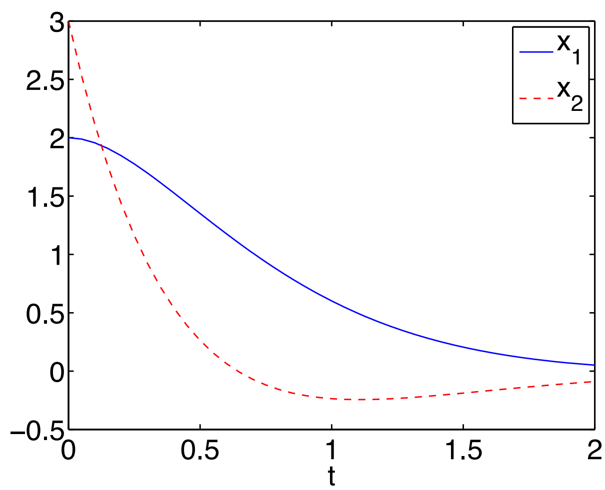


Figure 7: The solution x of $x'(t) = Ax(t)$ for the matrix in Example 24 oscillates since all eigenvalues of A are complex and decays, since the real parts of the eigenvalues are negative.

8 Dynamical Systems: Mechanical Systems

Earlier, we have derived equations for static mechanical systems such as planar trusses. The vector $x \in \mathbb{R}^n$ of displacements of the nodes is the solution of the system

$$Sx = f$$

of linear equations, where S is a symmetric matrix and f is the force vector. Now we allow the displacements to vary with time. The governing equations are derived from 'force equal mass times acceleration' and are given by

$$Mx''(t) + Sx(t) = f(t),$$

where M is a diagonal matrix with diagonal entires given by the masses $m_j > 0$, $j = 1, \ldots, n$, of the nodes. The right hand side $f(t)$ is the vector of external forces, which can vary with time. To compute the displacements, we need to specify the initial displacements $x(0)$ and the initial velocities $x'(0)$. The displacements of a time varying truss are computed as the solution of the second order dynamical system

$$\begin{aligned} Mx''(t) + Sx(t) &= f(t), \qquad t > 0, && \text{(19a)} \\ x(0) &= x_0, && \text{(19b)} \\ x'(0) &= x_1. && \text{(19c)} \end{aligned}$$

Since M is invertible, (19) is equivalent to

$$\begin{aligned} x''(t) &= -M^{-1}Sx(t) + M^{-1}f(t), \qquad t > 0, && \text{(20a)} \\ x(0) &= x_0, && \text{(20b)} \\ x'(0) &= x_1. && \text{(20c)} \end{aligned}$$

We will use the diagonalization of $-M^{-1}S$ to solve (20). Although S and M are symmetric, the matrix $-M^{-1}S$ is in general not symmetric. Therefore, it is not obvious that $-M^{-1}S$ can be diagonalized. We will show next that $-M^{-1}S$ is similar to a symmetric matrix and therefore can be diagonalized.

The matrix M is diagonal with diagonal entires $m_j > 0$, $j = 1, \ldots, n$. We define

$$M^{1/2} = \begin{pmatrix} m_1^{1/2} & & \\ & \ddots & \\ & & m_n^{1/2} \end{pmatrix}.$$

Clearly, $M^{1/2}M^{1/2} = M$. We set $M^{-1/2} = (M^{1/2})^{-1} = (M^{-1})^{1/2}$.

If λ is an eigenvalue of $M^{-1}S$ with corresponding eigenvector v, then

$$M^{-1}Sv = v\lambda.$$

This is equivalent to

$$M^{-1/2}SM^{-1/2}M^{1/2}v = M^{1/2}v\lambda.$$

Thus, λ is an eigenvalue of $M^{-1}S$ with corresponding eigenvector v if and only if λ is an eigenvalue of $M^{-1/2}SM^{-1/2}$ with corresponding eigenvector $M^{1/2}v$. (Note that $M^{1/2}$ is invertible and therefore $M^{1/2}v \neq 0$ if and only if $v \neq 0$.) Since S and $M^{-1/2}$ are symmetric the matrix $M^{-1/2}SM^{-1/2}$ is symmetric. Hence, $M^{-1/2}SM^{-1/2}$ has n real eigenvalues and n orthonormal eigenvectors. Consequently, $-M^{-1}S$ has n real eigenvalues and n linearly independent eigenvectors. That is there exists a nonsingular matrix $V \in \mathbb{R}^{n\times n}$ of eigenvectors and a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that

$$M^{-1}S = V\Lambda V^{-1}. \tag{21}$$

We will show that for the matrices M and S arising in trusses, the eigenvalues of $M^{-1}S$ are positive.

If we insert this into (20a) and define $z(t) = V^{-1}x(t)$ we obtain

$$x''(t) = -V\Lambda V^{-1}x(t) + M^{-1}f(t), \qquad t > 0$$

$$z''(t) = \frac{d^2}{dt^2}\left(V^{-1}x(t)\right) = V^{-1}x''(t) = -\Lambda z(t) + \underbrace{V^{-1}M^{-1}f(t)}_{=g(t)}, \qquad t > 0$$

Hence (20) is equivalent to

$$\begin{aligned} z''(t) &= -\Lambda z(t) + g(t), \qquad t > 0, && \text{(22a)}\\ z(0) &= z_0 \stackrel{\text{def}}{=} V^{-1}x_0, && \text{(22b)}\\ z'(0) &= z_1 \stackrel{\text{def}}{=} V^{-1}x_1. && \text{(22c)} \end{aligned}$$

Since the the eigenvalues of $M^{-1}S$ are positive, Λ is a diagonal matrix with positive diagonal entries.

The system (22) is a collection of scalar second order differential equations of the type

$$\begin{aligned} \xi''(t) &= -\lambda\xi(t) + \gamma(t), \qquad t > 0, && \text{(23a)}\\ \xi(0) &= \xi_0, && \text{(23b)}\\ \xi'(0) &= \xi_1, && \text{(23c)} \end{aligned}$$

where $\lambda > 0$. If $\gamma(t) = 0$, the solution of (23) is given by

$$\xi(t) = \sin(\sqrt{\lambda}t)\ \xi_1 + \cos(\sqrt{\lambda}t)\ \xi_0 \tag{24}$$

We can now apply the solution formula (24) to the n differential equations (22) to obtain $z(t)$. The solution x of (20) is then given by $x(t) = Vz(t)$.

Alternatively, we can convert (20) into a system of first order equations and then apply the techniques introduced in the previous section. We convert (20) into a system of first order equations by introducing the auxiliary unknown

$$y(t) = x'(t).$$

Inserting this variable into (20) we arrive at

$$x'(t) = y(t), \qquad t > 0, \tag{25}$$
$$y'(t) = -M^{-1}Sx(t) + M^{-1}f(t), \qquad t > 0, \tag{26}$$
$$x(0) = x_0, \tag{27}$$
$$y(0) = x_1, \tag{28}$$

or, in matrix-vector notation,

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ -M^{-1}S & 0 \end{pmatrix}}_{\stackrel{\text{def}}{=} B} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad t > 0, \tag{29a}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}. \tag{29b}$$

To solve (29) with the techniques we have discussed, we need to diagonalize $B \in \mathbb{R}^{2n\times 2n}$. We will show that the eigenvalues and eigenvectors of B are closely related to those of $M^{-1}S$. Let μ be an eigenvalue of B. Since B is a block 2×2 matrix, it is useful to write the eigenvector of B corresponding to μ as

$$\begin{pmatrix} w \\ z \end{pmatrix}$$

where w and z are vectors of length n. By definition of eigenvalues and eigenvectors we obtain

$$\begin{pmatrix} z \\ -M^{-1}Sw \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ -M^{-1}S & 0 \end{pmatrix}}_{\stackrel{\text{def}}{=} B} \begin{pmatrix} w \\ z \end{pmatrix} = \mu \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} \mu w \\ \mu z \end{pmatrix}.$$

If we insert the first equation $z = \mu w$ into the second equation $-M^{-1}Sw = \mu z$ we obtain

$$-M^{-1}Sw = \mu^2 w.$$

Thus we have shown that if μ is an eigenvalue of B with eigenvector $\begin{pmatrix} w \\ z \end{pmatrix}$, then μ^2 is an eigenvalue of $-M^{-1}S$ with eigenvector w. Since $-M^{-1}S$ is diagonalizable and has negative eigenvalues $-\lambda_1, \dots, -\lambda_n$ ($\lambda_1, \dots, \lambda_n > 0$) with corresponding eigenvectors $v_1, \dots, v_n$ (see (21)), the eigenvalues of B are

$$\mu_1 = \sqrt{\lambda_1}\, i, \ldots, \mu_n = \sqrt{\lambda_n}\, i, \mu_{n+1} = -\sqrt{\lambda_1}\, i, \ldots, \mu_{2n} = -\sqrt{\lambda_n}\, i,$$

with corresponding eigenvectors

$$\begin{pmatrix} v_1 \\ \sqrt{\lambda_1}\, i\, v_1 \end{pmatrix}, \ldots, \begin{pmatrix} v_n \\ \sqrt{\lambda_1}\, i\, v_n \end{pmatrix}, \begin{pmatrix} v_1 \\ -\sqrt{\lambda_1}\, i\, v_1 \end{pmatrix}, \ldots, \begin{pmatrix} v_n \\ -\sqrt{\lambda_1}\, i\, v_n \end{pmatrix}.$$

Example 25 As an example we consider the mass-spring system studied in Section 2.2 of the lecture notes. For this system

$$S = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$

We assume that at rest, the masses are positioned at 1, 2 and 3, respectively. The solution of (19) for masses $m_1 = 2, m_2 = 1, m_3 = 1$, spring stiffnesses $k_1 = \ldots k_4 = 1$, initial displacements $x_0 = (0, 0.5, 0)^T$ and initial velocities $x_1 = (0, 0, 0)^T$ is shown in Figure 8. $\diamond$

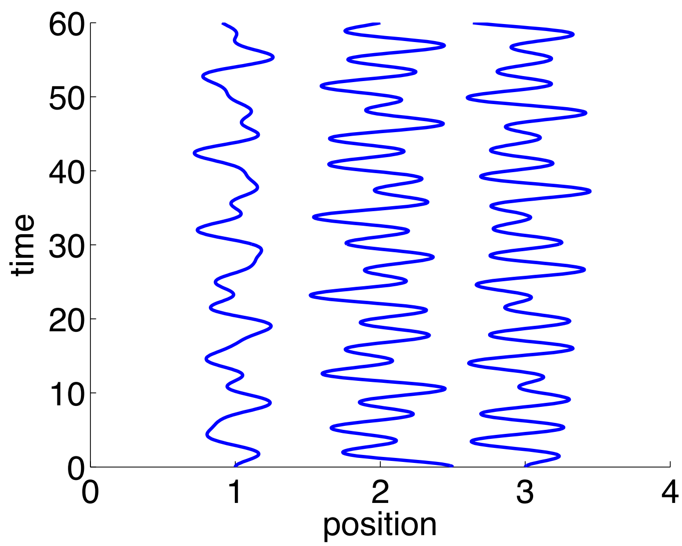


Figure 8: The positions $x_1(t) + 1$, $x_2(t) + 2$, $x_3(t) + 3$ of the three masses over time, where the displacements are computed by solving (19) with initial displacements $x_0 = (0, 0.5, 0)^T$ and initial velocities $x_1 = (0, 0, 0)^T$.

We can extend the techniques above to damped systems with certain damping matrices. A damped system corresponding to (19) is given by

$$M x''(t) + D x'(t) + S x(t) = f(t), \qquad t > 0, \tag{30a}$$
$$x(0) = x_0, \tag{30b}$$
$$x'(0) = x_1 \tag{30c}$$

where $D \in \mathbb{R}^{n\times n}$ represents the damping term. We assume that

$$D = \alpha M + \beta S \tag{31}$$

with some real coefficients $\alpha, \beta > 0$.

Write (30) as a system of first order differential equations

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ * & * \end{pmatrix}}_{\stackrel{\text{def}}{=} B} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad t > 0, \tag{32a}$$
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}. \tag{32b}$$

What is B for the damped system? Compute the eigenvalues and eigenvectors of B.

Example 26 We consider the mass spring system of Example 25 and add damping of the form $D = 0.01M + 0.01S$. The solution of the damped system (30) is shown in Figure 9. ◇

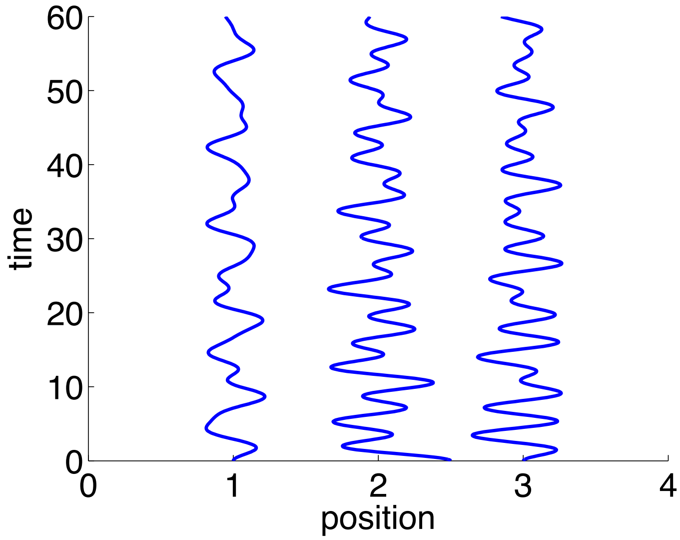


Figure 9: The positions $x_1(t)+1$, $x_2(t)+2$, $x_3(t)+3$ of the three masses over time, where the displacements are computed by solving (30) with damping $D = 0.01M+0.01S$, initial displacements $x_0 = (0, 0.5, 0)^T$ and initial velocities $x_1 = (0, 0, 0)^T$.

8.1 Positive (Semi-)Definite Matrices

Earlier, in this section we have claimed that the matrix $M^{-1}S$ has positive eigenvalues. This is due to the structure of the stiffness matrix S.

A symmetric matrix $S \in \mathbb{R}^{n\times n}$ is called *symmetric positive definite* if

$$v^T S v > 0 \quad \text{for all vectors } v \neq 0.$$

A symmetric matrix $S \in \mathbb{R}^{n\times n}$ is called *symmetric positive semidefinite* if

$$v^T S v \geq 0 \quad \text{for all vectors } v.$$

The positive [semi-]definiteness property of a symmetric matrix S is closely related to the properties of the eigenvalues of S: If λ is an eigenvalues of a symmetric positive [semi-]definite matrix S with corresponding eigenvector v, then $Sv = v\lambda$. Hence

$$0 \underset{[\leq]}{<} v^T S v = \|v\|_2^2 \lambda,$$

which implies that the eigenvalues of a symmetric positive [semi-]definite matrix are positive [non-negative].

On the other hand if all eigenvalues $\lambda_1, \ldots, \lambda_n$ of the symmetric matrix are positive [non-negative], then there exists an orthonormal matrix $Q \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ such that

$$S = Q\Lambda Q^T.$$

Hence for a nonzero vector v,

$$v^T S v = \underbrace{v^T Q}_{= z^T} \Lambda \underbrace{Q^T v}_{= z} = \sum_{j=1}^{m} \lambda_j z_j^2 \underset{[\geq]}{>} 0.$$

That is, if all eigenvalues $\lambda_1, \ldots, \lambda_n$ of the symmetric matrix are positive [non-negative], then the matrix is positive [semi-]definite.

Thus we have shown that a symmetric matrix S is positive [semi-]definite if and only if its eigenvalues are positive [non-negative].

The stiffness matrices arising in mechanical systems are given by

$$S = A^T K A,$$

where $K = \text{diag}(k_1, \ldots, k_m)$. For a vector v we find

$$v^T S v = v^T A^T K \underbrace{Av}_{= z} = \underbrace{(Av)^T}_{= z^T} K \underbrace{Av}_{= z} = z^T K z = \sum_{j=1}^{m} k_j z_j^2 \geq 0.$$

Thus, the stiffness matrix S is always symmetric positive semidefinite. If the truss is stable, then $\mathcal{N}(A) = \{0\}$. In particular, $z = Av \neq 0$ for $v \neq 0$. Hence, for a stable truss

$$v^T S v = v^T A^T K A v = \underbrace{(Av)^T}_{= z^T} K \underbrace{Av}_{= z} = z^T K z = \sum_{j=1}^{m} k_j z_j^2 > 0 \text{ for all } v \neq 0.$$

Thus, the stiffness matrix S for a stable truss is always symmetric positive definite and therefore has only positive eigenvalues. If S is symmetric positive definite, then $M^{-1/2} S M^{-1/2}$ is also positive definite and therefore has only positive eigenvalues. Since $M^{-1/2} S M^{-1/2}$ and $M^{-1} S$ have the same eigenvalues, all eigenvalues of $M^{-1} S$ are positive and the truss is stable.

9 Dynamical Systems: Electrical Circuits

Consider the RC circuit shown in figure 11 with two resistors and two capacitors. Ohm's Law tells us that the current through a resistor is proportional to the potential drop across it, i.e. $y = e/R$. A capacitor consists of a pair of charged plates, separated by a gap, often filled with a dielectric material. The charges create an electric field between the plates, and a corresponding voltage drop $e = x_1 - x_2$, shown in figure 10. Since the time derivative of the charge $Q'(t)$ is equal to the current

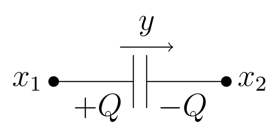


Figure 10: The capacitor obeys the equation $Q = Ce$, where $e = x_1 - x_2$ is the voltage drop across the capacitor, and C is the capacitance (measured in Farads), which depends on the geometry of the plates, and the properties of the material in the gap.

$y(t)$, the current through a capacitor is proportional to the time rate of change of the potential drop across it, i.e.

$$y(t) = Ce'(t),$$

where C is the capacitance of the capacitor, and the prime notation denotes differentiation with respect to time. Furthermore, the current stimulus $V(t)$ is now a function varying in time. Thus, we have moved from the static arena (constant in time) to the dynamic arena (varying in time). We now use the four steps outlined in Chapter 1 of the Course Notes to model the RC circuit.

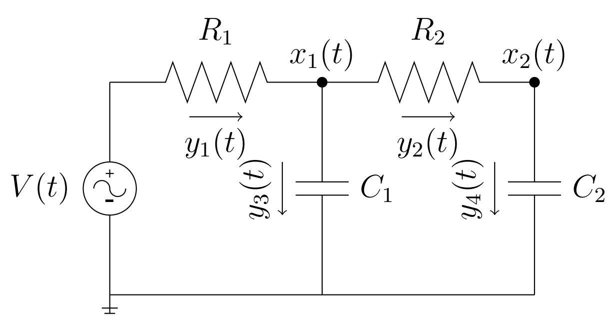


Figure 11: A Simple RC Circuit

Step 1. Potential Drops

The convention is to measure the potential drops across each element in the circuit as "tail-minus-head". Thus, we have the equations

$$
\begin{aligned}
e_1 &= V(t) - x_1 \\
e_2 &= x_1 - x_2 \\
e_3 &= x_1 - 0 \\
e_4 &= x_2 - 0
\end{aligned}
$$

These equations can be written in the form of $e(t) = b(t) - Ax(t)$ where

$$
A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} V(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

Step 2. Resistance and Capacitance

For the resistors, we apply Ohm's Law:

$$
\begin{aligned}
y_1 &= \frac{e_1}{R_1} \\
y_2 &= \frac{e_2}{R_2}
\end{aligned}
$$

For the capacitors, we have:

$$
\begin{aligned}
y_3 &= C_1 e_3' \\
y_4 &= C_2 e_4'
\end{aligned}
$$

Thus, we have $y(t) = Ge(t) + Ce'(t)$ where

$$
G = \begin{pmatrix} 1/R_1 & 0 & 0 & 0 \\ 0 & 1/R_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_2 \end{pmatrix}.
$$

Step 3: Kirchoff's Current Law

At each node, we balance the current going in with the current going out. Thus,

$$
\begin{aligned}
y_1 - y_2 - y_3 &= 0 \\
y_2 - y_4 &= 0
\end{aligned}
$$

or $A^T y(t) = 0$.

Step 4: Assemble previous steps.

