Circuits I

Abstract: In this lab a nerve fiber was modeled as a resistor network consisting of 12 compartments. Using the Strang Quartet method, the network was described using a system of linear equations, namely $A^T G Ax = f + A^T G b$. In the laboratory, a physical resistor network of the same format was the model was used to generate experimental data in order to test the models accuracy. These data were then compared to the values for the node potentials predicted by the model.

The model provided extremely accurate predictions of node potentials – the largest percent error was 0.30%, with an average error of 0.26% demonstrating that the Strang Quartet method of modeling a resistor network is accurate.

The network was also tested with half the original voltage so as to verify that the network is linear – both the model and experimental data agreed that this is true.

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1. Parameters

The resistor network consists of a voltage source, a ground, and several resistors connected as shown in Fig. 1:

For our model, there are 12 nodes, the axial resistance is $R_i = 1k\Omega$, and the membrane resistance is $R_m = 49.9k\Omega$. The voltage source supplies $v_0 = 12V$. For our purposes, the space between the voltage source and the first resistor is considered the 0th node, $x_0$ (in some cases it is in our best interest to begin counting at 0 rather than 1).

2. Modeling Method

The resistor network in question was modeled using the Strang Quartet Method – the linear systems of equations representing Kirchhoff’s Voltage Law (KVL), Ohm’s Law, and Kirchhoff’s Current Law (KCL) were developed, combined, and then solved to yield the potentials at each of the nodes.

KVL refers to the potential differences across resistors. To determine the potential difference $e$ across a resistor, the potential after the resistor is subtracted from the potential before the resistor. For example: $e_1 = x_0 - x_1$, $e_2 = x_1$, $e_3 = x_1 - x_2$, and so on, with the number of equations in the system depending on the number of compartments in the network. This system of linear equations can be compiled into matrix form – for example, a 3-compartment system would produce these matrices:
The resulting matrix equation can be written concisely as \( e = -Ax \), where \( e \) is the vector of potential differences, \( x \) is the vector of node potentials, and \( A \) is the matrix relating the node potentials to the potential differences (\( A \) is called the "adjacency matrix" – note that the values are the opposite sign of what would be expected and the minus sign is pulled out of the matrix, this is for reasons which will become obvious later). It is apparent that the adjacency matrix contains a repeating pattern, \( \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \), and we may use this fact to easily generate adjacency matrices for \( n \)-compartment resistor networks (given that the geometry of the networks remains the same). The \( e \) and \( x \) matrices can also easily be expanded to represent larger networks.

The resistor network we are modeling is a little more complex, however, since we are analyzing in terms of a voltage that is applied, as opposed to a current. Therefore, the value of \( x_0 \) is known to be \( v_0 \). As such, the first value of the vector of potentials is known, and the KCL equation changes thus (continuing with the 3-compartment example):

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6
\end{pmatrix} = \begin{pmatrix} -v_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

which can be written more concisely as \( e = b - Ax \). The first column of \( A \) has been stripped off and multiplied by \( v_0 \) to generate \( b \), which is incorporated into the equation to yield the KVL equation for this case. Again, each matrix in this equation is easily expanded to represent networks with \( n \) compartments.

Ohm’s Law states that the current \( y \) through a resistor is equal to the potential difference \( e \) divided by the resistance \( R \), or \( y = e/R \). This relationship, when expanded to a system, can easily be expressed in a matrix form thus (again, 3-compartment example):

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6
\end{pmatrix} = \begin{pmatrix} -v_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]
This diagonal matrix of the inverses of resistances is called $G$, and the full equation is written as $y = Ge$.

KCL states that the sum of the currents ($y$) into each node is zero. For example, $i_0 - y_1 = 0$, $y_1 - y_2 - y_3 = 0$, and so on. When compiled into matrices, the system of equations becomes (for a 3-compartment example):

$$
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6 \\
\end{pmatrix} =
\begin{pmatrix}
1/R_i & 0 & 0 & 0 & 0 & 0 \\
0 & 1/R_m & 0 & 0 & 0 & 0 \\
0 & 0 & 1/R_i & 0 & 0 & 0 \\
0 & 0 & 0 & 1/R_m & 0 & 0 \\
0 & 0 & 0 & 0 & 1/R_i & 0 \\
0 & 0 & 0 & 0 & 0 & 1/R_m \\
\end{pmatrix}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6 \\
\end{pmatrix}
$$

Closer inspection reveals that the first matrix is simply the transpose of $A$, therefore this system of equations can be written as $A^Ty = -f$.

