In this lab we investigate a second vibration problem: the motion of beads threaded on a taut string that is anchored at both ends. Like the last lab’s compound pendulum, this problem stimulated great developments in mechanics during the eighteenth century. As we take more and more masses and pack them ever closer together, we obtain a model of a continuous string, as you might find on a violin or guitar. It turns out that the eigenvalues associated with such a string explain the pleasant progression of musical notes—but that is a story best saved for a course in partial differential equations.

Here, we imagine $n$ point-masses $m_1, \ldots, m_n$ fixed horizontally from left to right on a taut massless string of total length $\ell$. The rest positions of the masses provide a natural partition of the string, so the length between the left support and the first bead is $\ell_0$, between the first and second bead is $\ell_1$, etc., as illustrated for $n = 4$ in Figure 10.1, so that $\ell = \sum_{k=0}^{n} \ell_k$.

Now we pluck the string lightly, so as to induce a vibration among the beads. As the beads are fixed on the string, they do not slide horizontally. For small vibrations, the dominant displacement of each mass will be in perpendicular to the rest state, i.e., vertical. We denote the displacement of mass $j$ at time $t$ by $y_j(t)$, as illustrated in Figure 9.2.

Our goal is to determine the displacements $y_j(t)$, given a description of the initial pluck and the various masses and lengths. In next week’s lab, the culminating project of the semester, we shall solve the inverse problem: Given knowledge of how a beaded string vibrates, how heavy are the beads and where are they on the string?

Unsurprisingly, the mathematics behind bead vibrations is essentially identical to that of the compound pendulum; the only difference is the horizontal arrangement of the masses and the fact that both ends are now fixed. For details on general systems of this sort, see Gantmacher and Krein, Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems (AMS, 2002). We begin by deriving formulas for the kinetic and potential energies, then set up a system of linear
differential equations.

▶ Kinetic and Potential Energies

As before, we use the Euler–Lagrange equation to derive a system of differential equations that describe the displacements. The formula for kinetic energy is unchanged from the pendulum case; adding up \( \frac{1}{2} m v^2 \) at each bead gives

\[
T(t) = \frac{1}{2} \sum_{k=1}^{n} m_k [y'_k(t)]^2.
\]

The potential energy requires a more subtle computation; again, it will be the sum of the potential energies in each segment, that is, the sum of the products of tension and elongation. Since the beads are arranged horizontally, rather than vertically as in the pendulum, we presume that the string is held in a constant tension \( \sigma \) throughout the entire ensemble. The elongation computation is identical to the calculation for the pendulum. Suppose we wish to measure the elongation of the \( k \)th segment, as illustrated in Figure 9.2. Assuming that the beads are only moving vertically, the stretched length of this segment is simply

\[
\sqrt{\ell_k^2 + (y_{k+1} - y_k)^2} - \ell_k,
\]

and as in the previous lab, we use the fact that \( \ell_k \) is much larger than the displacements \( y_k \) and \( y_{k+1} \) to argue that

\[
\sqrt{\ell_k^2 + (y_{k+1} - y_k)^2} - \ell_k = \ell_k \sqrt{1 + \frac{(y_{k+1} - y_k)^2}{\ell_k^2}} - \ell_k \\
\approx \ell_k \left( 1 + \frac{1}{2} \frac{(y_{k+1} - y_k)^2}{\ell_k^2} \right) - \ell_k \\
= \frac{(y_{k+1} - y_k)^2}{2\ell_k}.
\]
Some special attention must be paid to the end cases, \( k = 0 \) and \( k = n \). These can be handled just as easily as the top segment of the pendulum: since the ends of the string remain fixed, we define \( y_0 = y_{n+1} = 0 \), and the elongation formula works out just right. As we have \( n + 1 \) segments corresponding to \( \ell_0, \ldots, \ell_n \), the formula for the potential energy takes the form

\[
V(t) = \frac{\sigma}{2} \sum_{k=0}^{n} \frac{(y_{k+1} - y_k)^2}{\ell_k}.
\]

**Euler–Lagrange Equations**

We substitute our formulas into the Euler–Lagrange equation

\[
\frac{d}{dt} \frac{\partial T}{\partial y_j'} + \frac{\partial V}{\partial y_j} = 0, \quad j = 1, \ldots, n
\]

just as in the last lab. From the kinetic energy formula, we have

\[
\frac{d}{dt} \frac{\partial T}{\partial y_j'}(t) = m_j y_j''(t).
\]