We have

$$\begin{aligned} 0 &= A^T y(t) \\ &= A^T \left[Ge(t) + Ce'(t)\right] \\ &= A^T \left[G\left(b(t) - Ax(t)\right) + C\left(b'(t) - Ax'(t)\right)\right] \\ &= A^T Gb(t) - A^T GAx(t) + A^T Cb'(t) - A^T CAx'(t) \end{aligned}$$

which is equivalent to

$$A^T CAx'(t) = -A^T GAx(t) + A^T Gb(t) + A^T Cb'(t).$$

Simple matrix multiplication gives

$$A^T GA = \begin{pmatrix} 1/R_1 + 1/R_2 & -1/R_2 \\ -1/R_2 & 1/R_2 \end{pmatrix} \quad A^T Gb(t) = \begin{pmatrix} V(t)/R_1 \\ 0 \end{pmatrix}$$

$$A^T CA = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \qquad A^T Cb'(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can easily compute

$$(A^T CA)^{-1} = \begin{pmatrix} 1/C_1 & 0 \\ 0 & 1/C_2 \end{pmatrix}.$$

So

$$x'(t) = Bx(t) + g(t), \tag{33}$$

where

$$B = -(A^T CA)^{-1}(A^T GA) = \begin{pmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{C_1}\frac{1}{R_2} \\ \frac{1}{C_2}\frac{1}{R_2} & -\frac{1}{C_2}\frac{1}{R_2} \end{pmatrix}$$

and

$$g(t) = (A^T CA)^{-1} A^T Gb(t) = \begin{pmatrix} \frac{V(t)}{C_1 R_1} \\ 0 \end{pmatrix}.$$

9.1 Inductors

The present framework can easily be extended to include inductors. An inductor is made with a coil of wire, often surrounding a magnetic material. With constant current, it creates a magnetic field. Changes in the current cause the magnetic flux through the coil to change with time, which induces a back EMF (a potential difference, measured in volts) in the wire due to Faraday's law. It is shown in figure 12.

The steps for the Strang Quartet must be modified as follows to account for a circuit with inductors.

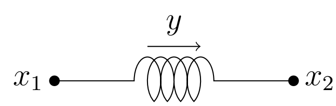


Figure 12: The inductor obeys the equation $e(t) = Ly'(t)$, where $e(t) = x_1(t) - x_2(t)$ is the voltage drop across the inductor, $y'(t)$ is the time derivative of the current through the inductor, and L is the inductance (measured in Henrys), a property determined by the geometry of the inductor, the number of coils, and by the magnetic properties of the material enclosed by the coils.

Step 1. Potential Drops

No change from the previous description; one still computes the voltage drop across inductor elements using "tail-minus-head." The result is $e(t) = b(t) - Ax(t)$, as usual.

Step 2. Inductance

Recall that we obtained for an RC circuit a matrix equation for the current of the form

$$y(t) = Ge(t) + Ce'(t).$$

If circuit element j is an inductor, we have $e_j(t) = L_j y_j'(t)$. Since

$$y_j(t) = y_j(0) + \int_0^t y_j'(\tau)\,d\tau = y_j(0) + \frac{1}{L_j}\int_0^t e_j(\tau)\,d\tau.$$

Thus, we can write

$$y'(t) = Ge'(t) + Ce''(t) + Le(t),$$

where L is a diagonal matrix whose jth diagonal entry is $1/L_j$ if circuit element j is an inductor, and 0 otherwise.

Step 3: Kirchoff's Current Law

Since $A^T y(t) = 0$, we also have that $A^T y'(t) = 0$.

Step 4: Assemble previous steps

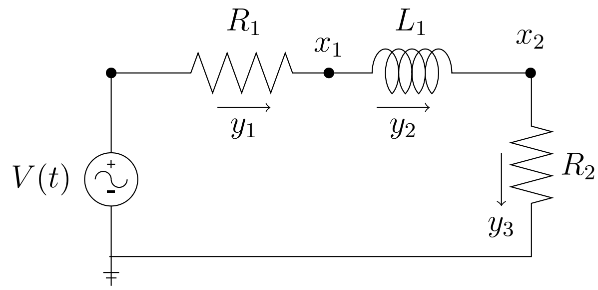
We have

$$\begin{aligned} 0 &= A^T y'(t) \\ &= A^T \left[Ge'(t) + Ce''(t) + Le(t)\right] \\ &= A^T \left[G\left(b'(t) - Ax'(t)\right) + C\left(b''(t) - Ax''(t)\right) + L\left(b(t) - Ax(t)\right)\right]. \end{aligned}$$

This is a second-order system in x. We can go about solving it using techniques similar to those used in section 8. See the next section for an example using the Laplace transform, a technique that we will learn later in the course.

9.2 Application to Power Transmission

Consider the following simple model of a transmission line:



The component on the far left is an AC voltage source. Its functional form is

$$V(t) = V_p \cos(\omega t).$$

In the U.S., $\omega = 120\pi$ rad/s since our AC power operates at a frequency of 60 Hz. R_1 and L_1 represent the impedance of the transmission line. Later we will assume that we have a 200 km transmission line, whose resistance is 0.05 Ω/km and inductance is 0.001 H/km (so that $R_1 = 10\Omega$ and $L_1 = 0.2$H). Finally, R_2 represents the load that the voltage source has been built to power. It could represent the electricity demands of a town, for instance.

The above system may be represented as

$$x'(t) = Bx(t) + g(t),$$

with

$$B = -(A^T GA)^{-1}(A^T LA),\ g(t) = (A^T GA)^{-1}(A^T Gb'(t) + A^T Lb(t)).$$

As usual,

$$e(t) = \underbrace{\begin{pmatrix} V(t) \\ 0 \\ 0 \end{pmatrix}}_{b(t)} - \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}}_{A} x(t).$$

Now with the currents we must account for the inductor along with the resistors. Since $y_2(t) = y_2(0) + \frac{1}{L_1}\int_0^t e(t)dt$ we have:

$$y(t) = \underbrace{\begin{pmatrix} 1/R_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/R_2 \end{pmatrix}}_{G} e(t) + \begin{pmatrix} 0 \\ y_2(0) \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/L_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{L} \int_0^t e(t)dt.$$

Now, Kirchoff's current law gives

$$A^T y(t) = 0.$$

Taking the derivative of both sides and substituting in for y and then e we get:

$$\begin{aligned} A^T y'(t) &= 0 \\ A^T G e'(t) + A^T L e(t) &= 0 \\ A^T G b'(t) - A^T G A x'(t) + A^T L b(t) - A^T L A x(t) &= 0 \end{aligned}$$

$$\boxed{A^T G A x'(t) = -A^T L A x(t) + A^T G b'(t) + A^T L b(t)}$$

and we are done as long as $A^T G A$ is invertible. In fact,

$$A^T G A = \begin{pmatrix} 1/R_1 & 0 \\ 0 & 1/R_2 \end{pmatrix}$$

which is certainly invertible for physically reasonable values of R_1 and R_2 (> 0 and $< \infty$).

We will analyze this circuit using the Laplace transform; see sections 6.1-6.4 in the course notes. Upon application of the Laplace transform to this first-order system of ODE's, we obtain the resolvent matrix $(sI - B)^{-1}$. The poles of the resolvent are the values s for which the resolvent matrix does not exist. These values are the eigenvalues of B.

$$B = -(A^T G A)^{-1} A^T L A = \begin{pmatrix} -R_1/L_1 & R_1/L_1 \\ R_2/L_1 & -R_2/L_1 \end{pmatrix}$$

$$sI - B = \begin{pmatrix} s + R_1/L_1 & -R_1/L_1 \\ -R_2/L_1 & s + R_2/L_1 \end{pmatrix}$$

$$\boxed{(sI - B)^{-1} = \frac{1}{s(sL_1 + R_1 + R_2)} \begin{pmatrix} sL_1 + R_2 & R_1 \\ R_2 & sL_1 + R_1 \end{pmatrix}}$$

Here, we have

$$\begin{aligned} \det(sI - B) &= (s + R_1/L_1)(s + R_2/L_1) - R_1 R_2 / L_1^2 \\ &= s\left(s + \frac{R_1 + R_2}{L_1}\right), \end{aligned}$$

such that the eigenvalues of B are $\boxed{0 \text{ and } -\frac{R_1+R_2}{L_1}}$.

Next, we calculate the Laplace transform $\mathcal{L}x$, assuming that

$$x_i(0) = \frac{V_p R_2}{R_1 + R_2}, \ i = 1, 2.$$

$$\mathcal{L}x(s) = (sI - B)^{-1}(\mathcal{L}g(s) + x(0))$$

We have everything except $\mathcal{L}g(s)$ ...

$$g(t) = \begin{pmatrix} V'(t) \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega V_p \sin(\omega t) \\ 0 \end{pmatrix} \Rightarrow \mathcal{L}g(s) = \begin{pmatrix} -\frac{\omega^2 V_p}{s^2+\omega^2} \\ 0 \end{pmatrix}.$$

Now assemble ...

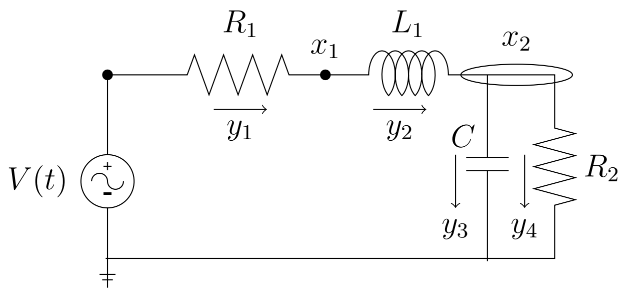
$$\begin{aligned}
\mathcal{L}x(s) &= \frac{1}{s(sL_1+R_1+R_2)} \begin{pmatrix} sL_1+R_2 & R_1 \\ R_2 & sL_1+R_1 \end{pmatrix} \left(\begin{pmatrix} -\frac{\omega^2 V_p}{s^2+\omega^2} \\ 0 \end{pmatrix} + \frac{V_p R_2}{R_1+R_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\
&= \frac{1}{s(sL_1+R_1+R_2)} \begin{pmatrix} \frac{V_p R_2 (sL_1+R_2)}{R_1+R_2} - \frac{\omega^2 V_p (sL_1+R_2)}{s^2+\omega^2} + \frac{V_p R_1 R_2}{R_1+R_2} \\ \frac{V_p R_2^2}{R_1+R_2} - \frac{\omega^2 V_p R_2}{s^2+\omega^2} + \frac{V_p R_2 (sL_1+R_1)}{R_1+R_2} \end{pmatrix} \\
&= \boxed{\frac{V_p R_2}{s(R_1+R_2)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{\omega^2 V_p}{s(sL_1+R_1+R_2)(s^2+\omega^2)} \begin{pmatrix} sL_1+R_2 \\ R_2 \end{pmatrix}}
\end{aligned}$$

The solution $x(t)$ can be calculated using Matlab's symbolic toolbox and the `ilaplace` function. See the code below:

```
%% No capacitor system.  Set-up.

syms R1 L1 R2 Vp om y20
syms t s

V = Vp*cos(om*t);
b = [V; 0; 0];
A = [1 0; -1 1; 0 -1];
G = diag([1/R1 0 1/R2]);
y0 = [0; y20; 0];
L = diag([0 1/L1 0]);

B = -inv(A'*G*A)*A'*L*A;
g = inv(A'*G*A)*(A'*G*diff(b,'t') + A'*L*b);

%% Transform.

R = inv(s*eye(2) - B);
Lg = laplace(g);
x0 = (Vp*R2/(R2 + R1))*[1; 1];
Lx = R*(Lg+x0);

%% Invert and analyze.
```

```
x1 = ilaplace(Lx);
e1 = b - A*x1;
y1 = G*e1 + y0 + L*int(e1,'t');
P1 = e1(3)*y1(3);

x1a = maple('eval',x1,'[R1 = 10, L1 = 0.2, R2 = 250, Vp = 100000, om = 2*pi*60]');
ezplot(x1a(2),[0,1]);
```

Now, consider adding a capacitor in parallel with the load:



This system may be represented as

$$Hz'(t) = Bz(t) + g(t),$$

where

$$z(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}, \; H = \begin{pmatrix} I & 0 \\ 0 & A^TCA \end{pmatrix}, \; B = \begin{pmatrix} 0 & I \\ -A^TLA & -A^TGA \end{pmatrix}$$

and

$$g(t) = \begin{pmatrix} 0 \\ A^TCb''(t) + A^TGb'(t) + A^TLb(t) \end{pmatrix}.$$

Now, we have four voltage drops:

$$e(t) = \underbrace{\begin{pmatrix} V(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{b(t)} - \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}}_{A} x(t),$$

and a capacitor in addition to our inductor and two resistances:

$$y(t) = \underbrace{\begin{pmatrix} 1/R_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/R_2 \end{pmatrix}}_{G} e(t) + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{C} e'(t) + \begin{pmatrix} 0 \\ y_2(0) \\ 0 \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/L_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{L} \int_0^t e(t)dt.$$

Again we use $A^T y'(t) = 0$:

$$\begin{aligned} A^T y'(t) &= 0 \\ A^T Ce''(t) + A^T Ge'(t) + A^T Le(t) &= 0 \\ A^T Cb''(t) - A^T CAx''(t) + A^T Gb'(t) - A^T GAx'(t) + A^T Lb(t) - A^T LAx(t) &= 0 \end{aligned}$$

$$A^T CAx''(t) = -A^T GAx'(t) - A^T LAx(t) + A^T Cb''(t) + A^T Gb'(t) + A^T Lb(t)$$

To transform this into a first-order system, set $z_1(t) = x(t)$, $z_2(t) = x'(t)$. Then we can write

$$\underbrace{\begin{pmatrix} I & 0 \\ 0 & A^T CA \end{pmatrix}}_{H} \underbrace{\begin{pmatrix} {z_1}'(t) \\ {z_2}'(t) \end{pmatrix}}_{z'(t)} = \underbrace{\begin{pmatrix} 0 & I \\ -A^T LA & -A^T GA \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}}_{z(t)} + \underbrace{\begin{pmatrix} 0 \\ A^T Cb''(t) + A^T Gb'(t) + A^T Lb(t) \end{pmatrix}}_{g(t)}.$$

Note that

$$A^T CA = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix},$$

which is not invertible. This is what prohibits us from putting this system into the form $x'(t) = Bx(t) + g(t)$.

Next, we compute the Laplace transform $\mathcal{L}z(s)$, given that $Hz'(t) = Bz(t) + g(t)$. We start by taking the Laplace transform of both sides and applying the properties discussed in class:

$$\begin{aligned} \mathcal{L}(Hz'(t)) &= \mathcal{L}(Bz(t) + g(t)) \\ H(s\mathcal{L}z(s) - z(0)) &= B\mathcal{L}z(s) + \mathcal{L}g(s) \end{aligned}$$

Now solve for $\mathcal{L}z(s)$:

$$(sH - B)\mathcal{L}z(s) = \mathcal{L}g(s) + Hz(0)$$

$$\boxed{\mathcal{L}z(s) = (sH - B)^{-1}(\mathcal{L}g(s) + Hz(0))}$$

We can once again use Matlab's symbolic toolbox to calculate the Laplace transform of z and then the value of $z(t)$ for this system, assuming that

$$x_i(0) = \frac{V_p R_2}{R_1 + R_2},\ x_i'(0) = 0,\ i = 1, 2.$$

See the code below:

```
%% System with capacitor.

syms R1 L1 R2 C Vp om y20
syms t s

V = Vp*cos(om*t);
```

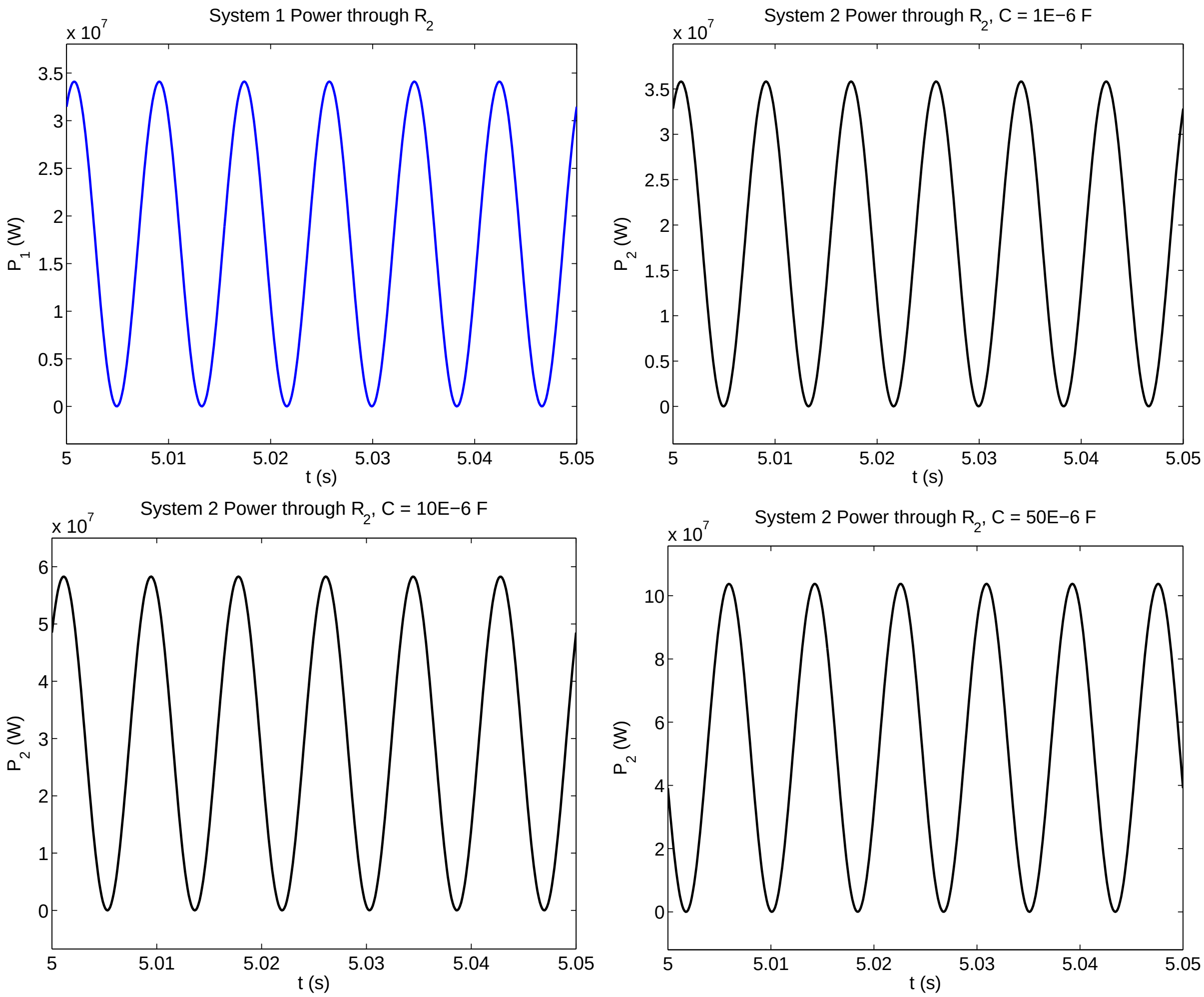
```
b = [V; 0; 0; 0];
A = [1 0; -1 1; 0 -1; 0 -1];
G = diag([1/R1 0 0 1/R2]);
Cm = diag([0 0 C 0]);
y0 = [0; y20; 0; 0];
L = diag([0 1/L1 0 0]);

H = [eye(2) zeros(2,2); zeros(2,2) A'*Cm*A];
B = [zeros(2,2) eye(2); -A'*L*A -A'*G*A];
g = [zeros(2,1); A'*Cm*diff(diff(b,'t'),'t') + A'*G*diff(b,'t') + A'*L*b];

%% Transform.

R = inv(s*H - B);
Lg = laplace(g);
z0 = [(Vp*R2/(R2 + R1)); (Vp*R2/(R2 + R1)); 0; 0];
Lz = R*(Lg + H*z0);

%% Invert and analyze.

z = ilaplace(Lz);
x2 = z(1:2);
e2 = b - A*x2;
y2 = G*e2 + Cm*diff(e2,'t') + y0 + L*int(e2,'t');
P2 = e2(4)*y2(4);
```

We will now analyze the power dissipated by R_2 (the amount consumed by the load we aim to supply), with some given values for the circuit parameters: $V_p = 100$ kV, $R_1 = 10\Omega$, L_1 = 0.2 H, $R_2 = 250\Omega$, $\omega = 120\pi$ rad/sec. We then plot the result for the first system for 5 sec $\leq t \leq$ $(5 + 1/20)$ sec, and make three plots for the second system over the same interval, one each with $C = 1\mu$F, $C = 10\mu$F and $C = 50\mu$F. This is done by the following code:

```
x2a = maple('eval',x2,'[R1 = 10, L1 = 0.2, R2 = 250, C = 1E-6, ...
            Vp = 100000, om = 2*pi*60]');
ezplot(x2a(2),[0,1]);

%% Plot power
P1a = maple('eval',P1,'[R1 = 10, L1 = 0.2, R2 = 250, Vp = 100000, om = 2*pi*60]');
P2a = maple('eval',P2,'[R1 = 10, L1 = 0.2, R2 = 250, Vp = 100000, om = 2*pi*60]');
figure
ezplot(P1a,[5,5+1/20]);
title('System 1 Power through R_2');
xlabel('t (s)');
```

```
ylabel('P_1 (W)');
figure
h = ezplot(maple('eval',P2a,'C=1E-6'),[5,5+1/20]);
title('System 2 Power through R_2, C = 1E-6 F');
xlabel('t (s)');
ylabel('P_2 (W)');
set(h,'Color','k');
figure
h = ezplot(maple('eval',P2a,'C=10E-6'),[5,5+1/20]);
title('System 2 Power through R_2, C = 10E-6 F');
xlabel('t (s)');
ylabel('P_2 (W)');
set(h,'Color','k');
figure
h = ezplot(maple('eval',P2a,'C=50E-6'),[5,5+1/20]);
title('System 2 Power through R_2, C = 50E-6 F');
xlabel('t (s)');
ylabel('P_2 (W)');
set(h,'Color','k');

Pavg = 20E6;
Pavg1 = int(60*P1a,'t',5,5+1/60);
Pavg2_1 = int(60*maple('eval',P2a,'C=1E-6'),'t',5,5+1/60);
Pavg2_10 = int(60*maple('eval',P2a,'C=10E-6'),'t',5,5+1/60);
Pavg2_50 = int(60*maple('eval',P2a,'C=50E-6'),'t',5,5+1/60);
Pavg1/Pavg
Pavg2_1/Pavg
Pavg2_10/Pavg
Pavg2_50/Pavg
```

with the plots shown below:



Average power dissipated by each system: $P_{1,\text{avg}} = 17.0MW$, $P_{2,1\mu F,\text{avg}} = 17.9MW$, $P_{2,10\mu F,\text{avg}} = 29.1MW$, and $P_{2,50\mu F,\text{avg}} = 51.9MW$.