After deriving the KVL, Ohm’s Law, and KCL equations, we combined them into one equation that we could use to solve for the potentials, following this process:

Eqn 1: $e = b - Ax$  \hspace{1cm}  Eqn 2: $y = Ge$  \hspace{1cm}  Eqn 3: $A^Ty = -f$

Eqn 2 $\rightarrow$ Eqn 3: \hspace{0.5cm} $yA^TGe = -f$ (Eqn 4)

Eqn 1 $\rightarrow$ Eqn 4: \hspace{0.5cm} $A^TGb - A^TGAx = -f$

\hspace{1cm} $A^TGAx = f + A^TGb$

While it is possible to derive and solve these systems of equations by hand, as the number of compartments in the network increases the systems rapidly become rather cumbersome. Therefore, a MATLAB program was written to do the work.
function lab1

N = 12; % # of nodes
R_i = 1e3; % Axial resistance, ohms
R_m = 49.9e3; % Membrane resistance, ohms
v_0 = 12; % Voltage, volts
m = 2*N; % # of resistors (edges)
n = N+1; % # of nodes
A = zeros(m,n); % scaffold for adjacency matrix

for j = 1:N % fill in adjacency matrix
    A((2*j)-1,1) = -1;
    A((2*j)-1, j+1) = 1;
    A(2*j, j+1) = -1;
end

b = v_0*A(:,1); % build b from 1st column of A
A = A(:,2:end); % remove 1st column of A
G = zeros(m,1); % build G
G(1:2:end,1) = 1/R_i;
G(2:2:end,1) = 1/R_m;
G = diag(G);
\[ S = A' \ast G \ast A; \] % compile matrices into one

\[ f = -A' \ast G \ast b; \] % stimulus vector

\[ x1 = S \backslash f; \] % solve for potentials
\[ x1 = [v_0; x1]; \] % add in the potential for the 0th node, which is
% the same as v_0

\[ x2 = [12.03; 10.54; 9.27; 8.18; 7.25; 6.47; 5.82; 5.28; 4.85; 4.51; 4.26; 4.10; 4.02]; \] % experimental data
% for 12 V

clf
\[ z = 0:N; \] % x-axis plotting points
\[ \text{plot}(z,x1,'bo-') \] % plot calculated potentials
hold on
\[ \text{plot}(z,x2,'rx:') \] % plot experimental data
\[ \text{xlabel('z (node)')}; \]
\[ \text{ylabel('Potential (V)')}; \]
\[ \text{tlab} = ['Predicted and Experimental potential along a '...\]
\[ \quad \text{num2str(N)} \ ' \text{compartment fiber, } v_0 = ' \text{num2str(v_0)}]; \]
\[ \text{title(tlab)} \]
\[ \text{legend('Predicted', 'Experimental')}; \]

(Note: This code contains data obtained via experiment - the experimental method will
be explained in section 4)

When executed, this code generates a plot of the predicted and experimentally
determined node potentials from the 12-compartment resistor network
under consideration (Fig. 2).
4. Experimental Method

In the laboratory, the equipment depicted in Fig. 3 was used to physically reproduce the 12-compartment network modeled above. The network was connected to the power supply (which was set to supply 12V), and the multimeter was used to read the potentials of each node relative to ground (tabulated in Table 1). These data were entered into the MATLAB code above and plotted next to the predicted values, as shown in Fig. 2.

Fig. 2 - Plot of predicted and experimentally determined node potentials on a 12-compartment resistor network ($v_0 = 12V$)
**Fig. 3 - Equipment used in this lab (clockwise from top left: Power supply, multimeter, and resistor network*)**

*NOTE: in this picture there is no resistor between \(x_0\) and \(x_1\), this was later altered so as to better fit the model

**image reproduced from lab 1 manual (http://www.caam.rice.edu/ caam335lab/labs/lab1.pdf)**

<table>
<thead>
<tr>
<th>Node</th>
<th>(x_0)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>(x_7)</th>
<th>(x_8)</th>
<th>(x_9)</th>
<th>(x_{10})</th>
<th>(x_{11})</th>
<th>(x_{12})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Potential (V)</td>
<td>12.03</td>
<td>10.54</td>
<td>9.27</td>
<td>8.18</td>
<td>7.25</td>
<td>6.47</td>
<td>5.82</td>
<td>5.28</td>
<td>4.85</td>
<td>4.51</td>
<td>4.26</td>
<td>4.10</td>
<td>4.02</td>
</tr>
</tbody>
</table>

Table 1 - Experimentally determined potentials at each node (\(v_0 = 12V\))

5. Analysis and Conclusion

Simple visual analysis of the plot in Fig. 2 and the data in Table 2 shows that the predicted and experimental values are very close; for a more quantitative description, a percent error analysis can be easily done:

\[
\text{% error} = 100 \times \frac{\text{experimental value} - \text{predicted value}}{\text{experimental value}}
\]

As stated in the Abstract, this analysis determines that the largest percent error was 0.3%, and the average error was 0.26%. As such, we may conclude
that the Strang Quartet method produces an accurate model of a resistor network.