The potential energy term,

\[
\frac{\partial V}{\partial y_j} = \frac{\sigma}{2} \sum_{k=0}^{n} \frac{\partial}{\partial y_j} \left[ \frac{(y_{k+1} - y_k)^2}{\ell_k} \right],
\]

again requires a little more work. The partial derivative within the sum on the right will generally be nonzero except in the cases where \( k = j - 1 \) or \( k = j \). That is,

\[
\frac{\partial V}{\partial y_j} = \frac{\sigma}{2} \left( \frac{\partial}{\partial y_j} \left[ \frac{(y_j - y_{j-1})^2}{\ell_{j-1}} \right] + \frac{\partial}{\partial y_j} \left[ \frac{(y_{j+1} - y_j)^2}{\ell_j} \right] \right)
\]

\[
= \frac{\sigma}{2} \left( \frac{2y_j - 2y_{j-1}}{\ell_{j-1}} + \frac{2y_j - 2y_{j+1}}{\ell_j} \right)
\]

\[
= \left( -\frac{\sigma}{\ell_{j-1}} y_{j-1} + \frac{\sigma}{\ell_j} y_j + \frac{\sigma}{\ell_j} y_{j+1} \right).
\]

The cases of \( j = 1 \) and \( j = n \) are special, as \( y_0 = y_{n+1} = 0 \), and hence

\[
\frac{\partial V}{\partial y_1} = \left( \frac{\sigma}{\ell_0} + \frac{\sigma}{\ell_1} \right) y_1 + \left( -\frac{\sigma}{\ell_1} \right) y_2
\]

\[
\frac{\partial V}{\partial y_n} = \left( -\frac{\sigma}{\ell_{n-1}} y_{n-1} + \frac{\sigma}{\ell_{n-1}} \right) y_n + \left( \frac{\sigma}{\ell_n} + \frac{\sigma}{\ell_n} \right) y_n.
\]

These partial derivatives couple each of the \( n \) masses to its nearest neighbors, and so we must solve the Euler–Lagrange equation at all masses simultaneously: these \( n \) coupled linear equations lead to a single differential equation involving \( n \times n \) matrices.
Matrix Formulation

Substituting the partial derivatives of kinetic and potential energies into the Euler–Lagrange equation gives

\[
m_j y_j''(t) = \begin{cases} 
\left( -\frac{\sigma}{\ell_0} - \frac{\sigma}{\ell_1} \right) y_1 + \left( \frac{\sigma}{\ell_1} \right) y_2, & j = 1; \\
\left( \frac{\sigma}{\ell_{j-1}} \right) y_{j-1} + \left( -\frac{\sigma}{\ell_{j-1}} - \frac{\sigma}{\ell_j} \right) y_j + \left( \frac{\sigma}{\ell_j} \right) y_{j+1}, & 1 < j < n; \\
\left( \frac{\sigma}{\ell_{n-1}} \right) y_{n-1} + \left( -\frac{\sigma}{\ell_{n-1}} - \frac{\sigma}{\ell_n} \right) y_n, & j = n.
\end{cases}
\]

or, in matrix form,

\[
\begin{bmatrix}
m_1 \\
m_2 \\
m_3 \\
\vdots \\
m_n
\end{bmatrix}
\begin{bmatrix}
y_1''(t) \\
y_2''(t) \\
y_3''(t) \\
\vdots \\
y_n''(t)
\end{bmatrix} =
\begin{bmatrix}
-\frac{\sigma}{\ell_0} - \frac{\sigma}{\ell_1} & \frac{\sigma}{\ell_1} & 0 & \cdots & 0 \\
\frac{\sigma}{\ell_1} & -\frac{\sigma}{\ell_1} - \frac{\sigma}{\ell_2} & \frac{\sigma}{\ell_2} & \cdots & 0 \\
0 & \frac{\sigma}{\ell_2} & -\frac{\sigma}{\ell_2} - \frac{\sigma}{\ell_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{\sigma}{\ell_{n-1}} - \frac{\sigma}{\ell_n}
\end{bmatrix}
\begin{bmatrix}
y_1(t) \\
y_2(t) \\
y_3(t) \\
\vdots \\
y_n(t)
\end{bmatrix}.
\]