Ratio of average power dissipated to P_{avg}, for each system: $\frac{P_{1,\text{avg}}}{P_{\text{avg}}} = 0.853$, $\frac{P_{2,1\mu F,\text{avg}}}{P_{\text{avg}}} = 0.896$, $\frac{P_{2,10\mu F,\text{avg}}}{P_{\text{avg}}} = 1.46$, $\frac{P_{2,50\mu F,\text{avg}}}{P_{\text{avg}}} = 2.59$.

Adding a capacitor in parallel with the load counteracts the effects of impedance (resistance and inductance) in the transmission line, at least in terms of the amount of power available to the load. In fact, if the capacitance is large enough, the amount of power available can be amplified to values significantly greater than what would be available if there was no impedance in the transmission line. However, we suspect that such amplification is not free (with regards to the input energy required to maintain the amplitude of $V(t)$) ...

10 The Leontief Input-Output Model and Eigenvalues of the Consumption Matrix

Let $C \in \mathbb{R}^{n \times n}$ be a diagonalizable matrix with nonzero entires.

$$C = V \Lambda V^{-1}.$$

We want to know under what conditions the inverse $(I - C)^{-1}$ exists and has nonnegative entries. The eigenvalues of C, more precisely the largest eigenvalue of C will give us the answer.

If C has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $I - C$ has eigenvalues $1 - \lambda_1, \ldots, 1 - \lambda_n$. In particular, $I - C$ is invertible if and only if $1 - \lambda_1, \ldots, 1 - \lambda_n \neq 0$, i.e., if one is not an eigenvalues of C.

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the eigenvalues of C and assume that they are ordered such that

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|.$$

11 The Jordan Normal Form

11.1 Computing the Jordan Normal Form

In the previous sections, we have studied several applications of the diagonalization of matrices $A \in \mathbb{R}^{n\times n}$. Unfortunately, not all matrices can be diagonalized as we have seen already in Example 6.

Example 27 The matrix

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

has eigenvalues $\lambda_1 = -2$, $\lambda_2 = \lambda_3 = 4$. The corresponding eigenspaces are

$$\mathcal{N}(-2I - A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{N}(4I - A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Since the dimension of the eigenspace $\mathcal{N}(4I - A)$ (= the geometric multiplicity of the eigenvalue 4) is less than the algebraic multiplicity of the eigenvalue 4, the matrix A is not diagonalizable. $\diamond$

In Example 27, we had only two linearly independent eigenvectors, so it was impossible to come up with a diagonal form for A via

$$A = V \Lambda V^{-1},$$

as we are "missing" a third vector for V. It turns out that we can always find vectors that can put A in an almost diagonal form if we are willing to relax our notion of the eigenvector. The rest of this section briefly develops these ideas into the Jordan Normal Form, which can also be seen as a consequence of the spectral representation developed in chapter 9 of the course notes.

Let λ_j be an eigenvalue of A with algebraic multiplicty m_j. Eigenvectors of A corresponding to λ_j are elements of

$$\mathcal{N}(A - \lambda_j I).$$

The dimension $\dim \mathcal{N}(A - \lambda_j I)$ is the geometric multiplicity of the eigenvalue λ_j. Note that

$$\mathcal{N}(A - \lambda_j I) \subset \mathcal{N}(A - \lambda_j I)^{m_j}.$$

Non-zero elements of the set on the right-hand side are called *generalized eigenvectors* of A, corresponding to the eigenvalue λ_j. Thus, eigenvectors are also generalized eigenvectors. Suppose that $v \neq 0$ satisfies $(A-\lambda I)^m v = 0$, and that $m > 0$ is the smallest integer such that this equation holds. From the vector v, we obtain a *cycle* of generalized eigenvectors

$$\left\{ v, (A-\lambda I)v, \ldots, \underbrace{(A-\lambda I)^{m-1}v}_{\text{only eigenvector}} \right\}.$$

The next theorem gives linear independence of this cycle. These cycles will turn out to contain the missing basis vectors required to complete the matrix V, and render A into an almost diagonal form.

Theorem 28 (Linear Independence Generalized Eigenvector Cycles) *Let $(A-\lambda I)^m v = 0$, i.e., let v be a generalized eigenvector of A with eigenvalue λ. Then the cycle of generalized eigenvectors*

$$\left\{ v, (A-\lambda I)v, \ldots, (A-\lambda I)^{m-1}v \right\}$$

is linearly independent.

Proof: By induction. Suppose that

$$\left\{ (A-\lambda I)^j v \right\}_{j=k}^{m-1}$$

is linearly independent, and that

$$\sum_{j=k-1}^{m-1} \alpha_j (A-\lambda I)^j v = 0.$$

Multiply by $(A-\lambda I)$ to obtain

$$\begin{aligned} 0 =&(A-\lambda I) \sum_{j=k-1}^{m-1} \alpha_j (A-\lambda I)^j v = \sum_{j=k-1}^{m-1} \alpha_j (A-\lambda I)^{j+1} v \\ =& \left(\sum_{j=k}^{m-1} \alpha_{j-1} (A-\lambda I)^j v \right) + \alpha_{m-1} \underbrace{(A-\lambda I)^m v}_{=0}. \end{aligned}$$

Thus, $\alpha_{k-1} = \ldots = \alpha_{m-2} = 0$, and therefore,

$$0 = \sum_{j=k-1}^{m-1} \alpha_j (A-\lambda I)^j v = \alpha_{m-1}(A-\lambda I)^{m-1} v \Rightarrow \alpha_{m-1} = 0.$$

□

For any matrix $A \in \mathbb{R}^{n\times n}$, it can be shown (see chapter 9 of the course notes) that a basis for $\mathbb{R}^n$ can be constructed that is composed of generalized eigenvectors of A. When the generalized eigenvectors are appropriately chosen, the matrix

$$V^{-1}AV$$

can be reduced to *Jordan Canonical Form* or *Jordan Normal Form*.

Theorem 29 (Jordan Normal Form) *For any square matrix $A \in \mathbb{R}^{n\times n}$ there exists a nonsingular matrix $V \in \mathbb{C}^{n\times n}$ such that*

$$V^{-1}AV = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & J_k \end{pmatrix} \stackrel{\text{def}}{=} J, \tag{34}$$

where

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ & & & & & \\ \vdots & & \ddots & \ddots & & \\ & & & & & \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{n_i\times n_i}, \quad i = 1, \dots, k,$$

are the so-called Jordan blocks, $\lambda_1, \dots, \lambda_k$ are eigenvalues of A, and $\sum_{i=1}^{k} n_i = n$. Moreover, let

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 & \dots & 0 \\ 0 & \Lambda_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \Lambda_k \end{pmatrix},$$

where

$$\Lambda_i = \begin{pmatrix} \lambda_i & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 0 & \dots & 0 & 0 \\ & & & & & \\ \vdots & & \ddots & \ddots & & \\ & & & & & \\ 0 & 0 & 0 & \dots & \lambda_i & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{n_i\times n_i}, \quad i = 1, \dots, k.$$

Then $A = S + N$, where $S = V\Lambda V^{-1}$ is diagonalizable, and the matrix N is nilpotent of order k ($N^k = 0$, $N^{k-1} \neq 0$) for some $k \le n$. Furthermore, $NS = SN$.

From

$$V^{-1}AV = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & J_k \end{pmatrix}$$

we find that

$$AV = V \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & J_k \end{pmatrix}.$$

To see how we can use this to compute the columns of V, we partition the columns of V according to the blocks in J. We set

$$V = \Big(v_{1,1}, \dots, v_{1,n_1}, \dots\dots, v_{k,1}, \dots, v_{k,n_k}\Big).$$

Note that $v_{1,1}$ is the first column of V.

If we insert this and use the rules for matrix-matrix multiplication, we see that

$$A\Big(v_{i,1}, \dots, v_{i,n_i}\Big) = \Big(v_{i,1}, \dots, v_{i,n_i}\Big)J_i$$

for $i = 1, \dots, k$. Using the structure of the Jordan block J_i we find that

$$\begin{aligned} Av_{i,1} &= \lambda_i v_{i,1}, \\ Av_{i,2} &= \lambda_i v_{i,2} + v_{i,1}, \\ &\vdots \\ Av_{i,n_i} &= \lambda_i v_{i,n_i} + v_{i,n_i-1}. \end{aligned}$$

The first identity states that $v_{i,1}$ is an eigenvector of A corresponding to the eigenvalue λ_i. The others are part of a generalized eigenvector cycle.

Computation of the Jordan Normal Form (in the case of real eigenvalues) can be summarized by the following procedure. We assume that $\lambda_j \in \mathbb{R}$ is an eigenvalue of A with algebraic multiplicity m_j. Let

$$\delta_k = \dim \mathcal{N}(A - \lambda I)^k.$$

1. Compute a basis $\{v_1, v_2, \dots, v_{\delta_1}\}$ for $\mathcal{N}(A - \lambda_j I)$. The geometric multiplity of λ_j is δ_1. Arrange the basis as the columns of the matrix V:

$$V = \begin{pmatrix} v_1 & v_2 & \cdots & v_{\delta_1} \end{pmatrix}.$$

2. If $\delta_1 < \delta_2$, find a basis $\{\widetilde{v_1}, \widetilde{v_2}, \ldots, \widetilde{v_{\delta_1}}\}$ for $\mathcal{N}(A - \lambda_j I)$, with
$$\tilde{V} = \begin{pmatrix} \widetilde{v_1} & \widetilde{v_2} & \cdots & \widetilde{v_k} \end{pmatrix} = VC,$$
($C \in \mathbb{R}^{k\times k}$ is an invertible matrix), such that there are $\delta_2 - \delta_1$ linearly independent solutions $w_1, \ldots, w_{\delta_2-\delta_1}$ to
$$(A - \lambda I)w_j = \widetilde{v_j}.$$
This leads to a matrix of generalized eigenvectors
$$\left(\begin{array}{cccccccc|ccc} \widetilde{v_1} & w_1 & \widetilde{v_2} & w_2 & \cdots & \widetilde{v}_{\delta_2-\delta_1} & w_{\delta_2-\delta_1} & & \widetilde{v}_{\delta_2-\delta_1+1} & \cdots & \widetilde{v}_{\delta_1} \end{array} \right).$$

3. If $\delta_2 < \delta_3$, repeat the process...

4. Continue until the matrix V contains a complete set of m_j generalized eigenvectors. Note that at each step where $\delta_k < \delta_{k+1}$, this will involve recombining the basis vectors in V, as described in step 2.

We illustrate the computation of the Jordan Normal Form in the following two examples.

Example 30 Consider the matrix A in Example 27. The eigenvalue -2 is simple. The eigenvalue 4 has a geometric multiplicity of 1 that is less than its algebraic multiplicity of 2. Therefore, the Jordan normal form of A is given by
$$\underbrace{\begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} -1 & 1 & * \\ 1 & -1 & * \\ 1 & 1 & * \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} -1 & 1 & * \\ 1 & -1 & * \\ 1 & 1 & * \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}}_{=J}.$$

The missing column, denoted by $v_{2,2}$ in the notation used above, is obtained from
$$\underbrace{\begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}}_{=A} v_{2,2} = 4v_{2,2} + \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{=v_{2,1}},$$
i.e., $(4I - A)v_{2,2} = -v_{2,1}$.

Solving $(4I - A)v_{2,2} = -v_{2,1}$ gives
$$v_{2,2} = \begin{pmatrix} 1+\alpha \\ -\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha v_{2,1}, \quad \alpha \in \mathbb{R}.$$

Note that $(4I - A)$ is singular and span$\{v_{2,1}\} = \mathcal{N}(4I - A)$, which can be seen from $v_{2,2} = (1,0,0)^T + \alpha v_{2,1}$. We find

$$\underbrace{\begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}}_{=J}.$$

◇

The second example indicates why the computation of the Jordan Normal Form is involved.

Example 31 The matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

has eigenvalue $\lambda = 2$ with algebraic multiplicity 3. The eigenspace is

$$\mathcal{N}(2I - A) = \text{span}\Big\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\widetilde{v}_1}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\widetilde{v}_2} \Big\}.$$

Since the dimension of the eigenspace $\mathcal{N}(2I - A)$ (= the geometric multiplicity of the eigenvalue 2) is less than the algebraic multiplicity of the eigenvalue 2, the matrix A is not diagonalizable.

From the geometric and algebraic multiplicity of the eigenvalue 2 we can deduce the structure of the matrix J in the Jordan normal form

$$\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}_{=J}.$$

We also know that the first column $v_{1,1}$ of V is an eigenvector corresponding to the eigenvalue 2 and that the second column $v_{2,1}$ of V is an eigenvector corresponding to the eigenvalue 2 that is linearly independent of $v_{1,1}$. However, the columns $v_{1,1}$ and $v_{2,1}$ are in general not equal to the basis vectors $\widetilde{v}_1$ and $\widetilde{v}_2$ for the eigenspace $\mathcal{N}(2I - A)$.

Since $v_{2,1}$ is an eigenvector corresponding to the eigenvalue 2, $v_{2,1} = c_1\widetilde{v}_1 + c_2\widetilde{v}_2$ for some scalars c_1, c_2. We need to find c_1, c_2 as well as a vector $v_{2,2}$ such that

$$(2I - A)v_{2,2} = c_1\widetilde{v}_1 + c_2\widetilde{v}_2,$$

i.e.,

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} v_{2,2} = \begin{pmatrix} c_1 \\ 0 \\ c_2 \end{pmatrix}.$$

This system only has a solution if $c_1 = -c_2$ and in this case the solutions are

$$v_{2,2} = \begin{pmatrix} 0 \\ -c_1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

Hence, we obtain

$$v_{2,1} = c_1 \widetilde{v}_1 - c_1 \widetilde{v}_2,$$

for any $c_1 \neq 0$ and

$$v_{2,2} = \begin{pmatrix} 0 \\ -c_1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for any $\alpha, \beta \in \mathbb{R}$. We set $c_1 = 1$ and $\alpha = \beta = 0$, which gives

$$v_{2,1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_{2,2} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

The vector $v_{1,1}$ is an eigenvector corresponding to the eigenvalue 2 that is linearly independent of $v_{2,1}$. That is we need to find

$$v_{1,1} \in \mathcal{N}(2I - A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

with $v_{1,1} \neq \gamma v_{2,1}$ for some $\gamma \in \mathbb{R}$. The vector $v_{1,1} = (1, 0, 0)^T$ satisfies these criteria. (There are other choices, such as $v_{1,1} = (0, 0, 1)^T$.)

Thus, the Jordan Normal Form of A is given by

$$\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}_{=J} \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}}_{=V^{-1}}.$$

◇

The computation of the Jordan Normal Form can be complicated and, except for special cases, we will not compute the Jordan Normal Form. We use the existence of the Jordan Normal Form and basic properties of the Jordan blocks to learn about the behavior of solutions of dynamical systems and other problems. Next we will study the matrix exponential of Jordan blocks J_i and of A, which are important quantities for the solution of dynamical systems with constant coefficients.

11.2 Computing the Matrix Exponential using the Jordan Normal Form

The Jordan Normal Form

$$A = V \underbrace{\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & J_k \end{pmatrix}}_{=J} V^{-1}.$$

can be used to compute powers of A and evaluate matrix functions, such as $\exp(A)$. For example, the ℓth power of A is

$$A^\ell = VJV^{-1}VJV^{-1}\dots VJV^{-1} = VJ^\ell V^{-1}.$$

Since J is block diagonal,

$$J^\ell = \begin{pmatrix} J_1^\ell & 0 & \dots & 0 \\ 0 & J_2^\ell & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & J_k^\ell \end{pmatrix}.$$

To compute powers of a Jordan block we write

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 & 0 \\ & & & & & & \\ & & \ddots & \ddots & & & \\ & & & & & & \\ 0 & 0 & 0 & \dots & \lambda_i & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda_i \end{pmatrix} = \lambda_i I + N_i,$$

where

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ & & & & & & \\ & & \ddots & \ddots & & & \\ & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}.$$

One computes

$$N_i^2 = \begin{pmatrix} 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ & & & & & & \\ & & \ddots & \ddots & & & \\ & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}, \dots, N_i^{n_i-1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ & & & & & & \\ & & \ddots & \ddots & & & \\ & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n_i \times n_i},$$

and $N_i^{n_i} = 0 \in \mathbb{R}^{n_i \times n_i}$. For example if $n_i = 4$,

$$N_i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; N_i^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; N_i^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; N_i^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$J_i^\ell = (\lambda_i I + N_i)^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} \lambda_i^{\ell-j} N_i^j = \sum_{j=0}^{\min\{\ell, n_i-1\}} \binom{\ell}{j} \lambda_i^{\ell-j} N_i^j, \tag{35}$$

where

$$\binom{\ell}{j} = \frac{\ell!}{j!(\ell-j)!}.$$

If we use the fact that for two matrixes A, B that commute, i.e., $AB = BA$, it holds $\exp(A + B) = \exp(A)\exp(B)$, then the expression for $\exp(J_i t)$ can be derived easily (of course we have avoided the difficulty of actually showing that $\exp(A + B) = \exp(A)\exp(B)$ for two matrices A, B that commute). Since the matrices $\lambda_i I$ and N_i commute we have

$$\begin{aligned} \exp(J_i t) &= \exp((\lambda_i I + N_i)t) = \exp(\lambda_i t I)\exp(N_i t) \\ &= \exp(\lambda_i t)\exp(N_i t) = \exp(\lambda_i t)\left(\sum_{j=0}^{\infty} \frac{t^j}{j!} N_i^j\right) \\ &= \exp(\lambda_i t)\left(\sum_{j=0}^{n_i-1} \frac{t^j}{j!} N_i^j\right). \end{aligned} \tag{36}$$

(In Section 11.3 we will derive (36) without using the identity $\exp(A + B) = \exp(A)\exp(B)$ for two matrixes A, B that commute.)