6. Appendix: Verification of Linearity

In theory, if the resistor network is linear, when the applied voltage is changed, the potentials at each of the nodes should change proportionally. To test this, we ran the experiment again, this time with \( v_0 = 6 \text{V} \). The MATLAB code is designed so that this change is easily made; line 17 of the code simply needs to be changed to:

\[
v_0 = 6; \quad \text{% Voltage, volts}
\]

The physical experiment was run again, this time with the power supply set to 6V, and data were gathered using the same method as before. This data was entered into the MATLAB code, changing vector \( x_2 \) in line 47 thus:

\[
x_2 = [6; 5.26; 4.62; 4.08; 3.62; 3.22; 2.90;... \quad \text{% experimental data}
2.63; 2.41; 2.25; 2.12; 2.04; 2.00]; \quad \text{% for 6 V}
\]

When the code is executed and the experiment run, the results are as shown in Table 3 and Fig. 4.
<table>
<thead>
<tr>
<th>Node</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
<th>$x_{10}$</th>
<th>$x_{11}$</th>
<th>$x_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Predicted Potential (V)</td>
<td>6.00</td>
<td>5.26</td>
<td>4.62</td>
<td>4.07</td>
<td>3.61</td>
<td>3.22</td>
<td>2.89</td>
<td>2.62</td>
<td>2.41</td>
<td>2.24</td>
<td>2.12</td>
<td>2.04</td>
<td>2.00</td>
</tr>
<tr>
<td>Experimental Potential (V)</td>
<td>6.00</td>
<td>5.26</td>
<td>4.62</td>
<td>4.08</td>
<td>3.62</td>
<td>3.22</td>
<td>2.90</td>
<td>2.63</td>
<td>2.41</td>
<td>2.25</td>
<td>2.12</td>
<td>2.04</td>
<td>2.00</td>
</tr>
</tbody>
</table>

Table 3 - Predicted and Experimentally determined potentials at each node ($v_0 = 6V$)

Once again, the predicted and experimental data line up quite nicely, and if we combine the data from the 12V and 6V runs, we can show that the potentials for the 6V run are nearly, if not exactly, half those of the 12V run (Table 4).
<table>
<thead>
<tr>
<th>Node</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
<th>$x_{10}$</th>
<th>$x_{11}$</th>
<th>$x_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Predicted Potential ($v_0 = 12V$)</td>
<td>12.00</td>
<td>10.51</td>
<td>9.24</td>
<td>8.14</td>
<td>7.22</td>
<td>6.44</td>
<td>5.78</td>
<td>5.25</td>
<td>4.81</td>
<td>4.48</td>
<td>4.23</td>
<td>4.07</td>
<td>3.99</td>
</tr>
<tr>
<td>$\frac{v_0 = 12V \text{ Predictions}}{2}$</td>
<td>6.00</td>
<td>5.25</td>
<td>4.62</td>
<td>4.07</td>
<td>3.61</td>
<td>3.22</td>
<td>3.89</td>
<td>2.62</td>
<td>2.41</td>
<td>2.24</td>
<td>2.12</td>
<td>2.04</td>
<td>2.00</td>
</tr>
<tr>
<td>Predicted Potential ($v_0 = 6V$)</td>
<td>6.00</td>
<td>5.26</td>
<td>4.62</td>
<td>4.07</td>
<td>3.61</td>
<td>3.22</td>
<td>2.89</td>
<td>2.62</td>
<td>2.41</td>
<td>2.24</td>
<td>2.12</td>
<td>2.04</td>
<td>2.00</td>
</tr>
<tr>
<td>Experimental Potential ($v_0 = 12V$)</td>
<td>12.03</td>
<td>10.54</td>
<td>9.27</td>
<td>8.18</td>
<td>7.25</td>
<td>6.47</td>
<td>5.82</td>
<td>5.28</td>
<td>4.85</td>
<td>4.51</td>
<td>4.26</td>
<td>4.10</td>
<td>4.02</td>
</tr>
<tr>
<td>$\frac{v_0 = 12V \text{ Experimental}}{2}$</td>
<td>6.02</td>
<td>5.27</td>
<td>4.64</td>
<td>4.09</td>
<td>3.63</td>
<td>3.24</td>
<td>2.91</td>
<td>2.64</td>
<td>2.43</td>
<td>2.26</td>
<td>2.13</td>
<td>2.05</td>
<td>2.01</td>
</tr>
<tr>
<td>Experimental Potential ($v_0 = 6V$)</td>
<td>6.00</td>
<td>5.26</td>
<td>4.62</td>
<td>4.08</td>
<td>3.62</td>
<td>3.22</td>
<td>2.90</td>
<td>2.63</td>
<td>2.41</td>
<td>2.25</td>
<td>2.12</td>
<td>2.04</td>
<td>2.00</td>
</tr>
</tbody>
</table>

**Table 4** - Predicted and Experimentally determined potentials (in V) for $v_0 = 12V$ and $v_0 = 6V$, as well as the $v_0 = 12V$ potentials divided by 2.

Percent error analysis of these data shows that the largest error in the predicted values is 0.19%, while in the experimental data the largest error is 0.82%; therefore these data verify that the circuit is linear.