This differential equation

\[
y''(t) = -M^{-1}Ky(t)
\]

has the same form as the equation that described motion of the compound pendulum masses, and we can solve it in the same manner. Let \((\omega_j^2, u_j)\) denote pairs of eigenvalues and eigenvectors of \(M^{-1}K\), i.e., \(M^{-1}Ku_j = \omega_j^2 u_j\) for \(j = 1, \ldots, n\). Stacking each of these \(n\) eigenvalue–eigenvector equations side by side results in the matrix equation

\[
M^{-1}KU = U\Lambda,
\]

where the \(n \times n\) matrices \(U\) and \(\Lambda\) are defined as

\[
U = \begin{bmatrix}
u_1 & u_2 & \cdots & u_n
\end{bmatrix}, \quad \Lambda = \begin{bmatrix}
\omega_1^2 & 0 & \cdots & 0 \\
0 & \omega_2^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega_n^2
\end{bmatrix}.
\]

The differential equation (9.1) can thus be translated to

\[
y''(t) = -U\Lambda U^{-1}y(t),
\]

whereby premultiplication by \(U^{-1}\) gives

\[
U^{-1}y''(t) = -\Lambda U^{-1}y(t).
\]
Let us focus for a moment on this vector \( U^{-1}y(t) \), which we define as

\[
\hat{y}(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix} := U^{-1}y(t).
\]

Notice that

\[
y(t) = UU^{-1}y(t) = U\hat{y}(t) = \sum_{j=1}^{n} \gamma_j(t)u_j,
\]

so the entries of \( \hat{y}(t) \) are simply the coefficients we need to express \( y(t) \) in the basis of eigenvectors. Moreover, equation (9.2) is simply

\[
\hat{y}''(t) = -\Lambda\hat{y}(t),
\]

and since \( \Lambda \) is a diagonal matrix, this \( n \times n \) system reduces to \( n \) independent scalar equations:

\[
\gamma''_j(t) = -\omega_j^2 \gamma_j(t),
\]

each of which has a very simple general solution,

\[
\gamma_j(t) = \gamma_j(0) \cos(\omega_j t) + \left( \frac{\gamma'_j(0)}{\omega_j} \right) \sin(\omega_j t),
\]

in terms of the initial state \( \gamma_j(0) \) and velocity \( \gamma'_j(0) \).

The general solution (9.4) simplifies if we presume that the string begins with zero initial velocity,

\[
y'(0) = 0 \implies \hat{y}'(0) = 0 \implies \gamma'_j(0) = 0,
\]

in which case

\[
\gamma_j(t) = \gamma_j(0) \cos(\omega_j t).
\]

Substituting these equations into the expansion of \( y(t) \) in eigenvector coordinates (9.3) reveals a beautiful solution:

\[
y(t) = \sum_{j=1}^{n} \gamma_j(0) \cos(\omega_j t)u_j.
\]

Enjoy this equation for a moment: it says that the masses are displaced in concert as the superposition of \( n \) independent vectors vibrating at distinct frequencies. Can we detect these frequencies in the laboratory?

▶ Experimental Set-Up

The CAAM String Lab features a high-precision monochord that we shall use for this lab, illustrated in Figure 9.3. Tension is measured with a force transducer placed at the end of the string. The string then passes through a collet, which itself is mounted in a collet vise that, when tightened,
holds the end of the string tight. The string then passes through a photodetector that is used to measure the string vibrations at a single point along the string. Brass beads are threaded onto the string, which then passes through a second collet. (These beads have been carefully machined so as to snugly fit onto the properly chosen metal string.) Finally, the string is wound upon a spindle, which is used to apply tension to the string. (This tensioning requires a find hand: even metal strings will snap!) Close-ups of a collet in a vise and brass beads on a string are shown in Figure 9.4.

Notice that we do not measure the displacements of any one bead: the photodetector only measures the displacement of a single point along the string, but this point must vibrate at the same frequencies as the individual beads. As explained below, you will record displacements at a single point in time, then compute the Fourier transform of this data. The MATLAB code given below will produce a plot with horizontal axis in Hertz, i.e., cycles per second. As discussed in the last lab, a peak at $f$ Hz corresponds to a frequency of

$$\omega_j = 2\pi f.$$
Setting Up a Beaded String

We describe the process starting with no string in the device.