For example if

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 \\ 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix} \in \mathbb{R}^{4\times 4},$$

then

$$\exp(J_i t) = \exp(\lambda_i t) \begin{pmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{4\times 4}.$$

The functions that appear in the entries of $\exp(J_i t)$ are shown in Figure 13 for $\lambda_i = -1$. The functions $\exp(-t)t$, $\exp(-t)t^2/2$, and $\exp(-t)t^3/6$ grow initially until the term $\exp(-t)$ dominates the powers of t and the functions decrease monotonically.

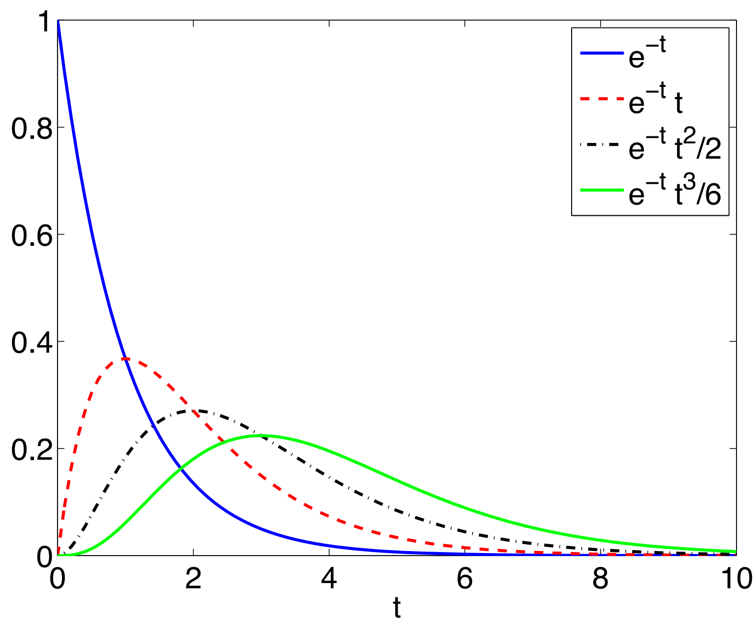


Figure 13: While the function $\exp(-t)$ decreases monotonically, the functions $\exp(-t)t$, $\exp(-t)t^2/2$, and $\exp(-t)t^3/6$ grow initially until the $\exp(-t)$ term dominates the powers of t and the functions decrease monotonically.

If (34) holds, the matrix exponential of At for a scalar t is given by

$$\exp(At) = V \begin{pmatrix} \exp(J_1 t) & 0 & \dots & 0 \\ 0 & \exp(J_2 t) & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \exp(J_k t) \end{pmatrix} V^{-1}.$$

Example 32 Suppose that the concentrations as a function of time for three chemical species are given by $x_1(t), x_2(t), x_3(t)$. Furthermore, suppose the rate of the reaction $x_1 \to x_2$ is proportional to x_1 and the rate of the reaction $x_2 \to x_3$ is proportional to x_2. Finally, suppose that the initial concentrations are $x_1(0) = 1$, $x_2(0) = x_3(0) = 1$.

The mathematical model for the chemical reaction is

$$\begin{aligned} x_1'(t) &= -k_1 x_1(t), \\ x_2'(t) &= k_1 x_1(t) - k_2 x_2(t), \\ x_3'(t) &= k_2 x_2(t), \end{aligned}$$

and $x(0) = (1, 0, 0)^T$, where k_1, k_2 are the rate constants for the reaction. In matrix vector form,

$$\begin{aligned} \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} &= \begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \\ \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The solution of the differential equation $x'(t) = Ax(t)x(t)$, $t > 0$, with initial conditon $x(0) = x_0$ is given by

$$x(t) = \exp(At)x_0.$$

We study two cases, $k_1 = k_2$ and $k_1 \neq k_2$.

$k_1 = k_2$: If $k_1 = k_2 \neq 0$, the eigenvalues of

$$A = \begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_1 & 0 \\ 0 & k_1 & 0 \end{pmatrix}$$

are $-k_1$ with algebraic multiplicity 2 and 0 with algebraic multiplicity 1. The corresponding eigenspaces are

$$\mathcal{N}(-k_1 I - A) = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{N}(-A) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Since the dimension of the eigenspace $\mathcal{N}(-k_1 I - A)$ (= the geometric multiplicity of the eigenvalue $-k_1$) is less than the algebraic multiplicity of the eigenvalue $-k_1$, the matrix A is not diagonalizable.

The previous calculations show that the Jordan normal form is

$$\underbrace{\begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_1 & 0 \\ 0 & k_1 & 0 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} 0 & * & 0 \\ -1 & * & 0 \\ 1 & * & 1 \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} 0 & * & 0 \\ -1 & * & 0 \\ 1 & * & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} -k_1 & 1 & 0 \\ 0 & -k_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=J}.$$

The missing column, denoted by $v_{1,2}$ in the notation used above, is obtained from

$$\underbrace{\begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_1 & 0 \\ 0 & k_1 & 0 \end{pmatrix}}_{=A} v_{1,2} = -k_1 v_{1,2} + \underbrace{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}_{=v_{1,1}},$$

i.e., $(-k_1 I - A)v_{1,2} = -v_{1,1}$.

Solving $(-k_1 I - A)v_{1,2} = -v_{1,1}$ gives

$$v_{1,2} = \begin{pmatrix} -1/k_1 \\ 1/k_1 - \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} -1/k_1 \\ 1/k_1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/k_1 \\ 1/k_1 \\ 0 \end{pmatrix} + \alpha v_{1,1}, \quad \alpha \in \mathbb{R}.$$

Choosing $\alpha = 0$, we find

$$\underbrace{\begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_1 & 0 \\ 0 & k_1 & 0 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} 0 & -1/k_1 & 0 \\ -1 & 1/k_1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{=V} = \underbrace{\begin{pmatrix} 0 & -1/k_1 & 0 \\ -1 & 1/k_1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} -k_1 & 1 & 0 \\ 0 & -k_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=J}$$

or, equivalently,

$$\underbrace{\begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_1 & 0 \\ 0 & k_1 & 0 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} 0 & -1/k_1 & 0 \\ -1 & 1/k_1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} -k_1 & 1 & 0 \\ 0 & -k_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=J} \underbrace{\begin{pmatrix} -1 & -1 & 0 \\ -k_1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{=V^{-1}}.$$

We have

$$\begin{aligned} &\exp\left(\begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_1 & 0 \\ 0 & k_1 & 0 \end{pmatrix} t \right) \\ &= \underbrace{\begin{pmatrix} 0 & -1/k_1 & 0 \\ -1 & 1/k_1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} \exp(-k_1 t) & \exp(-k_1 t)t & 0 \\ 0 & \exp(-k_1 t) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=\exp(Jt)} \underbrace{\begin{pmatrix} -1 & -1 & 0 \\ -k_1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{=V^{-1}}. \end{aligned}$$

The right plot in Figure 14 shows the solution $x(t) = \exp(At)x_0$ of the differential equation $x'(t) = Ax(t)$, $t > 0$, where A s constructed with $k_1 = k_2 = 1$, and with initial condition $x_0 = (1, 0, 0)^T$. The left plot shows the solution $z(t) = \exp(Jt)z_0$ of of the differential equation $z'(t) = Jz(t)$, $t > 0$, with initial condition $z_0 = V^{-1}x_0$.

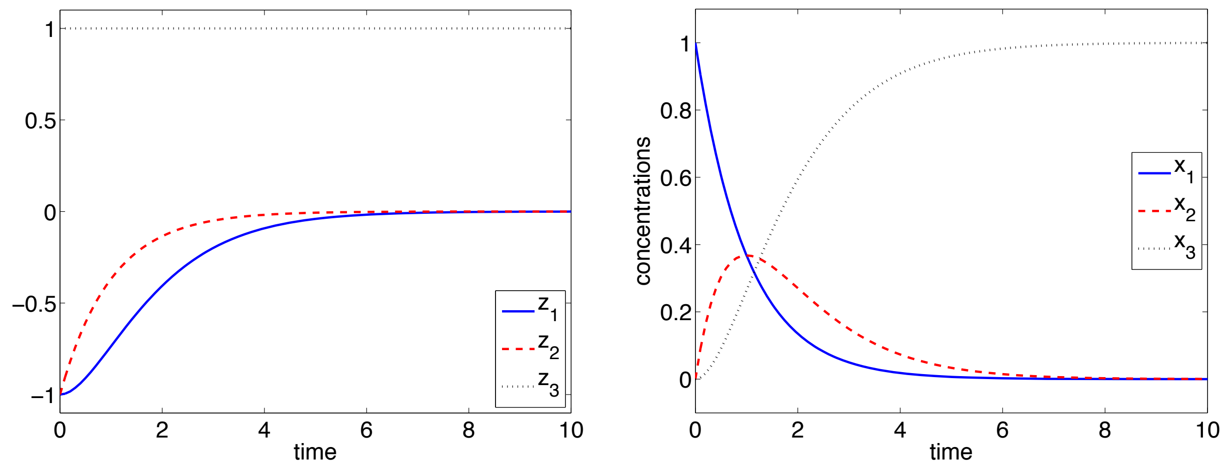


Figure 14: The solution z of $z'(t) = Jz(t)$ (left plot) and the solution z of $z'(t) = Ax(t)$ of with $k_1 = k_2 = 1$ and initial value $x_0 = (1, 0, 0)^T$.

$k_1 \neq k_2$: If $k_1 \neq k_2$ and $k_1 \neq 0$, $k_2 \neq 0$, then eigenvalues of

$$A = \begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{pmatrix}$$

are $-k_1, -k_2, 0$, each with algebraic multiplicity 1. The corresponding eigenspaces are

$$\mathcal{N}(-k_1 I - A) = \text{span}\left\{ \begin{pmatrix} (k_1 - k_2)/k_2 \\ -k_1/k_2 \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{N}(-k_2 I - A) = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{N}(-A) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Since A has three distinct eigenvalues, it is diagonalizable. In particular,

$$\underbrace{\begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} (k_1-k_2)/k_2 & 0 & 0 \\ -k_1/k_2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} -k_1 & 0 & 0 \\ 0 & -k_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} k_2/(k_1-k_2) & 0 & 0 \\ -k_1/(k_1-k_2) & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{=V^{-1}}.$$

The right plot in Figure 15 shows the solution $x(t) = \exp(At)x_0$ of the differential equation $x'(t) = Ax(t)$, $t > 0$, where A s constructed with $k_1 = k_2 = 1$, and with initial condition $x_0 = (1, 0, 0)^T$. The left plot shows the solution $z(t) = \exp(\Lambda t)z_0$ of of the differential equation $z'(t) = \Lambda z(t)$, $t > 0$, with initial condition $z_0 = V^{-1}x_0$.

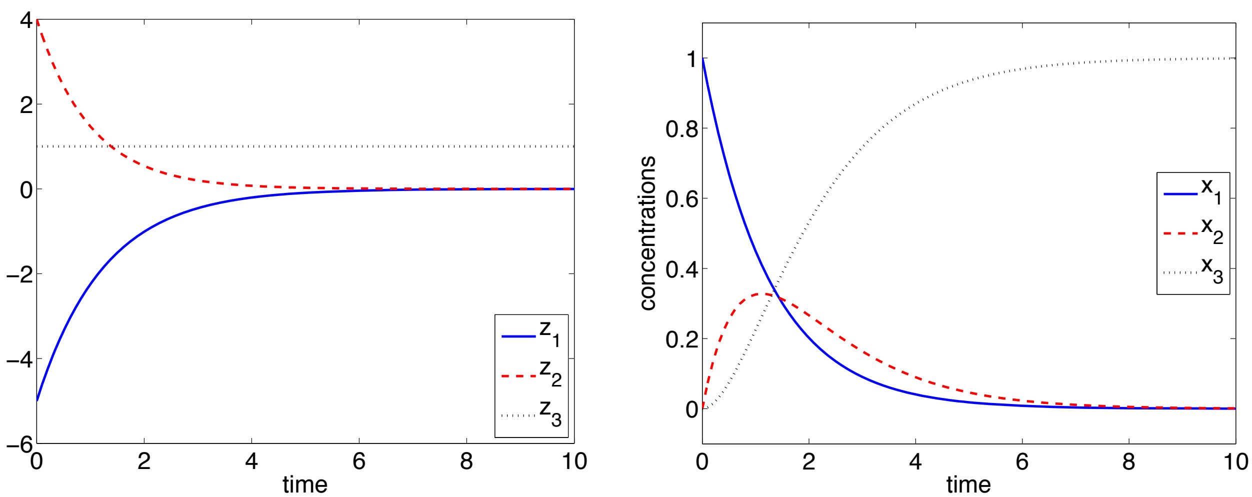


Figure 15: The solution z of $z'(t) = \Lambda z(t)$ (left plot) and the solution z of $z'(t) = Ax(t)$ of with $k_1 = 0.8$, $k_2 = 1$ and initial value $x_0 = (1, 0, 0)^T$.

$\diamond$

11.3 Computing the Matrix Exponential of a Jordan Block

In the previous section we have computed the matrix exponential of a Jordan block, $\exp(J_i t)$, using the identity $\exp((\lambda_i I + N_i)t) = \exp(\lambda_i t I)\exp(N_i t)$. This identity is true, but it relies in the fact that for two matrixes A, B that commute, i.e., $AB = BA$, it holds $\exp(A + B) = \exp(A)\exp(B)$, which we haven't proven. For completeness, we show (36) using elementary computations.

Using the expression for J_i^ℓ derived in (35) we obtain

$$
\begin{aligned}
\exp(J_i) &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} J_i^\ell = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j=0}^{\min(\ell, n_i-1)} \left(\begin{array}{c} \ell \\ j \end{array} \right) \lambda_i^{\ell-j} N_i^j \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\lambda_i^\ell I + \sum_{j=1}^{\min(\ell, n_i-1)} \left(\begin{array}{c} \ell \\ j \end{array} \right) \lambda_i^{\ell-j} N_i^j \right) \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \lambda_i^\ell I + \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j=1}^{\min(\ell, n_i-1)} \left(\begin{array}{c} \ell \\ j \end{array} \right) \lambda_i^{\ell-j} N_i^j \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \lambda_i^\ell I + \sum_{j=1}^{n_i-1} \frac{1}{\ell!} \sum_{\ell=0}^{\ell} \left(\begin{array}{c} \ell \\ j \end{array} \right) \lambda_i^{\ell-j} N_i^j + \sum_{\ell=n_i}^{\infty} \frac{1}{\ell!} \sum_{j=1}^{n_i-1} \left(\begin{array}{c} \ell \\ j \end{array} \right) \lambda_i^{\ell-j} N_i^j \\
&= \sum_{\ell=0}^{\infty} \frac{\lambda_i^\ell}{\ell!} I + \sum_{j=1}^{n_i-1} \sum_{\ell=j}^{n_i-1} \frac{\lambda_i^{\ell-j}}{(\ell-j)!} \frac{1}{j!} N_i^j + \sum_{j=1}^{n_i-1} \sum_{\ell=n_i}^{\infty} \frac{\lambda_i^{\ell-j}}{(\ell-j)!} \frac{1}{j!} N_i^j \\
&= \sum_{\ell=0}^{\infty} \frac{\lambda_i^\ell}{\ell!} I + \sum_{j=1}^{n_i-1} \sum_{\ell=0}^{n_i-1-j} \frac{\lambda_i^\ell}{\ell!} \frac{1}{j!} N_i^j + \sum_{j=1}^{n_i-1} \sum_{\ell=n_i-j}^{\infty} \frac{\lambda_i^\ell}{\ell!} \frac{1}{j!} N_i^j \\
&= \underbrace{\sum_{\ell=0}^{\infty} \frac{\lambda_i^\ell}{\ell!}}_{=\exp(\lambda_i)} I + \sum_{j=1}^{n_i-1} \underbrace{\sum_{\ell=0}^{\infty} \frac{\lambda_i^\ell}{\ell!}}_{=\exp(\lambda_i)} \frac{1}{j!} N_i^j \\
&= \exp(\lambda_i) \sum_{j=0}^{n_i-1} \frac{1}{\ell!} N_i^j.
\end{aligned}
$$

12 The Singular Value Decomposition

12.1 The Singular Value Decomposition

For any matrix $A \in \mathbb{R}^{m\times n}$ there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ and a 'diagonal' matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & 0 & \dots & 0 \\ & \ddots & & & & & & \\ & & \sigma_r & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{m\times n} \qquad \text{if } m \le n$$

or

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \end{pmatrix} \in \mathbb{R}^{m\times n} \qquad \text{if } m > n,$$

with diagonal entries

$$\sigma_1 \ge \ldots \ge \sigma_r > \sigma_{r+1} = \ldots = \sigma_{\min\{m,n\}} = 0$$

such that

$$A = U\Sigma V^T. \tag{37}$$

The decomposition (37) is called the *singular value decomposition (SVD)* of $A \in \mathbb{R}^{m\times n}$. The scalars $\sigma_1, \ldots, \sigma_{\min\{m,n\}}$ are called the singular values of A. The SVD can be computed using the MATLAB command `svd`.

If the columns of V are $v_1, \ldots, v_n$ and the columns of U are $u_1, \ldots, u_m$, then the SVD (37) can also be written as

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T. \tag{38}$$

12.2 The SVD and the Fundamental Theorem of Linear Algebra

From the SVD (38) we can easily determine the range and null-spaces of A and A^T.

- The range space of A is $\mathcal{R}(A) = \text{span}\{u_1, \ldots, u_r\}$.
  ($\mathcal{R}(A) = \{0\}$ if $r = 0$, i.e., if all singular values of A are zero (i.e., A is the zero matrix).)

  Proof: Since $Ax = \sum_{i=1}^r u_i(\sigma_i v_i^T x) \in \text{span}\{u_1, \ldots, u_r\}$ for any $x \in \mathbb{R}^n$, we have $\mathcal{R}(A) \subset \text{span}\{u_1, \ldots, u_r\}$. On the other hand, the orthogonality of the vectors $v_1, \ldots, v_n$ implies $Av_j/\sigma_j = \sum_{i=1}^r \sigma_i(\sigma_i v_i^T v_j/\sigma_j) = u_j$ for $j = 1, \ldots, r$, i.e, $u_j \in \mathcal{R}(A)$ for $j = 1, \ldots, r$. Consequently, $\text{span}\{u_1, \ldots, u_r\} \subset \mathcal{R}(A)$. □

- The null space of A is $\mathcal{N}(A) = \text{span}\{v_{r+1}, \ldots, v_n\}$.
  ($\mathcal{N}(A) = \{0\}$ if $r = n$, i.e., if all singular values of A are positive and $m \geq n$.)

  Proof: If $x \in \mathcal{N}(A)$, then $Ax = \sum_{i=1}^r \sigma_i(\sigma_i v_i^T x) = 0$. If we multiply by u_j^T, $j \in \{1, \ldots, r\}$, then $0 = u_j^T Ax = \sum_{i=1}^r u_j^T u_i(\sigma_i v_i^T x) = \sigma_j v_j^T x$. Since $\sigma_j > 0$ for $j \in \{1, \ldots, r\}$, we obtain $v_j^T x = 0$, $j = 1, \ldots, r$. By orthogonality of the vectors $v_1, \ldots, v_n$, $x \in \text{span}\{v_{r+1}, \ldots, v_n\}$. On the other hand, the orthogonality of the vectors $v_1, \ldots, v_n$ implies that for $j > r$, $Av_j = \sum_{i=1}^r \sigma_i(\sigma_i v_i^T v_j) = 0$ i.e, $v_j \in \mathcal{N}(A)$ for $j = r+1, \ldots, n$. Consequently, $\text{span}\{v_1, \ldots, v_r\} \subset \mathcal{N}(A)$. □

- Since
  $$A^T = \left(\sum_{i=1}^r \sigma_i u_i v_i^T\right)^T = \sum_{i=1}^r \left(\sigma_i u_i v_i^T\right)^T = \sum_{i=1}^r \sigma_i v_i u_i^T$$
  the same arguments as above can be used to show that
  $$\mathcal{R}(A^T) = \text{span}\{v_1, \ldots, v_r\}$$
  and
  $$\mathcal{N}(A^T) = \text{span}\{u_{r+1}, \ldots, u_m\}.$$

12.3 The SVD and Data Compression

A black and white image can be represented as a $m \times n$ matrix of pixels, where each entry a_{ij} of the matrix is the grey value of the pixel ij. (A color image can be stored stored as three $m \times n$ matrices, where the ijth entry of the first matrix is the ”red” value of the pixel ij, the ijth entry of the second matrix is the ”green” value of the pixel ij, and ijth entry of the second matrix is the ”blue” value of the pixel ij.) If stored this way, the representation of a matrix requires mn reals.

We can compute the singular values decomposition
$$A = U\Sigma V^T = \sum_{i=1}^{\min\{m,n\}} \sigma_i u_i v_i^T.$$

Recall that the singular values are ordered

$$\sigma_1 \geq \ldots \geq \sigma_{\min\{m,n\}} \geq 0.$$

If a singular value σ_{k+1} is much smaller than the first k singular values, then the contributions of $\sigma_{k+1} u_{k+1} v_{k+1}^T$, $\sigma_{k+2} u_{k+2} v_{k+2}^T$, ..., are smaller than the contributions of $\sigma_1 u_1 v_1^T, \ldots, \sigma_k u_k v_k^T$. Therefore we may approximate

$$A = U\Sigma V^T = \sum_{i=1}^{\min\{m,n\}} \sigma_i u_i v_i^T \approx \sum_{i=1}^{k} \sigma_i u_i v_i^T \stackrel{\text{def}}{=} A_k$$

Although A_k is also an $m \times n$ matrix, we only need to store k vectors u_i is size m, k vectors v_i is size n, and k scalars σ_i. Hence, the storage of A_k requires $(m+n+1)k$ reals. If k is small relative to m and n, $(m+n+1)k \ll mn$.

See `implot.m` for an example.

How well does the compressed image approximate the original one? We can provide a simple expression for the error $A - A_k$ in the Frobenius norm.

Recall that the Frobenius norm of an $m \times n$ matrix A with entries a_{ij} is given by

$$\|A\|_F = \left(\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}^2 \right)^{1/2}.$$

Before we analyze the error $\|A - A_k\|_F$, we prove a few properties of the Frobenius norm of a matrix.

- If $Q \in \mathbb{R}^{m \times m}$ is an orthogonal matrix, then $\|QA\|_F = \|A\|_F$.

  Proof: If $A \in \mathbb{R}^{m \times n}$ has columns $a_1, \ldots, a_n$, then the square of the Frobenius norm of A is equal to the sum of the squares of the 2-norms of the columns of A, i.e.,

  $$\|A\|_F = \left(\sum_{j=1}^{n} \|a_j\|_2^2 \right)^{1/2}.$$

  Since the columns of QA are $Qa_1, \ldots, Qa_n$ we have

  $$\|QA\|_F = \left(\sum_{j=1}^{n} \|Qa_j\|_2^2 \right)^{1/2}.$$

  Since Q is orthogonal, $\|Qv\|_2^2 = v^T Q^T Q v = v^T v = \|v\|_2^2$ for any vector $v \in \mathbb{R}^m$. Hence,

  $$\|QA\|_F = \left(\sum_{j=1}^{n} \|Qa_j\|_2^2 \right)^{1/2} = \left(\sum_{j=1}^{n} \|Qa_j\|_2^2 \right)^{1/2} = \|A\|_F.$$

  $\square$

- If $P \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\|AP\|_F = \|A\|_F$.

  Proof: Note that the Frobenius norm of the transpose of a matrix is equal to the Frobenius norm of the matrix, i.e., $\|M\|_F = \|M^T\|_F$ for any matrix M. Therefore,

$$\|AP\|_F = \|P^T A^T\|_F = \|A^T\|_F = \|A\|_F.$$

$\square$

- If $\sigma_1, \ldots, \sigma_{\min\{m,n\}}$ are the singular values of $A \in \mathbb{R}^{m \times n}$, then

$$\|A\|_F = \left(\sigma_1^2 + \ldots + \sigma_{\min\{m,n\}}^2\right)^{1/2}$$

  Proof: Let $A = U\Sigma V^T$ be the singular value decomposition of A. Then

$$\|A\|_F = \|U\Sigma V^T\|_F = \|\Sigma V^T\|_F = \|\Sigma\|_F = \left(\sigma_1^2 + \ldots + \sigma_{\min\{m,n\}}^2\right)^{1/2}.$$

$\square$

Now we can provide an estimate for the error $A - A_k$. We consider the case $m \geq n$, but $m < n$ can be handled the same way. We have

$$A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$

and

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T = U\Sigma_k V^T$$

where

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_k & & & \\ & & & \sigma_{k+1} & & \\ & & & & \ddots & \\ 0 & & & & & \sigma_n \\ \vdots & & & & & \vdots \\ 0 & & & & & 0 \end{pmatrix}, \quad \Sigma_k = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & & 0 \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

Hence,

$$\|A - A_k\|_F = \|U(\Sigma - \Sigma_k)V^T\|_F = \|\Sigma - \Sigma_k\|_F = \left(\sigma_{k+1}^2 + \ldots + \sigma_{\min\{m,n\}}^2\right)^{1/2}.$$

12.4 Linear Least Squares Problems and the SVD

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. We are interested in the solution of the linear least squares problem

$$\min_x \|Ax - b\|_2^2.$$

We assume that $m \geq n$, but everything can be easily adjusted for the case $m < n$.

Recall that for an orthogonal matrix Q, we have

$$\|Qz\|_2^2 = z^T Q^T Q z = z^T z = \|z\|_2^2$$

that is multiplication of a vector by an orthogonal matrix preserves the length of the vector. Now, using the orthogonality of U and V we find that

$$\begin{aligned}
\|Ax - b\|_2^2 &= \|U^T(Ax - b)\|_2^2 \\
&= \|U^T(AVV^Tx - b)\|_2^2 \\
&= \|\Sigma V^T x - U^T b\|_2^2 \\
&= \sum_{i=1}^r (\sigma_i z_i - u_i^T b)^2 + \sum_{i=r+1}^m (u_i^T b)^2,
\end{aligned}$$

where we have set $z = V^T x$. Thus,

$$\min_x \|Ax - b\|_2^2 = \min_{z=V^Tx} \sum_{i=1}^r (\sigma_i z_i - u_i^T b)^2 + \sum_{i=r+1}^m (u_i^T b)^2.$$

The solutions are obviously given by

$$\begin{aligned}
z_i &= \frac{u_i^T b}{\sigma_i}, & i &= 1, \ldots, r, \\
z_i &= \text{arbitrary}, & i &= r+1, \ldots, n.
\end{aligned}$$

and

$$x = Vz = \sum_{i=1}^n v_i z_i.$$

Moreover,

$$\min_x \|Ax - b\|_2^2 = \sum_{i=r+1}^m (u_i^T b)^2.$$

Since V is orthogonal, we find that

$$\|x\|_2 = \|VV^T x\|_2 = \|V^T x\|_2 = \|z\|_2$$

Hence, the minimum norm solution of the linear least squares problem is given by

$$x^\dagger = V z^\dagger,$$

where $z^\dagger \in \mathbb{R}^n$ is the vector with entries

$$\begin{aligned} z_i^\dagger &= \frac{u_i^T b}{\sigma_i}, \quad i = 1, \dots, r, \\ z_i^\dagger &= 0, \qquad i = r+1, \dots, n, \end{aligned}$$

i.e.

$$x^\dagger = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i. \tag{39}$$

Given the singular value decomposition $A = U\Sigma V^T = \sum_{i=1}^r \sigma_i u_i v_i^T$ of $A \in \mathbb{R}^{m\times n}$, the pseudo inverse of A is given by

$$A^\dagger = V \Sigma^\dagger U^T = \sum_{i=1}^{r} \sigma_i^{-1} v_i u_i^T,$$

where

$$\Sigma^\dagger = \begin{pmatrix} \sigma_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \sigma_r^{-1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & & 0 \end{pmatrix} \in \mathbb{R}^{n\times m} \qquad \text{if } m \le n$$

or

$$\Sigma^\dagger = \begin{pmatrix} \sigma_1^{-1} & & & & & 0 & \dots & 0 \\ & \ddots & & & & & & \\ & & \sigma_r^{-1} & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{n\times m} \qquad \text{if } m > n$$

The SVD approach to solving a linear least squares problem gives valuable insight into the solution (39) of the linear least squares problem. Suppose that the data b are

$$b = b^{\text{ex}} + \delta b,$$

where $b^{\rm ex}$ is the exact data and δb represents the measurement error. In this case (39) can be written as

$$x^\dagger = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i}\, v_i = \sum_{i=1}^{r} \left(\frac{u_i^T b^{\rm ex}}{\sigma_i} + \frac{u_i^T (\delta b)}{\sigma_i} \right)\, v_i. \tag{40}$$

We are really interested in

$$x^{\rm ex} = \sum_{i=1}^{r} \frac{u_i^T b^{\rm ex}}{\sigma_i}\, v_i$$

but because of the presence of measurement errors can only compute (40). The error

$$\|x^\dagger - x^{\rm ex}\|_2 = \Big\| \sum_{i=1}^{r} \frac{u_i^T (\delta b)}{\sigma_i}\, v_i \Big\|_2 = \sum_{i=1}^{r} \frac{(u_i^T (\delta b))^2}{\sigma_i^2}$$

depends not only on the size $\|\delta b\|_2$ of the measurement error, but also on the singlar values. If a singular value σ_i is small, then $u_i^T (\delta b)/\sigma_i$ could be large, even if $u_i^T (\delta b)$ is small. This shows that errors δb in the data can be magnified by small singular values σ_i.

Example 33

```
% Compute A
t = 10.^(0:-1:-10)';
A = [ ones(size(t))  t   t.^2  t.^3  t.^4  t.^5];

% compute SVD of A
[U,D,V] = svd(A);
sigma = diag(D);

% compute exact data
xex     = ones(6,1);
bex     = A*xex;

for i = 1:10
   % data perturbation
   deltab   = 10^(-i)*(0.5-rand(size(bex))).*bex;
   b        = bex+deltab;
   w        = U'*b;
   % solution of perturbed linear least squares problem
   x        = V * (w(1:6) ./ sigma);
   errx(i+1) = norm(x - xex);
   errb(i+1) = norm(deltab);
end

loglog(errb,errx,'*')
ylabel(' || x^{ex} - x ||_2 ')
xlabel(' || \delta b ||_2 ')
```

The singular values of A are given by

$$\sigma_1 \approx 3.4 \qquad \sigma_4 \approx 7.2 * 10^{-4}$$
$$\sigma_2 \approx 2.1 \qquad \sigma_5 \approx 6.6 * 10^{-7}$$
$$\sigma_3 \approx 8.2 * 10^{-2} \qquad \sigma_6 \approx 5.5 * 10^{-11}$$

The error $\|x^{\text{ex}} - x\|_2$ for different values of $\|\delta b\|_2$ are shown in Figure 16. Figure 16 shows that small perturbations δb in the measurements can lead to large errors in the solution x of the linear least squares problem if the singular values of A are small.

◇

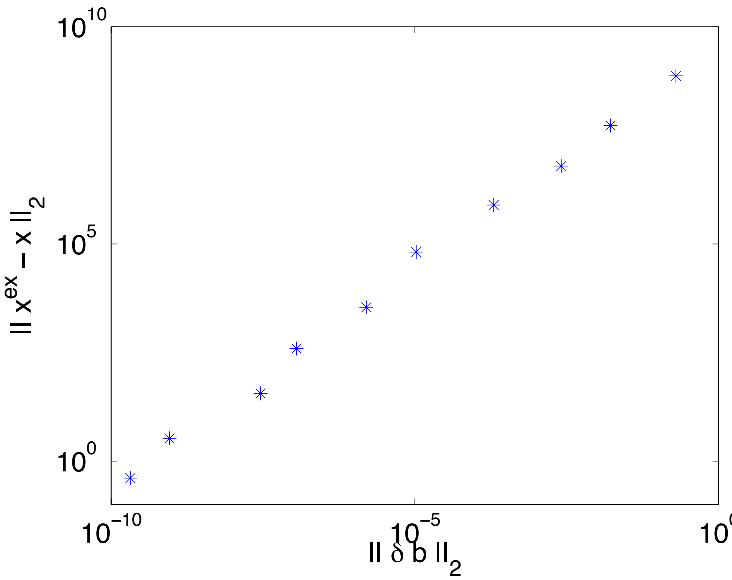


Figure 16: Dependency of the least squares solution on data perturbations

13 Fourier Series and the DFT

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π periodic function. i.e, $f(t) = f(t + 2\pi)$ for all t. The Fourier series of f is given by

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt), \tag{41a}$$

where

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(kt) dt, \qquad k = 0, 1, \ldots, \tag{41b}$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(kt) dt, \qquad k = 1, 2, \ldots. \tag{41c}$$

We will study the truncated Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kt) + b_k \sin(kt).$$

Lemma 34 *i. If we define the complex numbers*

$$\widetilde{c}_k = \frac{1}{2}(a_k - ib_k), \quad \widetilde{c}_{-k} = \overline{\widetilde{c}_k} = \frac{1}{2}(a_k + ib_k), \quad k = 0, \ldots, n, \tag{42}$$

then

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kt) + b_k \sin(kt) = \sum_{k=-n}^{n} \widetilde{c}_k e^{ikt}. \tag{43}$$

and

$$\widetilde{c}_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt, \quad k = 0, \ldots, n. \tag{44}$$

ii. If we define $N = 2n + 1$ *and the complex numbers*

$$\begin{aligned} \widetilde{c}_k &= \tfrac{1}{2}(a_k - ib_k), & k = 0, \ldots, n, \\ \widetilde{c}_{N-k} &= \overline{\widetilde{c}_k} = \tfrac{1}{2}(a_k + ib_k), & k = 1, \ldots, n, \end{aligned} \tag{45}$$

then

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kt) + b_k \sin(kt) = \sum_{k=0}^{N-1} \widetilde{c}_k e^{ikt}. \tag{46}$$

Now, we find approximations for the Fourier coefficients

$$\widetilde{c}_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{ikt}dt, \quad k = 0, \dots, n.$$

We sample f at $N = 2n+1$ points

$$t_j = j\frac{2\pi}{N}, \quad j = 0, \dots, N-1.$$

Let

$$f_j = f(t_j), \quad j = 0, \dots, N-1.$$

Using the fact that f is 2π periodic, we find that $f_N = f_0$. Applying the composite trapezoidal rule to approximate $\widetilde{c}_k$, we obtain

$$\widetilde{c}_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikx}dx \approx \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-ijk2\pi/N}.$$

We set

$$c_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-ijk2\pi/N} \quad k = 0, \dots, N. \tag{47}$$

Given (47) approximations of the Fourier coefficients (41b,c) are given by

$$\begin{aligned} a_k &\approx 2\Re(\widetilde{c}_k) \quad & k = 1, \dots, n = (N-1)/2, \\ b_k &\approx -2\Im(\widetilde{c}_k) \quad & k = 1, \dots, n = (N-1)/2, \end{aligned} \tag{48}$$

see Lemma 34.

We note that the approximate Fourier coefficients defined by (47) maintain the property that $\widetilde{c}_{N-k} = \overline{\widetilde{c}_k}, \quad k = 0, \dots, N-1$ (cf. Lemma 34).

Lemma 35 *Let $f_j \in \mathbb{R}$, $j = 0, \dots, N-1$ be given. If c_k, $k = 1, \dots N$, are defined by (47), then*

$$c_{N-k} = \overline{c_k}, \quad k = 0, \dots, N-1.$$

Proof:

$$c_{N-k} = \sum_{j=0}^{N-1} f_j e^{-ij(N-k)2\pi/N} = \sum_{j=0}^{N-1} f_j \underbrace{e^{ijk2\pi/N}}_{=\overline{e^{-ijk2\pi/N}}} \underbrace{e^{-ij2\pi}}_{=1} = \overline{c_k}.$$

□

13.1 The Discrete Fourier Transform (DFT)

Given N real numbers

$$f_0, \dots, f_{N-1}$$

we want to compute

$$c_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-ijk2\pi/N}, \quad k = 0, \dots, N-1. \tag{49}$$

The mapping of f_j, $j = 0, \dots, N-1$, into $c_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-ijk2\pi/N}$, $k = 0, \dots, N-1$, is called the *Discrete Fourier Transform* (DFT).

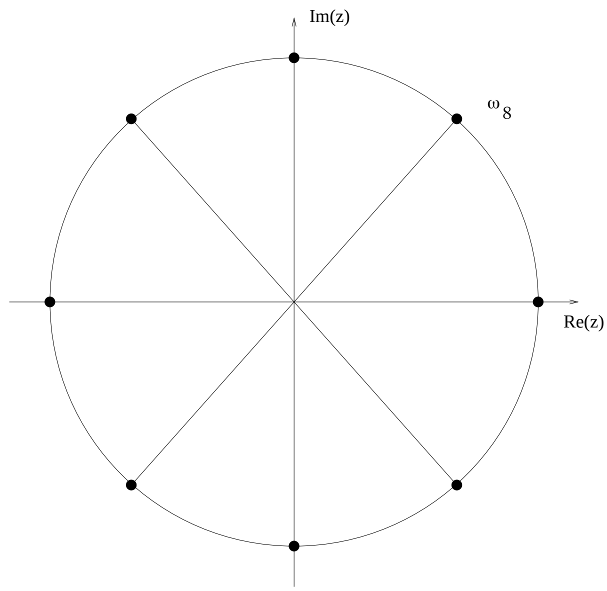


Figure 17: 8th root of unity

Let

$$\omega_N = e^{i2\pi/N} \in \mathbb{C}$$

be the Nth root of unity. With this notation we can write

$$\begin{aligned} c_k &= \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-ijk2\pi/N}, \\ &= \frac{1}{N}\sum_{j=0}^{N-1} f_j \omega_N^{-jk}, \\ &= \frac{1}{N}(1, \omega_N^{-k}, \omega_N^{-2k}, \dots, \omega_N^{-(N-1)k}) \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}, \quad k = 0, \dots, n-1. \end{aligned}$$

Thus,

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{pmatrix} = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \dots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & \dots & \omega_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_N^{-(N-1)} & \omega_N^{-2(N-1)} & \dots & \omega_N^{-(N-1)^2} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}. \tag{50}$$

The matrix

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N^{1} & \omega_N^{2} & \dots & \omega_N^{N-1} \\ 1 & \omega_N^{2} & \omega_N^{4} & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)^2} \end{pmatrix} \tag{51}$$

is called the *Fourier matrix*. Since

$$\omega_N^{-l} = e^{-il2\pi/N} = \overline{e^{il2\pi/N}} = \overline{\omega_N}^{l},$$

the matrix in (50) is equal to $\overline{F_N}$, the conjugate complex of F_N. We can write (50) as

$$\begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix} = \frac{1}{N}\overline{F_N} \begin{pmatrix} f_0 \\ \vdots \\ f_{N-1} \end{pmatrix}. \tag{52}$$

Thus, the application of the DFT (49) to a vector f is a matrix-vector multiplication $c = \overline{F_N} y$.

Lemma 36 *Let F_N be Fourier matrix given by (51). It holds that*

$$\overline{F_N} F_N = NI,$$

i.e.,

$$F_N^{-1} = \frac{1}{N}\overline{F_N}, \quad (\overline{F_N})^{-1} = \frac{1}{N} F_N.$$

Proof: For $l, k \in \{0, \dots, N-1\}$,

$$\begin{aligned}
(\overline{F_N} F_N)_{lk} &= \sum_{j=0}^{N-1} \omega_N^{-lj} \omega_N^{jk} \\
&= \sum_{j=0}^{N-1} e^{-ilj2\pi/N} e^{ikj2\pi/N} \\
&= \sum_{j=0}^{N-1} e^{i(k-l)j2\pi/N} \\
&= \sum_{j=0}^{N-1} \left(e^{i(k-l)2\pi/N}\right)^j \\
&= \begin{cases} N & \text{if } k = l, \\ \frac{1-(e^{i(k-l)2\pi/N})^N}{1-e^{i(k-l)2\pi/N}} = 0 & \text{if } k \neq l. \end{cases}
\end{aligned}$$

□

The *inverse discrete Fourier transform* maps c into f and is given by

$$\begin{pmatrix} f_0 \\ \vdots \\ f_{N-1} \end{pmatrix} = F_N \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix} \tag{53}$$

or, equivalently,

$$f_k = \sum_{j=0}^{N-1} c_j e^{ijk2\pi/N}, \quad k = 0, \dots, N-1. \tag{54}$$

In a straight forward implementation, the computations of matrix-vector products

$$\overline{F_N}\, c \quad \text{and} \quad \frac{1}{N} F_N\, f$$

require N^2 multiplications. However, using the special structure of F_N, these multiplications can be done much more efficiently using the so-called *Fast Fourier Transform* (FFT). This will be discussed in Section **??**. Matlab's `fft(f)` uses the fast Fourier transform to compute $\overline{F_N} f$ and Matlab's `ifft(c)` uses the fast Fourier transform to compute $(1/N)F_N c$. (*Warning: Note the difference in scale by N:* `fft(f)` *computes $\overline{F_N} f$, not $\frac{1}{N}\overline{F_N} f$ and* `ifft(f)` *computes $(1/N)F_N c$, not $F_N c$.*)

13.2 Application: Modal Analysis

We consider a mass-spring system with three masses and four springs. The displacement of the ith mass at time t is denoted by $x_i(t)$. It can be shown that the displacement obey the following second order differential equation

$$\begin{aligned} M\ddot{x}(t) + Kx(t) &= 0, \\ \dot{x}(0) &= x^{(1)}, \\ x(0) &= x^{(0)}, \end{aligned} \tag{55}$$

where

$$K = A^T C A,$$

with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{pmatrix},$$

and

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$

The stiffnesses c_i are positive numbers and we assume that the mass matrix $M = I$.