- Place the beads on the string in the desired order.
- Run one end through the photodetector, the collet, the force transducer, and the eye bolt (which prevents the string from slipping through the force transducer). Run the other end through the second collet.
- On both sides, run the string through the clamps and tighten them.
- On the side of the tensioner, attach the bolt sticking out of the clamp through the hole in the spindle. On the other side, take up slack so the tensioner will not need to be turned so far.
- Turn the knob on the top of the tensioner until the string is straight.
- Now turn the handles on the collet holders to lock them down on the string.

**Positioning the beads.** As the table is currently set up, the center of the string is between two grid holes. These holes are at one inch intervals. The best way to position the beads is to place your eye perpendicular to the table above a hole and then position the bead over it. Be sure to place the heavier beads to the outside, otherwise the vibrations of the lowest frequency mode are likely to be larger than the range of the photodetector.

Taking Measurements

- Turn on the power to the two power supplies located on the optical table. The larger one should read 15 volts on both displays; the other 10 volts.
- Now run the `stringWatch` script located in the `C:\CAAMlab\Scripts` directory.
- You should now see one plot. The two lines represent the displacements in either direction as a function of time. The displacement measurements are in arbitrary units. Tension in the string is indicated at the bottom of the plot.
- Now using the micrometer dials, center the string in the sensor moving both traces such that they are centered at zero.
- Unlock the collets by turning their handles, increase tension to the desired amount (between 150 and 220 Newtons is safe), then lock the collets back. Note there will be some ‘creep’ in the string, illustrated by a constantly decreasing tension for a short time. Let this effect settle down before taking measurements.
• Practice plucking the string so that vibrations stay bounded within the ±10 in displacement and do not exhibit noise. Clipping, when the string travels outside of the range of the sensor, will create unwanted artifacts in the Fourier transform. Noise can also occur if the vibrations get too close to the boundary; this will manifest itself in jagged lines when the string approaches the boundary. If you have a hard time keeping the vibrations within the limits, increase the tension on the string, or pluck more gently.

• Stop the **scope** script by pressing `Control-C`.

• Now that you have practiced plucking, you are prepared to record a pluck. Recall that a rule of thumb for the resolution of a Fourier transform is that the Fourier transform of a signal of length $T$ will have $T$ points per Hertz. Since most of the features we want to resolve are between 10 and 200 Hertz, we require a sample that covers something like 10 or 100 seconds. We advise a sample rate of $5 \times 10^4$ samples per second, which is fast enough to prevent aliasing problems. Record the data using the command

\[
[x, sr] = \text{getData}(5e4, 10);
\]

Remember the semicolon or else you will have quite a few numbers spewing forth.

• **Analyzing the data.** As we saw in the last lab, the peaks in the Fourier transform relate to the eigenvalues of the system. To obtain a plot of the Fourier transform, try the following:

\[
samp\_per\_sec = 5e4;
\]
\[
duration = 10;
\]
\[
[x, sr] = \text{getData}(samp\_per\_sec, duration);
\]
\[
N = \text{length}(x);
\]
\[
\text{semilogy}([0:N-1]/duration, \text{abs}(\text{fft}(x)));
\]
\[
xlim([0 200]);
\]

Noise in the Fourier transform comes in the form of a jittery magnitudes. Typically the first channel (blue) is cleaner than the second (green), an artifact of the particular photodetectors used in our assembly.

• **Which peaks?** Our measurements do not actually reflect a discretely beaded string, but a continuous string with non-uniform mass distribution. Because of that, we see many more peaks than we expect. Peaks coming from the beaded component tend to have larger magnitudes and wider bases (because they decay). However, one good way to tell which peaks are the right ones is to cheat: compare the experimental data to the frequencies you predict from the eigenvalues of $M^{-1}K$.

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Miscellaneous Notes on Using the String Lab

- Please do not move the collet plates, as they are delicately placed.
- The current string length is 45.25 inches.
- If your string breaks and another is not available, contact the lab assistant who will make another one. (The string diameter so closely fits the beads that the crimping that inevitably occurs when you cut a string must be milled off.)
- Occasionally MATLAB will pick up an error when you close stringWatch, and this will prevent you from using any other data acquisition script. If this happens, restart MATLAB.

Laboratory Instructions

1. Select $n = 2$ beads, weigh them on the lab scale, (confirming the measurement on the box) set them up as described above.
2. Pluck the string several times, each time recording the eigenvalues and the fundamental frequencies obtained from the getData script.
3. Compare these values to the eigenvalues that you would expect from the mathematical derivation described above.
4. Repeat steps 1–3 for $n = 4$ beads.