The solution of (55) is given by

$$x(t) = \sum_{k=1}^{3} [a_k \cos(\omega_k t) + b_k \sin(\omega_k t)] v_k, \tag{56}$$

where

$$K = V \Lambda V^T$$

is the spectral decomposition of K, v_k is the kth column of V, $\omega_k = \sqrt{\lambda_k}$,

$$a = V^T x^{(0)}, \quad \tilde{b} = V^T x^{(1)},$$

and $b_k = \tilde{b}_k / \omega_k$, $k = 1, 2, 3$.

For $c_1 = \ldots = c_4 = 1000$, the frequencies $\omega_1, \omega_2, \omega_3$ are given by

$$\omega_1 = 24.2030, \quad \omega_2 = 44.7214, \quad \omega_3 = 58.4313.$$

Now, suppose we do not know the frequencies $\omega_1, \omega_2, \omega_3$. We want to determine the frequencies from measurements of the displacement of the first mass.

We sample the displacement of the first mass at times

$$t_j = (j-1)\Delta t, \quad j = 1, \ldots, N$$

where

$$N = 2^p, \ p = 11.$$

This gives a vector $\hat{x} \in \mathbb{R}^{2^p}$. See Figure 18.

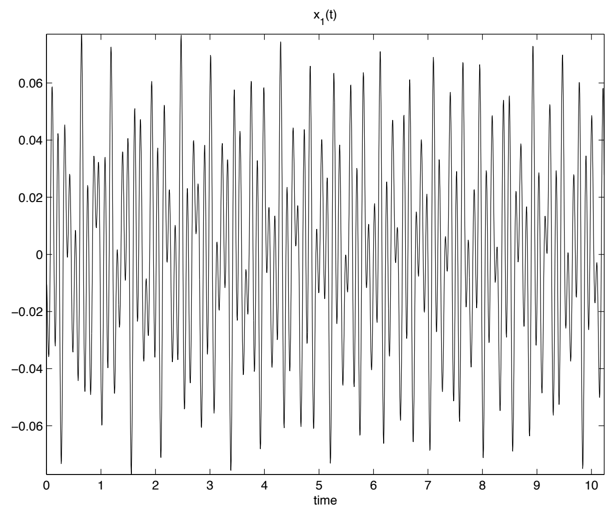


Figure 18: Measurements

Next, we take the Fast Fourier Transform of $\hat{x}$ to get the Fourier coefficients c_k. The absolute values $|c_k|$ are shown in Figure 19. The first $2^{11}/2 = 1024$ values in Figure 19 correspond to the the Fourier coefficients $c_0, \ldots, c_{1024}$, the last $2^{11}/2 = 1024$ values correspond to the the Fourier coefficients $c_{-1024}, \ldots, c_{-1}$,

Next, we want to match the coefficients c_k to the frequencies. By definition of the DFT:

$$\hat{x}_j = x_1((j-1)\Delta t) = \sum_{k=0}^{N-1} c_k e^{ikj\frac{2\pi}{N}} = \sum_{k=0}^{N-1} c_k e^{i(j\Delta t)\frac{k2\pi}{N\Delta t}}.$$

On the other hand, (56) shows that

$$x_1(t) = \sum_{k=1}^{3} a_k v_{1,k} \cos(\omega_k t) + b_k v_{1,k} \sin(\omega_k t) = \sum_{k=1}^{3} \tilde{c}_k e^{i\omega_k t} + \overline{\tilde{c}_k} e^{-i\omega_k t}$$

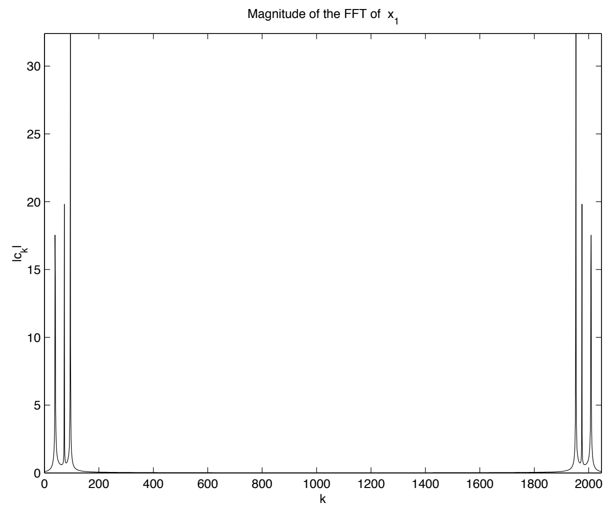


Figure 19: The absolute values $|c_k|$ of the Fourier coefficients

for some $\tilde{c}_k$. In particular,

$$
\begin{aligned}
x_1(t_j) &= \sum_{k=1}^{3} \tilde{c}_k e^{i\omega_k t_j} + \overline{\tilde{c}_k} e^{-i\omega_k t_j} \\
&\approx \sum_{k=0}^{N-1} c_k e^{i(j\Delta t)\frac{k2\pi}{N\Delta t}}.
\end{aligned}
$$

To identify the frequencies $\omega_1, \omega_2, \omega_3$ we plot the absolute values of the computed Fourier coefficients $c_0, \ldots, c_{1024}$. againts against $\frac{k2\pi}{N\Delta t}$. See Figure 20. In the bottom plot of Figure 20 we focus on the frequencies ω between 0 and 70. This plot shows three peaks around $\omega_1 = 24.2030, \omega_2 = 44.7214, \omega_3 = 58.4313$.

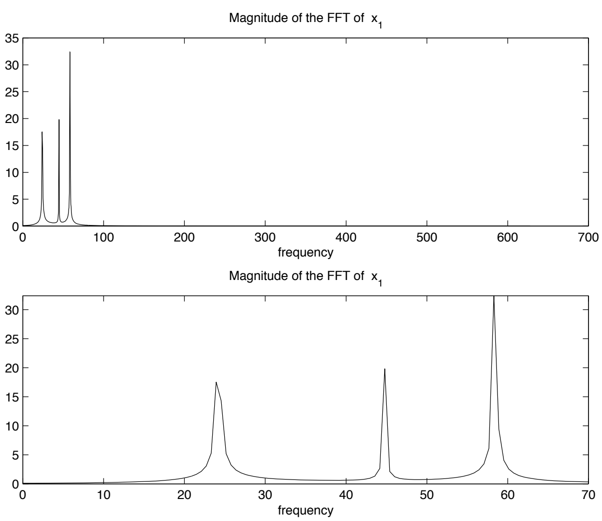


Figure 20: Results of the modal analysis

14 Laplace Transform

Example 37

(a) We compute the Laplace transform of e^t.

$$\begin{aligned}\mathcal{L}(e^t)(s) &= \int_0^\infty e^t e^{-st} dt = \int_0^\infty e^{(1-s)t} dt = \lim_{R\to\infty} \int_0^R e^{(1-s)t} dt \\ &= \lim_{R\to\infty} \frac{1}{1-s} e^{(1-s)t}\Big|_0^R = \lim_{R\to\infty} \frac{1}{1-s}\left(e^{(1-s)R} - 1\right) \\ &= \frac{1}{s-1}\end{aligned}$$

if $\mathrm{Re}(1-s) < 0$, i.e, $\mathrm{Re}(s) > 1$.

(b) We compute the Laplace transform of te^{-t}.

$$\begin{aligned}\mathcal{L}(te^{-t})(s) &= \int_0^\infty te^{-t}e^{-st} dt = \int_0^\infty te^{(-1-s)t} dt = \lim_{R\to\infty} \int_0^R te^{(-1-s)t} dt \\ &= \lim_{R\to\infty} \left(\frac{-1}{1+s} te^{(-1-s)t}\Big|_0^R - \int_0^R \frac{-1}{1+s} e^{(-1-s)t} dt \right) \\ &= \lim_{R\to\infty} \left(\frac{-1}{1+s} te^{(-1-s)t}\Big|_0^R - \frac{1}{(1+s)^2} e^{(-1-s)t}\Big|_0^R \right) \\ &= (0-0) - (0 - 1/(1+s)^2) = 1/(1+s)^2\end{aligned}$$

if $\mathrm{Re}(-1-s) < 0$, i.e, $\mathrm{Re}(s) > -1$.

$\diamond$

15 Complex Functions

Every complex function $f : \mathbb{C} \to \mathbb{C}$ with argument $z = x + iy$ can be written as

$$f(x + iy) = u(x, y) + iv(x, y).$$

Definition 38 *A complex function $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z_0 if the limit $\lim_{z\to 0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. In this case the derivative of f at z_0 is given by*

$$f'(z_0) = \lim_{z\to 0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Example 39

1. Let $f(z) = \text{Re}(z) = x$ and $z_0 = x_0+iy_0$. We consider the difference quotient with $z = z_0+h$ and the difference quotient with $z = z_0 + ih$. We find

   $$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{z_0 + h - z_0} = \lim_{h\to 0} \frac{x_0 + h - x_0}{h} = 1$$

   and

   $$\lim_{h\to 0} \frac{f(z_0 + ih) - f(z_0)}{z_0 + ih - z_0} = \lim_{h\to 0} \frac{x_0 - x_0}{ih} = 0.$$

   Since the two limits do not coincide, $\lim_{z\to 0} \frac{f(z)-f(z_0)}{z-z_0}$ does not exist and, consequently, the function $f(z) = \text{Re}(z)$ is not differentiable at any $z \in \mathbb{C}$.

$\diamond$

The previous example hints at conditions that need to be satisfied when f is differentiable at z_0. In this case for every sequence $\{z_n\}$ of points converging to z_0 the limits $\lim_{n\to\infty} \frac{f(z_n)-f(z_0)}{z_n-z_0}$ must coincide. In particular if we look at sequences of points converging to z_0 along the real axis and sequences of points converging to z_0 along the imaginary axis, their limits must coincide. That is, if the function f is differentiable at the point z_0, then

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{z_0 + h - z_0} = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

and

$$\lim_{h\to 0} \frac{f(z_0 + ih) - f(z_0)}{z_0 + ih - z_0} = \lim_{h\to 0} \frac{f(z_0 + ih) - f(z_0)}{ih} = \lim_{h\to 0} -i \frac{f(z_0 + ih) - f(z_0)}{h} = f'(z_0).$$

If we insert $f(x+iy) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, we obtain

$$
\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \\
&= \lim_{h\to 0} \frac{u(x_0+h,y_0) - u(x_0,y_0)}{h} + i\,\frac{v(x_0+h,y_0) - v(x_0,y_0)}{h} \\
&= \frac{\partial}{\partial x} u(x_0,y_0) + i\,\frac{\partial}{\partial x} v(x_0,y_0)
\end{aligned}
$$

and

$$
\begin{aligned}
f'(z_0) &= \lim_{h\to 0} -i\,\frac{f(z_0+ih) - f(z_0)}{h} \\
&= \lim_{h\to 0} -i\,\frac{u(x_0,y_0+h) - u(x_0,y_0)}{h} + \frac{v(x_0,y_0+h) - v(x_0,y_0)}{h} \\
&= -i\,\frac{\partial}{\partial y} u(x_0,y_0) + \frac{\partial}{\partial y} v(x_0,y_0).
\end{aligned}
$$

Comparing the real and imaginary parts, we conclude that

$$\frac{\partial}{\partial x} u(x,y) = \frac{\partial}{\partial y} v(x,y), \tag{57a}$$

$$\frac{\partial}{\partial y} u(x,y) = -\frac{\partial}{\partial x} v(x,y) \tag{57b}$$

hold at $z_0 = x_0 + iy_0$. The equations (57) are the *Cauchy-Riemann equations*. Thus we have proven the following result.

Theorem 40 *If the function $f(x+iy) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$, then the Cauchy-Riemann equations (57) hold at $z_0 = x_0 + iy_0$.*

Note that the Cauchy-Riemann equations are a necessary condition for differentiability. If a function is differentiable at a point, then the Cauchy-Riemann equations must be satisfied at that point. Conversely, if the Cauchy-Riemann equations are not satisfied at a point, then the function is not differentiable at that point. However, just because the Cauchy-Riemann equations hold at a point, does not allow one to conclude that the function is differentiable at that point. We illustrate this using some examples.

Example 41

1. In Example 39 we have already shown that the function $f(z) = \mathrm{Re}(z) = x$ is nowhere differentiable. Here we derive the same result using the Cauchy-Riemann equations. The

partial derivatives of the real and imaginary part of f are

$$\frac{\partial}{\partial x}u(x,y) = 1, \qquad \frac{\partial}{\partial y}v(x,y) = 0,$$
$$\frac{\partial}{\partial y}u(x,y) = 0, \qquad \frac{\partial}{\partial x}v(x,y) = 0.$$

Hence the Cauchy-Riemann equations are nowhere satisfied and the function $f(z) = \text{Re}(z)$ is nowhere differentiable.

2. Consider $f(z) = (x^2 + y) + i(y^2 - x)$. We have

$$\frac{\partial}{\partial x}u(x,y) = 2x, \qquad \frac{\partial}{\partial y}v(x,y) = 2y,$$
$$\frac{\partial}{\partial y}u(x,y) = 1, \qquad \frac{\partial}{\partial x}v(x,y) = -1.$$

Hence the Cauchy-Riemann equations are only satisfied at $z = x + ix$. The function $f(z) = (x^2 + y) + i(y^2 - x)$ is not differentiable at any point $z \notin \{x + ix \ : \ x \in \mathbb{R}\}$.

What about the differentiability at points $z \in \{x + ix \ : \ x \in \mathbb{R}\}$? The Cauchy-Riemann equations are only a necessary condition for differentiability. Thus, we *cannot* conclude that because the Cauchy-Riemann equations are satisfied the function is differentiable. In particular, Theorem 40 does not tell us anything about the differentiability at points $z \in \{x + ix \ : \ x \in \mathbb{R}\}$.

To check the differentiability of f at $z_0 = x_0 + ix_0$ we have to consider difference quotients with $z = z_0 + h + ik$, i.e., we have to consider

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{f(z_0+h+ik)-f(z_0)}{z_0+h+ik-z_0} \\
&= \lim_{(h,k)\to(0,0)} \frac{((x_0+h)^2+(x_0+k))+i((x_0+k)^2-(x_0+h))-[(x_0^2+x_0)+i(x_0^2-x_0)]}{h+ik} \\
&= \lim_{(h,k)\to(0,0)} \frac{2x_0h+h^2+k+i(2x_0k+k^2-h)}{h+ik} \\
&= \lim_{(h,k)\to(0,0)} \frac{2x_0(h+ik)+(h^2+k^2)+(k-ih)}{h+ik} \\
&= \lim_{(h,k)\to(0,0)} \frac{2x_0(h+ik)+(h^2+k^2)-i(h+ik)}{h+ik} \\
&= 2x_0 - i + \lim_{(h,k)\to(0,0)} \frac{h^2+k^2}{h+ik} \\
&= 2x_0 - i + \lim_{(h,k)\to(0,0)} \frac{(h^2+k^2)(h-ik)}{h^2+k^2} \\
&= 2x_0 - i + 0.
\end{aligned}$$

Thus, the function $f(z) = (x^2+y)+i(y^2-x)$ is differentiable at $z_0 = x_0 + ix_0$ and the derivative is

$$f'(z_0) = 2x_0 - i, \quad \text{for } z_0 = x_0 + ix_0$$

3. Consider

$$f(z) = \begin{cases} \dfrac{x^{4/3}y^{5/3} + ix^{5/3}y^{4/3}}{x^2+y^2}, & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

At $z = 0$, we have

$$\frac{\partial}{\partial x}u(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h-0} = \lim_{h\to 0} \frac{1}{h}\frac{h^{4/3}0^{5/3}}{h^2} = 0$$

and, similarly,

$$\frac{\partial}{\partial y}u(0,0) = \frac{\partial}{\partial x}v(0,0) = \frac{\partial}{\partial y}v(0,0) = 0.$$

The Cauchy-Riemann equations are satisfied at $z_0 = 0$, however, the function is not differentiable at $z_0 = 0$. To see this consider the difference quotient with $z = z_0 + h$ and the difference quotient with $z = z_0 + h + ih$. We find that

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h-0} = \lim_{h\to 0} \frac{1}{h}\frac{h^{4/3}0^{5/3} + ih^{5/3}0^{4/3}}{h^2+0^2} = 0$$

and

$$\lim_{h\to 0}\frac{f(z_0+h+ih)-f(z_0)}{h+ih} = \lim_{h\to 0}\frac{1}{(1+i)h}\frac{h^{4/3}h^{5/3}+ih^{5/3}h^{4/3}}{h^2+h^2}$$
$$= \lim_{h\to 0}\frac{1}{1+i}\frac{h^3+ih^3}{2h^3} = \frac{1}{2}.$$

Since the two limits do not coincide, f is not differentiable at $z_0 = 0$.

◇

A sufficient condition for the differentiability of a function is given in the following theorem.

Theorem 42 *Let $f(x+iy) = u(x,y)+iv(x,y)$ be defined on some open set G including the point $z_0 = x_0+iy_0$. If the partial derivatives of u and v exist in G, are continuous at (x_0, y_0), and satisfy the Cauchy-Riemann equations (57) at z_0, then f is differentiable at z_0.*

Example 43

1. Consider $f(z) = (x^2 + y) + i(y^2 - x)$. In Example 41 we have established that f is differentiable at any $z = x + ix$. We will argue that f is differentiable at any $z = x + ix$ using Theorem 42. We have

$$\frac{\partial}{\partial x}u(x,y) = 2x, \qquad \frac{\partial}{\partial y}v(x,y) = 2y,$$
$$\frac{\partial}{\partial y}u(x,y) = 1 \qquad \frac{\partial}{\partial x}v(x,y) = -1.$$

   The partial derivatives of u and v exist in $\mathbb{C}$, are continuous at any point in $\mathbb{C}$. Since the Cauchy-Riemann equations are satisfied at $z = x + ix$, Theorem 42 implies that f is differentiable at any $z = x + ix$.

2. Consider

$$f(z) = \begin{cases} \dfrac{x^{4/3}y^{5/3} + ix^{5/3}y^{4/3}}{x^2+y^2}, & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

   In Example 41 we have shown that f is not differentiable at $z_0 = 0$. Compute the partial derivatives of u and v and show that they are not continuous at $(x_0, y_0) = (0, 0)$.

◇

The following well-known identities for the differentiation of real functions also hold for the differentiation of complex functions.

$$
\begin{aligned}
(f+g)'(z_0) &= f'(z_0) + g'(z_0), \\
(af)'(z_0) &= af'(z_0) \qquad \text{for a constant } a \in \mathbb{C}, \\
(fg)'(z_0) &= f'(z_0)g(z_0) + f(z_0)g'(z_0), \\
\left(\frac{f}{g}\right)'(z_0) &= \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.
\end{aligned}
$$

16 Partial Fraction Expansions

The solution of dynamical systems

$$\begin{aligned} x'(t) &= Ax(t) + h(t), \qquad t > 0, \\ x(0) &= x_0. \end{aligned}$$

using the Laplace transform leads to

$$(\mathcal{L}x)(s) = (sI - A)^{-1}\Big(x_0 + (\mathcal{L}h)(s)\Big).$$

The inverse $(sI - A)^{-1}$ can be computed using Gaussian elimination and executing Gaussian elimination shows that $(sI - A)^{-1}$ is a matrix with entries given by rational functions of the type

$$\frac{f_{ij}(s)}{\det(sI - A)},$$

where $p_A(s) = \det(sI - A)$ is the characteristic polynomial of degree n of A and $f_{ij}(s)$ is a polynomial of degree less than n.

Example 44

- Consider

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2 - 4s + 3}\begin{pmatrix} s-2 & -1 \\ -1 & s-2 \end{pmatrix}.$$

  The characteristic polynomial of A is $p_A(s) = \det(sI - A) = s^2 - 4s + 3 = (s-1)(s-3)$. Consequently, the eigenvalues are given by $\lambda_1 = 3$ and $\lambda_2 = 1$.

- Consider

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

  This is matrix in the reaction kinetics ODE (see Example 32) with $k_1 = k_2 = 1$.

$$(sI - A)^{-1} = \frac{1}{s(1+s)^2}\begin{pmatrix} s(s+1) & 0 & 0 \\ s & s(s+1) & 0 \\ 1 & (s+1) & (s+1)^2 \end{pmatrix}.$$

  The characteristic polynomial of A is $p_A(s) = \det(sI - A) = (s+1)^2 s$ and the eigenvalues of A are given by $\lambda_1 = -1$ (with algebraic multiplicity 2) and $\lambda_2 = 0$.

$\diamond$

To compute the inverse Laplace transform we have to compute complex integrals with integrands of the type $f_{ij}(s)/\det(sI-A)$ or $f_{ij}(s)h_j(s)/\det(sI-A)$, where h_j is a component of the right hand side function h in the dynamical system. To compute these complex integrals, it will be beneficial to expand the rational functions $f_{ij}(s)/\det(sI-A)$ in a partial fraction expansion. We first consider the case in which all roots of the denominator polynomial are simple.

Let $g:\mathbb{C}\to\mathbb{C}$ be a complex polynomial of order m with roots $\lambda_1,\ldots,\lambda_n$ of multiplicity one, i.e.,

$$g(z) = c_n(z-\lambda_1)\ldots(z-\lambda_n)$$

Furthermore, let $f:\mathbb{C}\to\mathbb{C}$ be a complex polynomial of order at most $n-1$. The rational function f/g has a partial fraction expansion of the form

$$\frac{f(z)}{g(z)} = \frac{q_1}{z-\lambda_1} + \frac{q_2}{z-\lambda_2} + \ldots + \frac{q_n}{z-\lambda_n}.$$

To compute the coefficients $q_1,\ldots,q_n$ we proceed as follows. Multiply f/g by $z-\lambda_j$. This gives

$$(z-\lambda_j)\frac{f(z)}{g(z)} = (z-\lambda_j)\frac{q_1}{z-\lambda_1} + \ldots + q_j + \ldots + (z-\lambda_j)\frac{q_n}{z-\lambda_n}.$$

If we take the limit

$$\lim_{z\to\lambda_j}(z-\lambda_j)\frac{f(z)}{g(z)} = \lim_{z\to\lambda_j}(z-\lambda_j)\frac{q_1}{z-\lambda_1} + \ldots + q_j + \ldots + (z-\lambda_j)\frac{q_n}{z-\lambda_n} = q_j,$$

we obtain the jth coefficient.

Example 45 Consider

$$\frac{1}{(z-i)(z-1)} = \frac{q_1}{z-i} + \frac{q_2}{z-1}.$$

Multiply by $z-i$:

$$\frac{1}{z-1} = \frac{z-i}{(z-i)(z-1)} = (z-i)\frac{q_1}{z-i} + (z-i)\frac{q_2}{z-1} = q_1 + (z-i)\frac{q_2}{z-1}.$$

Evaluate at $z=i$:

$$\frac{-i-1}{2} = \frac{(-i-1)}{(i-1)(-i-1)} = \frac{1}{i-1} = q_1.$$

Multiply by $z-1$:

$$\frac{1}{z-i} = \frac{z-1}{(z-i)(z-1)} = (z-1)\frac{q_1}{z-i} + (z-1)\frac{q_2}{z-1} = (z-1)\frac{q_1}{z-i} + q_2.$$

Evaluate at $z = 1$:
$$\frac{1+i}{2} = \frac{1+i)}{(1-i)(1+i)} = \frac{1}{1-i} = q_2.$$
Thus, the partial fraction expansion of $1/((z-i)(z-1))$ is
$$\frac{1}{(z-i)(z-1)} = \frac{-\frac{1}{2}-\frac{1}{2}i}{z-i} + \frac{\frac{1}{2}+\frac{1}{2}i}{z-1}.$$

◇

Example 46 Consider
$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
The inverse $(sI-A)^{-1}$ can be computed using Gaussian elimination or using Matlab.

```
>> syms s
>> A = [ 2 -1; -1 2];
>> inv( s*eye(2)-A)

ans =
[ (s - 2)/(s^2 - 4*s + 3),       -1/(s^2 - 4*s + 3)]
[       -1/(s^2 - 4*s + 3), (s - 2)/(s^2 - 4*s + 3)]
>>
```

$$(sI-A)^{-1} = \frac{1}{s^2-4s+3}\begin{pmatrix} s-2 & -1 \\ -1 & s-2 \end{pmatrix}.$$
The characteristic polynomial of A is $p_A(s) = \det(sI-A) = s^2-4s+3 = (s-1)(s-3)$ and the eigenvalues are given by $\lambda_1 = 3$ and $\lambda_2 = 1$.

We now compute partial fraction expansions of the entries of $(sI-A)^{-1}$.

- $$\frac{s-2}{(s-3)(s-1)} = \frac{q_1}{s-3} + \frac{q_2}{s-1}.$$
  Multiply by $s-3$ to obtain $(s-2)/(s-1) = q_1 + (s-3)q_2/(s-1)$. Evaluate at $s=3$ to get $q_1 = 1/2$.
  Multiply by $s-1$ to obtain $(s-2)/(s-3) = (s-1)q_1/(s-3) + q_2$. Evaluate at $s=1$ to get $q_2 = 1/2$.

- $$\frac{-1}{(s-3)(s-1)} = \frac{q_1}{s-3} + \frac{q_2}{s-1}.$$

Multiply by $s-3$ to obtain $-1/(s-1) = q_1 + (s-3)q_2/(s-1)$. Evaluate at $s=3$ to get $q_1 = -1/2$.
Multiply by $s-1$ to obtain $-1/(s-3) = (s-1)q_1/(s-3) + q_2$. Evaluate at $s=1$ to get $q_2 = 1/2$.

Insert partial fraction expansion into $(sI-A)^{-1}$

$$\frac{1}{s^2-4s+3}\begin{pmatrix} s-2 & -1 \\ -1 & s-2 \end{pmatrix} = \frac{1}{s-3}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{s-1}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

We can compute this partial fraction expansion also using Matlab. For example the partial fraction expansion of

$$\frac{s-2}{s^2-4s+3}$$

can be computed using

```
>> [r,p,k]=residue([0,1,-2],[1,-4,3])
r =
    0.5000
    0.5000
p =
     3
     1
k =
     []
>>
```

p contains the roots of s^2-4s+3 and r contains the scalars q_k corresponding to the roots λ_k. Thus

$$\frac{s-2}{s^2-4s+3} = \frac{1/2}{s-3} + \frac{1/2}{s-1},$$

which is of course identical to what we have computed by hand. $\diamond$

Next we consider the case in which the denominator polynomial has multiple roots. Let $g : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree n with roots $\lambda_1, \dots, \lambda_k$ of multiplicities $m_1, \dots, m_k$, i.e.,

$$g(z) = c_n(z-\lambda_1)^{m_1} \dots (z-\lambda_k)^{m_k}$$

and $\sum_{j=1}^k m_j = n$. Furthermore, let $f : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree at most $n-1$. The rational function $q(z) = f(z)/g(z)$ can be written in the form

$$q(z) = \sum_{j=1}^{k} \sum_{\ell=1}^{m_j} \frac{q_{j,\ell}}{(z-\lambda_j)^\ell}.$$

We first consider an example on how to compute the coefficients $q_{j,\ell}$. Let

$$\frac{s^2}{(s+1)^3} = \frac{q_{1,1}}{s+1} + \frac{q_{1,2}}{(s+1)^2} + \frac{q_{1,3}}{(s+1)^3}.$$

We multiply by $(s+1)^3$ and take the limit $s \to -1$. This gives

$$1 = \lim_{s\to -1} s^2 = \lim_{s\to -1} q_{1,1}(s+1)^2 + q_{1,2}(s+1) + q_{1,3} = q_{1,3}.$$

To compute the other coefficients, we note that $s^2 = q_{1,1}(s+1)^2 + q_{1,2}(s+1) + q_{1,3}$ must hold for all s and therefore

$$\frac{d}{ds}s^2 = \frac{d}{ds}\Big(q_{1,1}(s+1)^2 + q_{1,2}(s+1) + q_{1,3}\Big)$$

and

$$\frac{d^2}{ds^2}s^2 = \frac{d^2}{ds^2}\Big(q_{1,1}(s+1)^2 + q_{1,2}(s+1) + q_{1,3}\Big)$$

Thus,

$$-2 = \lim_{s\to -1} 2s = \lim_{s\to -1}\frac{d}{ds}s^2 = \lim_{s\to -1}\frac{d}{ds}(q_{1,1}(s+1)^2 + q_{1,2}(s+1) + q_{1,3}) = \lim_{s\to -1} 2q_{1,1}(s+1) + q_{1,2} = q_{1,2}$$

and

$$2 = \lim_{s\to -1} 2 = \lim_{s\to -1}\frac{d^2}{ds^2}s^2 = \lim_{s\to -1}\frac{d^2}{ds^2}\Big(q_{1,1}(s+1)^2 + q_{1,2}(s+1) + q_{1,3}\Big) = \lim_{s\to -1} 2q_{1,1} = 2q_{1,1}.$$

The partial fraction expansion is

$$\frac{s^2}{(s+1)^3} = \frac{1}{s+1} + \frac{-2}{(s+1)^2} + \frac{1}{(s+1)^3}.$$

More generally, the rational function $q(z) = f(z)/g(z)$ can be written in the form

$$q(z) = \sum_{j=1}^{k}\sum_{\ell=1}^{m_j}\frac{q_{j,\ell}}{(z-\lambda_j)^\ell}.$$

where

$$q_{j,\ell} = \lim_{z\to\lambda_j}\frac{1}{(m_j-\ell)!}\frac{d^{m_j-\ell}}{dz^{m_j-\ell}}\left[(z-\lambda_j)^{m_j}q(z)\right].$$

Example 47 Consider

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

This is matrix in the reaction kinetics ODE (see Example 32) with $k_1 = k_2 = 1$. We have

$$(sI - A)^{-1} = \frac{1}{s(1+s)^2} \begin{pmatrix} s(s+1) & 0 & 0 \\ s & s(s+1) & 0 \\ 1 & (s+1) & (s+1)^2 \end{pmatrix}.$$

The characteristic polynomial of A is $p_A(s) = \det(sI - A) = s+1)^2 s$ and the eigenvalues of A are given by $\lambda_1 = -1$ (with algebraic multiplicity 2) and $\lambda_2 = 0$.

We compute the partial fractions of the entries of $(sI - A)^{-1}$. For

$$\frac{1}{s(s+1)^2} = \frac{q_1}{s} + \frac{q_{2,1}}{s+1} + \frac{q_{2,2}}{(s+1)^2}.$$

we find

$$q_1 = \frac{1}{(s+1)^2}|_{s=0} = 1.$$
$$q_{2,1} = \left(\frac{1}{s}\right)' |_{s=-1} = \frac{-1}{s^2}|_{s=-1} = -1.$$
$$q_{2,2} = \frac{1}{s}|_{s=-1} = -1.$$

Check:

$$\frac{1}{s} + \frac{-1}{s+1} + \frac{-1}{(s+1)^2} = \frac{(s+1)^2}{s(s+1)^2} - \frac{s(s+1)}{s(s+1)^2} - \frac{s}{s(s+1)^2} = \frac{(s^2+2s+1)-(s^2+s)-s}{s(s+1)^2} = \frac{1}{s(s+1)^2}.$$

We can compute this partial fraction expansion also using Matlab

```
>> [r,p,k]=residue([0,0,0,1],[1,2,1,0])

r =
    -1
    -1
     1


p =
    -1
    -1
     0


k =
     []
>>
```

For
$$\frac{1}{(s+1)^2} = \frac{q_{1,1}}{s+1} + \frac{q_{1,2}}{(s+1)^2}. \tag{58}$$
we find
$$\begin{aligned} q_{1,1} &= (1)' \,|_{s=-1} = 0|_{s=-1} = 0, \\ q_{1,2} &= 1|_{s=-1} = 1. \end{aligned}$$

Of course, we could have read off these coefficients directly from (58).

The partial fraction expansions for the other entries of $(sI-A)^{-1}$ can be computed analogously. Inserting the partial fraction expansion into $(sI-A)^{-1}$ gives
$$\begin{aligned} &(sI-A)^{-1} \\ &= \frac{1}{s} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{=P_1} + \frac{1}{s+1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}}_{=P_2} + \frac{1}{(s+1)^2} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{=D_2}. \end{aligned}$$

Note that $P_1^2 = P_1$, $P_2^2 = P_2$, $P_1 P_2 = 0$, $P_1 D_2 = D_2 P_1 = 0$, $P_2 D_2 = D_2 P_2 = D_2$. $\diamond$

17 Complex Integration

Let

$$C = \{z(t) \ : \ t \in [t_1, t_2]\}$$

where z is continuously differentiable and one-to-one. Such a set C is called a directed smooth curve. The integral of a continuous function $f : \mathbb{C} \to \mathbb{C}$ on the directed smooth curve is defined as

$$\int_C f(z)dz \stackrel{\text{def}}{=} \int_{t_1}^{t_2} f(z(t))z'(t)dt.$$

Example 48 Let $C = \{t + it^2 \ : \ t \in [0, 1]\}$ and $f(z) = z^2$. Then

$$\begin{aligned}
\int_C f(z)dz &= \int_0^1 (t + it^2)^2(1 + 2ti)dt \\
&= \int_0^1 (t^2 - t^4 + 2t^3 i)(1 + 2ti)dt \\
&= \int_0^1 (t^2 - 5t^4) + i(4t^3 - 2t^5)dt \\
&= (t^3/3 - t^5) + i(t^4 - t^6/3)\Big|_0^1 = -2/3 + 2/3\, i.
\end{aligned}$$

The orientation of the curve matters. Changing the orientation switches the sign of the integral. Let $C_- = \{(1 - t) + i(1 - t)^2 \ : \ t \in [0, 1]\}$ and $f(z) = z^2$. Then

$$\begin{aligned}
\int_{C_-} f(z)dz &= \int_0^1 ((1 - t) + i(1 - t)^2)^2 \, (-1 - 2(1 - t)i) \; dt \\
&= \int_0^1 -((1 - t)^2 - 5(1 - t)^4) - i(4(1 - t)^3 - 2(1 - t)^5)dt \\
&= ((1 - t)^3/3 - (1 - t)^4) + i((1 - t)^4 - (1 - t)^6/3)\Big|_0^1 = 2/3 - 2/3\, i.
\end{aligned}$$

$\diamond$

Example 49 Let $C(a,r) = \{a + re^{it} \ : \ t \in [0, 2\pi]\}$ be the circle around a with radius r.

$$\begin{aligned}
\int_{C(a,r)} (z-a)^m dz &= \int_0^{2\pi} (a + re^{it} - a)^m rie^{it} dt \\
&= r^{m+1} i \int_0^{2\pi} e^{i(m+1)t} dt \\
&= r^{m+1} i \left(\int_0^{2\pi} \cos((m+1)t) dt + i \int_0^{2\pi} \sin((m+1)t) dt \right) \\
&= \begin{cases} 2\pi i & \text{if } m = -1 \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

$\diamond$

Theorem 50 (Cauchy's Theorem) *If the function f is differentiable on and inside the closed curve C, then*

$$\int_C f(z) dz = 0.$$

Theorem 51 (Curve Replacement Lemma) *Let C_1 and C_2 be two closed curves with the same orientation such that C_1 lies in the inside of C_2. If the function f is differentiable on the curves C_1 and C_2 and in the region between the curves C_1 and C_2, then*

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz.$$

Example 52 In Example 47 we have computed the partial fraction expansion

$$\frac{1}{z(z+1)^2} = \frac{1}{z} + \frac{-1}{z+1} + \frac{-1}{(z+1)^2}.$$

If C is a closed curve encircling the points 0 and -1, then, by the Curve Replacement Lemma

$$\int_C \frac{1}{z(z+1)^2} dz = \int_{C(0,1/3)} \frac{1}{z(z+1)^2} dz + \int_{C(-1,1/3)} \frac{1}{z(z+1)^2} dz,$$

where $C(0, 1/3)$ and $C(-1, 1/3)$ are the circles with center 0 and -1, respectively, and radii $1/3$. The integrals can be computed using the partial fraction expansion, Cauchy's Theorem, and

Example 49. We have

$$
\begin{aligned}
\int_C \frac{1}{z(z+1)^2}dz &= \int_{C(0,1/3)} \frac{1}{z(z+1)^2}dz + \int_{C(-1,1/3)} \frac{1}{z(z+1)^2}dz \\
&= \underbrace{\int_{C(0,1/3)} \frac{1}{z}dz}_{=2\pi i \text{ by Ex. } 49} + \underbrace{\int_{C(0,1/3)} \frac{-1}{z+1}dz}_{=0 \text{ by Cauchy's Thm.}} + \underbrace{\int_{C(0,1/3)} \frac{-1}{(z+1)^2}dz}_{=0 \text{ by Cauchy's Thm.}} \\
&\quad + \underbrace{\int_{C(-1,1/3)} \frac{1}{z}dz}_{=0 \text{ by Cauchy's Thm.}} + \underbrace{\int_{C(-1,1/3)} \frac{-1}{z+1}dz}_{=-2\pi i \text{ by Ex. } 49} + \underbrace{\int_{C(-1,1/3)} \frac{-1}{(z+1)^2}dz}_{=0 \text{ by Ex. } 49} \\
&= 2\pi i - 2\pi i = 0.
\end{aligned}
$$

◇

The previous example indicates how to integrate rational functions in partial fraction expansion. Let

$$
q(z) = \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{q_{j,k}}{(z-\lambda_j)^k}
$$

and let C be a closed curve encircling all poles $\lambda_1, \ldots, \lambda_h$ of the rational function q. If C_j is the circle around λ_j with radius small enough so that all other poles are outside C_j and the circles $C_1, \ldots, C_h$ do not intersect, see Figure 21, then

$$
\begin{aligned}
\int_C q(z)dz &= \int_C \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{q_{j,k}}{(z-\lambda_j)^k}dz \\
&= \sum_{\ell=1}^{h} \int_{C_\ell} \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{q_{j,k}}{(z-\lambda_j)^k}dz \\
&= \sum_{j=1}^{h} \sum_{k=1}^{m_j} \sum_{\ell=1}^{h} q_{j,k} \underbrace{\int_{C_\ell} \frac{1}{(z-\lambda_j)^k}dz}_{\substack{=2\pi i \text{ if } \ell=j \text{ and } k=1; \\ =0 \text{ else}}} = 2\pi i \sum_{j=1}^{h} q_{j,k}
\end{aligned}
$$

Theorem 53 (Cauchy's Integral Formula) *If the function f is differentiable on and inside the closed curve C, then*

$$
f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}dz
$$

for each a lying inside C.

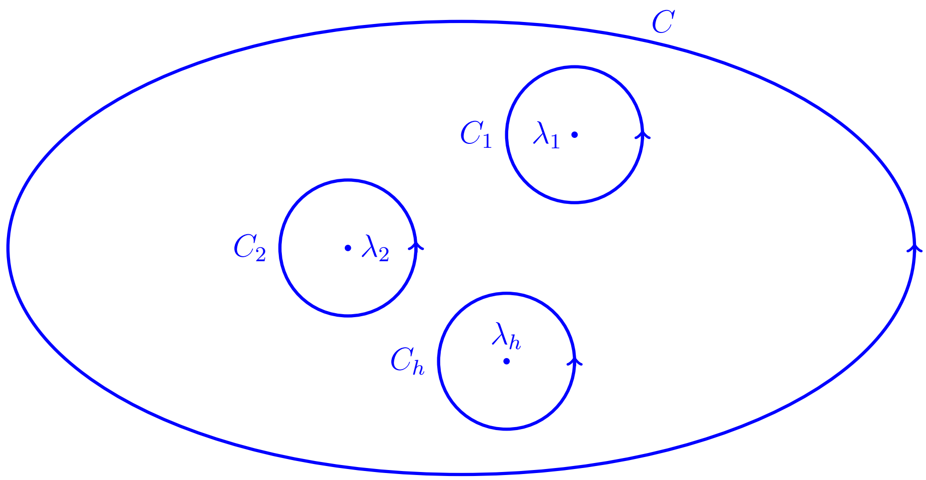


Figure 21: The poles $\lambda_1, \ldots, \lambda_h$ of the rational function q, the curve C and the circles used to compute $\int_C q(z)dz$.

Proof: Let a be a point inside C and let $r > 0$ such that $C(a,r)$ is inside C. By the Curve Replacement Lemma

$$\begin{aligned}\int_C \frac{f(z)}{z-a}dz &= \lim_{r\to 0}\int_{C(a,r)} \frac{f(z)}{z-a}dz \\ &= \lim_{r\to 0}\int_{C(a,r)} \frac{f(a)}{z-a}dz + \int_{C(a,r)} \frac{f(z)-f(a)}{z-a}dz \\ &= f(a)\,2\pi i + \lim_{r\to 0}\int_{C(a,r)} \frac{f(z)-f(a)}{z-a}dz.\end{aligned}$$

Since f is differentiable at a, $\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a)$. In particular, $\frac{f(z)-f(a)}{z-a}$ is bounded for all z sufficiently close to a, i.e., there exists $M > 0$ and $\epsilon > 0$ such that

$$\left|\frac{f(z)-f(a)}{z-a}\right| \le M$$

for all z with $|z-a| \le \epsilon$. This implies

$$\left|\int_{C(a,r)} \frac{f(z)-f(a)}{z-a}dz\right| \le M 2\pi r$$

for all $r \le \epsilon$. Hence,

$$\int_C \frac{f(z)}{z-a}dz = \lim_{r\to 0}\int_{C(a,r)} \frac{f(z)}{z-a}dz = f(a)\,2\pi i + \lim_{r\to 0}\int_{C(a,r)} \frac{f(z)-f(a)}{z-a}dz = f(a)\,2\pi i.$$

□

The Cauchy Integral Formula provides an alternative representation of f inside the circle. For any a lying inside C we have

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz. \tag{59}$$

The right hand side in (59) can be differentiated with respect to a:

$$\frac{d}{da} \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \int_C \frac{d}{da} \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz.$$

Therefore, because of the identity (59), f is differentiable and

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz. \tag{60}$$

The right hand side in (60) can be differentiated with respect to a:

$$\frac{d}{da} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz = \frac{1}{2\pi i} \int_C \frac{d}{da} \frac{f(z)}{(z-a)^2} dz = \frac{2}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz.$$

Therefore, because of the identity (60), f is twice differentiable and

$$f''(a) = \frac{2}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz. \tag{61}$$

We can repeat this process to obtain

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \tag{62}$$

for any integer n and any a inside the curve C. In particular, we have shown that if the function f is differentiable on and inside the closed curve C, if is infinitely often differentiable inside the closed curve C!

In the next example we will use curve Replacement Lemma and Cauchy's Integral Formula, Theorem 53, to compute complex integrals.

Example 54

1. Let $C = \{3e^{it} \; : \; t \in [0, 2\pi)\}$. Compute the integral

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz.$$

Let C_1 be the circle around $a = 1$ with radius $r = 1/2$ and let C_2 be the circle around $a = 2$ with radius $r = 1/2$. By the Curve Replacement Lemma

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = \int_{C_1} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz + \int_{C_2} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz.$$

The function $f(z) = \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)}$ is differentiable on C_1 and inside C_1. Hence, by the Cauchy Integral Formula (59),

$$\int_{C_1} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = 2\pi i \frac{\sin(\pi 1^2) + \cos(\pi 1^2)}{(1-2)} = -2\pi i(\sin(\pi) + \cos(\pi)) = 2\pi i.$$

The function $f(z) = \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)}$ is differentiable on C_2 and inside C_2. Hence, by the Cauchy Integral Formula (59),

$$\int_{C_2} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = 2\pi i \frac{\sin(\pi 2^2) + \cos(\pi 2^2)}{(2-1)} = 2\pi i(\sin(4\pi) + \cos(4\pi)) = 2\pi i.$$

Consequently,

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = 2\pi i + 2\pi i = 4\pi i.$$

2. Let $C = \{3e^{it} \ : \ t \in [0, 2\pi)\}$. Compute the integral

$$\int_C \frac{e^{2z}}{(z+1)^4} dz.$$

We use equation (62) with $a = -1$, $n = 3$, $f(z) = e^{2z}$. We can apply this formula, since $a = -1$ is inside C and $f(z) = e^{2z}$ is differentiable on C and inside C.

$$f(z) = e^{2z}, \quad f'(z) = 2e^{2z}, \quad f''(z) = 4e^{2z}, \quad f'''(z) = 8e^{2z}.$$

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} 8e^{2(-1)} = \frac{16\pi i}{6} e^{-2} = \frac{8\pi i}{3} e^{-2}.$$

$\diamond$

Our approach to compute integrals shown in the previous examples can be generalized. This will lead to the so-called Residue Theorem, which is an immediate consequence of the Curve Replacement Lemma and Cauchy's Integral Formula, Theorem 53. Next, we derive the Residue Theorem.

Let g be a polynomial with roots $\lambda_1, \ldots, \lambda_k$ of degree $m_1, \ldots, m_h$, respecively, i.e., let

$$g(z) = c(\lambda_1 - z)^{m_1} \ldots (\lambda_h - z)^{m_h}$$

with come constant c. Furthermore let C be a closed curve encircling each of the roots $\lambda_1, \ldots, \lambda_k$ and let the function f be differentiable on and inside C. We want to compute

$$\int_C \frac{f(z)}{g(z)} dz.$$

We proceed as in the computation of the integral of the partial fraction expansion of a rational function. Let C_j be a circle around λ_j with radius small enough so that all other roots are outside C_j and the circles $C_1, \ldots, C_h$ do not intersect. By the curve replacement lemma

$$\begin{aligned}\int_C \frac{f(z)}{g(z)} dz &= \sum_{j=1}^{h} \int_{C_j} \frac{f(z)}{g(z)} dz \\ &= \sum_{j=1}^{h} \int_{C_j} \frac{f(z)}{c(\lambda_1 - z)^{m_1} \ldots (\lambda_{j-1} - z)^{m_{j-1}} (\lambda_j - z)^{m_j} (\lambda_{j+1} - z)^{m_{j+1}} \ldots (\lambda_h - z)^{m_h}} dz \\ &= \sum_{j=1}^{h} \int_{C_j} \frac{f(z)}{c(\lambda_1 - z)^{m_1} \ldots (\lambda_{j-1} - z)^{m_{j-1}} (\lambda_{j+1} - z)^{m_{j+1}} \ldots (\lambda_h - z)^{m_h}} \; \frac{1}{(\lambda_j - z)^{m_j}} dz.\end{aligned}$$

If we define

$$\widetilde{f}_j(z) \stackrel{\text{def}}{=} \frac{f(z)}{c(\lambda_1 - z)^{m_1} \ldots (\lambda_{j-1} - z)^{m_{j-1}} (\lambda_{j+1} - z)^{m_{j+1}} \ldots (\lambda_h - z)^{m_h}} = (\lambda_j - z)^{m_j} \frac{f(z)}{g(z)},$$

then

$$\int_C \frac{f(z)}{g(z)} dz = \sum_{j=1}^{h} \int_{C_j} \frac{f(z)}{g(z)} dz = \sum_{j=1}^{h} \int_{C_j} \frac{\widetilde{f}_j(z)}{(\lambda_j - z)^{m_j}} dz.$$

Furthermore,

$$\int_{C_j} \frac{\widetilde{f}_j(z)}{(\lambda_j - z)^{m_j}} dz = \frac{2\pi i}{(m_j - 1)!} \frac{d^{m_j - 1}}{dz^{m_j - 1}} \widetilde{f}_j(\lambda_j) = \frac{2\pi i}{(m_j - 1)!} \lim_{z \to \lambda_j} \frac{d^{m_j - 1}}{dz^{m_j - 1}} \left((z - \lambda_j)^{m_j} \frac{f(z)}{g(z)} \right).$$

Thus we have proven the Residue Theorem.

Theorem 55 (The Residue Theorem) *If g is a polynomial with roots $\lambda_1, \ldots, \lambda_k$ of degree $m_1, \ldots, m_h$, respecively, and C is a closed curve encircling each of the roots $\lambda_1, \ldots, \lambda_k$ and f is a differentiable on an inside C, then*

$$\int_C \frac{f(z)}{g(z)} dz = 2\pi i \sum_{j=1}^{h} \textit{res}(\lambda_j; f/g),$$

where

$$\textit{res}(\lambda_j; f/g) = \lim_{z \to \lambda_j} \frac{1}{(m_j - 1)!} \frac{d^{m_j - 1}}{dz^{m_j - 1}} \left((z - \lambda_j)^{m_j} \frac{f(z)}{g(z)} \right)$$

is the residue of f/g at λ_j.

Example 56

1. Let $C = \{3e^{it} \ : \ t \in [0, 2\pi)\}$. In Example 54 we have shown that

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = 4\pi i.$$

We apply the Residue Theorem directly to recompute the integral The residues are

$$res(1) = \lim_{z \to 1} (z-1) \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} = \frac{\sin(\pi) + \cos(\pi)}{-1} = 1$$

$$res(2) = \lim_{z \to 2} (z-2) \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} = \frac{\sin(4\pi) + \cos(4\pi)}{1} = 1.$$

Hence

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = 2\pi i (1+1) = 4\pi i.$$

2. Let $C = \{3e^{it} \ : \ t \in [0, 2\pi)\}$. In Example 54 we have shown that

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i}{3} e^{-2}.$$

We apply the Residue Theorem directly to recompute the integral. The residue is

$$res(-1) = \lim_{z \to -1} \frac{1}{(4-1)!} \frac{d^3}{dz^3} e^{2z} = \lim_{z \to -1} \frac{1}{6} 2^3 e^{2z} = \frac{4}{3} e^{-2}.$$

Hence

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = 2\pi i \ res(-1) = \frac{8\pi i}{3} e^{-2}.$$

◇

Example 57

1. Let $C(1,3) = \{1 + 3e^{it} \ : \ t \in [0, 2\pi)\}$. Compute

$$\int_{C(1,3)} \frac{z^2}{(z+1)(z-1)^2}.$$

We apply the residue theorem. The roots of $(z+1)(z-1)^2$ are $\lambda_1 = -1$ with multiplicity $m_1 = 1$ and $\lambda_2 = 1$ with multiplicity $m_2 = 2$. Both roots are inside $C(1,3)$. We apply the residue theorem with

$$f(z) = z^2, \quad g(z) = (z+1)(z-1)^2.$$

The residues are

$$\begin{aligned}
\text{res}(-1) &= \lim_{z\to -1} \frac{1}{(1-1)!}\left((z+1)\frac{z^2}{(z+1)(z-1)^2}\right) \\
&= \lim_{z\to -1}\left(\frac{z^2}{(z-1)^2}\right) = 1/4, \\
\text{res}(1) &= \lim_{z\to 1} \frac{1}{(2-1)!}\frac{d}{dz}\left((z-1)^2\frac{z^2}{(z+1)(z-1)^2}\right) \\
&= \lim_{z\to 1}\frac{d}{dz}\left(\frac{z^2}{(z+1)}\right) \\
&= \lim_{z\to 1}\left(\frac{2z(z+1)-z^2}{(z+1)^2}\right) = 3/4.
\end{aligned}$$

Hence

$$\int_{C(1,3)} \frac{z^2}{(z+1)(z-1)^2} = 2\pi i\frac{1}{4} + 2\pi i\frac{3}{4} = 2\pi i.$$

2. Now, let $C(1,1) = \{1+e^{it} \ : \ t \in [0,2\pi)\}$. Compute

$$\int_{C(1,1)} \frac{z^2}{(z+1)(z-1)^2}.$$

Again, we apply the residue theorem. The roots of $(z+1)(z-1)^2$ are $\lambda_1 = -1$ with multiplicity $m_1 = 1$ and $\lambda_2 = 1$ with multiplicity $m_2 = 2$. ONLY the root $\lambda_2 = 1$ is inside $C(1,1)$. Hence we apply the residue theorem with

$$f(z) = \frac{z^2}{(z+1)}, \quad g(z) = (z-1)^2.$$

$$\begin{aligned}
\text{res}(1) &= \lim_{z\to 1} \frac{1}{(2-1)!}\frac{d}{dz}\left((z-1)^2\frac{z^2}{(z+1)(z-1)^2}\right) \\
&= \lim_{z\to 1}\frac{d}{dz}\left(\frac{z^2}{(z+1)}\right) \\
&= \lim_{z\to 1}\left(\frac{2z(z+1)-z^2}{(z+1)^2}\right) = 3/4.
\end{aligned}$$

(The calculation is identical to the computation of $res(1)$ in the previous part.) Hence

$$\int_{C(1,1)} \frac{z^2}{(z+1)(z-1)^2} = 2\pi i\frac{3}{4} = \pi\frac{3}{2}i.$$

$\diamond$

18 The Inverse Laplace Transform

An important application of the Residue Theorem is the computation of the Inverse Laplace Transform. The inverse Laplace transform of a function q is given by

$$(\mathcal{L}^{-1}q)(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(a+iy)t}q(a+iy)dy.$$

We compute the inverse Laplace transform of rational functions

$$q(z) = \frac{f(z)}{g(z)},$$

where $g : \mathbb{C} \to \mathbb{C}$ is a complex polynomial of order n with roots $\lambda_1, \dots, \lambda_k$ with multiplicities $m_1, \dots, m_k$, i.e.,

$$g(z) = c_n(z-\lambda_1)^{m_1}\dots(z-\lambda_k)^{m_k}$$

and $\sum_{j=1}^k m_j = n$, and $f : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree at most $n-1$.

We will show that the inverse Laplace transform of a rational function can be computed from

$$(\mathcal{L}^{-1}q)(t) = \sum_{j=1}^{h} \text{res}(\lambda_j; e^{zt}q(z)) \quad \text{for } t > 0. \tag{63}$$

Before we indicate why this identity is true, we apply it to compute the inverse Laplace transform of a few functions.

Example 58

(a) We compute the inverse Laplace transform of $q(s) = (s-1)^{-1}$. We have $\mathcal{L}^{-1}((s-1)^{-1})(t) = \text{res}(1; e^{st}q(s))$ and $\text{res}(1; e^{st}q(s)) = e^t$. Hence,

$$\mathcal{L}^{-1}((s-1)^{-1})(t) = e^t \quad \text{for } t > 0.$$

Note that in the Lecture on the Laplace transform we have computed $\mathcal{L}(e^t)(s) = (s-1)^{-1}$.

(b) We compute the inverse Laplace transform of $q(s) = (1+s)^{-2}$. We have $\mathcal{L}^{-1}((s+1)^{-2})(t) = \text{res}(-1; e^{st}q(s))$ and

$$\text{res}(-1; e^{st}q(s)) = \frac{d}{ds}e^{st}\Big|_{s=-1} = te^{st}\Big|_{s=-1} = te^{-t}.$$

Hence,

$$\mathcal{L}^{-1}((1+s)^{-2})(t) = te^{-t}. \quad \text{for } t > 0.$$

Note that in the Lecture on the Laplace transform we have computed $\mathcal{L}(te^t)(s) = (1+s)^{-2}$.

(c) We compute the inverse Laplace transform of $q(s) = s/(s^2+1)$. The roots of s^2+1 are $s = i, -i$. We have $\mathcal{L}^{-1}(s/(s^2+1))(t) = \text{res}(i; e^{st}q(s)) + \text{res}(-i; e^{st}q(s))$ and

$$\text{res}(i; e^{st}q(s)) = \lim_{s\to i}(s-i)e^{st}s/(s^2+1) = e^{st}s/(s+i)\Big|_{s=i} = \tfrac{1}{2}e^{it},$$
$$\text{res}(-i; e^{st}q(s)) = \lim_{s\to -i}(s+i)e^{st}s/(s^2+1) = e^{st}s/(s-i)\Big|_{s=-i} = \tfrac{1}{2}e^{-it},$$

Hence,

$$\mathcal{L}^{-1}((1+s)^{-2})(t) = \tfrac{1}{2}(e^{it}+e^{-it}) = \cos(t). \quad \text{for } t>0.$$

Note that in the Lecture on the Laplace transform we have computed $\mathcal{L}(\cos)(s) = s/(s^2+1)$.

◇

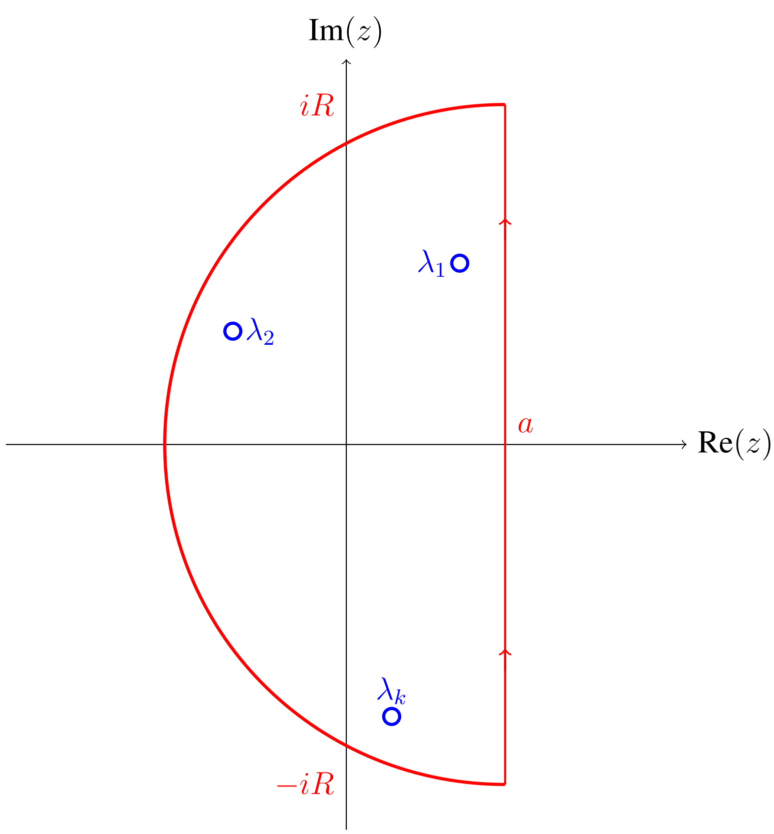


Figure 22: The roots $\lambda_1, \dots, \lambda_k$ of g and the curve C_R.

Now we prove (63). Consider the curve C_R depicted in Figure 22. The integral

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{g(z)}e^{zt}dz$$

can be computed using the Residue Theorem 55:

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{g(z)}e^{zt}dz=\sum_{j=1}^{h}\text{res}(\lambda_j;e^{zt}f(z)/g(z)),$$

Furthermore, since all poles of $q=f/g$ are inside C_R, the value of the integral does not change when the curve increases. In particular,

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{g(z)}e^{zt}dz=\sum_{j=1}^{h}\text{res}(\lambda_j;e^{zt}f(z)/g(z)). \tag{64}$$

We split C_R into the line

$$\gamma_R=\{a+iy\ :\ y\in[-R,R]\}$$

and the semi-circle

$$\Gamma_R=\left\{a+Re^{iy}\ :\ y\in[\pi/2,3\pi/2]\right\}.$$

We have

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{g(z)}e^{zt}dz=\frac{1}{2\pi i}\int_{\gamma_R}\frac{f(z)}{g(z)}e^{zt}dz+\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(z)}{g(z)}e^{zt}dz \tag{65}$$

By definition of the complex integral

$$\frac{1}{2\pi i}\int_{\gamma_R}\frac{f(z)}{g(z)}e^{zt}dz=\frac{1}{2\pi i}\int_{-R}^{R}\frac{f(a+iy)}{g(a+iy)}e^{(a+iy)t}\,i\,dy=\frac{1}{2\pi}\int_{-R}^{R}\frac{f(a+iy)}{g(a+iy)}e^{(a+iy)t}\,dy.$$

Hence,

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_R}\frac{f(z)}{g(z)}e^{zt}dz=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(a+iy)}{g(a+iy)}e^{(a+iy)t}\,dy=\left(\mathcal{L}^{-1}q\right)(t). \tag{66}$$

One can show that for $t>0$,

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(z)}{g(z)}e^{zt}dz=0. \tag{67}$$

Combining equations (64) to (67) gives (63).