CAAM 336 · DIFFERENTIAL EQUATIONS IN SCI AND ENG

Fall 2017

Examination 2

Instructions:

1. Time limit: 3 uninterrupted hours.

2. There are three questions worth a total of 100 points. Please do not look at the questions until you begin the exam.

3. Please answer the questions throughly (but succinctly!) and justify all your answers. Show your work for partial credit.

4. Each problem should have its own dedicated page.

5. Print your name on the line below:

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

7. Staple this page to the front of your exam.
1. [12 points]

(a) [3 points] Let \( V \) denote a vector space with an inner product \((\cdot, \cdot)\). Define what it means for the set of vectors \( \{v_1, \ldots, v_n\} \) in \( V \) to be mutually orthogonal.

(b) [6 points] Let \( W \) denote the vector space with inner product \( (u, v) = \int_{-1}^{1} u(x)v(x)dx \) spanned by the set of functions \( \{1, x, x^2\} \). Create a mutually orthogonal set of functions that span \( W \).

(c) [3 points] Verify that the functions you created in part (b) are mutually orthogonal.

Solution:

(a) \( \{v_1, \ldots, v_n\} \) is mutually orthogonal if both the following are satisfied:

(i) \( (v_i, v_j) = 0 \) for all \( j \neq j \in \{1, \ldots, n\} \);

(ii) \( (v_i, v_i) \neq 0 \) for all \( i \in \{1, \ldots, n\} \).

(b) Apply Gram-Schmidt process to \( \{v_1 = 1, v_2 = x, v_3 = x^2\} \):

\[
\hat{v}_1 = v_1 = 1;
\]

\[
\hat{v}_2 = v_2 - \frac{(v_2, \hat{v}_1)}{(\hat{v}_1, \hat{v}_1)} \hat{v}_1 = x - \frac{\int_{-1}^{1} x \cdot 1 dx}{\int_{-1}^{1} 1 \cdot 1 dx} = x;
\]

\[
\hat{v}_3 = v_3 - \frac{(v_3, \hat{v}_2)}{(\hat{v}_2, \hat{v}_2)} \hat{v}_2 - \frac{(v_3, \hat{v}_1)}{(\hat{v}_1, \hat{v}_1)} \hat{v}_1 = x^2 - \frac{\int_{-1}^{1} x^2 \cdot 1 dx}{\int_{-1}^{1} 1 \cdot 1 dx} = x^2 - \frac{1}{3};
\]

\( \{1, x, x^2 - \frac{1}{3}\} \) is a mutually orthogonal set of functions spanning \( W \).

(c) Check by calculating the inner products:

\[
(\hat{v}_1, \hat{v}_2) = \int_{-1}^{1} 1 \cdot x dx = \frac{1}{2} x^2 \bigg|_{-1}^{1} = 0
\]

\[
(\hat{v}_1, \hat{v}_3) = \int_{-1}^{1} 1 \cdot \left(x^2 - \frac{1}{3}\right) dx = \frac{1}{3} x^3 - \frac{1}{3} x \bigg|_{-1}^{1} = 0
\]

\[
(\hat{v}_2, \hat{v}_3) = \int_{-1}^{1} x \cdot \left(x^2 - \frac{1}{3}\right) dx = x^4 - \frac{1}{3} x^2 \bigg|_{-1}^{1} = 0
\]

(By symmetry of inner product, we have \( (\hat{v}_2, \hat{v}_1) = (\hat{v}_3, \hat{v}_1) = (\hat{v}_3, \hat{v}_2) = 0 \).)

\[
(\hat{v}_1, \hat{v}_1) = \int_{-1}^{1} 1 \cdot 1 dx = x \bigg|_{-1}^{1} = 2 \neq 0
\]

\[
(\hat{v}_2, \hat{v}_2) = \int_{-1}^{1} x \cdot x dx = \frac{1}{3} x^3 \bigg|_{-1}^{1} = \frac{2}{3} \neq 0
\]

\[
(\hat{v}_3, \hat{v}_3) = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right) \cdot \left(x^2 - \frac{1}{3}\right) dx = \frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{1}{9} x \bigg|_{-1}^{1} = \frac{8}{45} \neq 0
\]
2. [20 points]

Consider the Neumann boundary value problem

\[-\frac{d^2 u}{dx^2} = f(x) \quad 0 < x < 1 \]
\[u'(0) = a \quad u'(1) = b \]
\[\int_0^1 u(x)dx = 0 \]

(1)

where \(a, b \in \mathbb{R}\) are constants.

The differential operator associated with this problem when \(a = b = 0\) is \(L_A u = -\frac{d^2 u}{dx^2}\) where \(L_A : C_A^2[0, 1] \rightarrow C[0, 1]\) and

\[C_A^2[0, 1] = \{u \in C^2[0, 1], u'(0) = u'(1) = 0, \int_0^1 u(x)dx = 0\}\]

(a) [4 points] Prove that \(L_A\) is a symmetric operator with respect to the \(L^2\) inner product and that \(L_A\) has positive eigenvalues.

(b) [4 points] \(\phi_j(x) = \cos(j\pi x)\) and \(\lambda_j = j\pi^2\) for \(j = 1, \ldots\) are the eigenfunctions and corresponding eigenvalues for this operator. Does the solution to the boundary value problem exist when \(f(x) = \sin(\pi x)\)? Justify your response.

(c) [8 points] Consider equation (1) when \(a \neq 0\) and \(b \neq 0\). Use the spectral method to write the general solution to the Neumann boundary value problem.

**Hint:** If you are struggling to start this problem, start problem 3.

(d) [4 points] Use the spectral method to solve the Neumann boundary value problem (1) when \(f(x) = \cos(2\pi x) - 1\), \(a = 1\) and \(b = 2\).

**Additional information:**
\[\cos^2(v) = \frac{1 + \cos(2v)}{2}\]
\[\cos(v)\cos(w) = \frac{1}{2}(\cos(v - w) + \cos(v + w))\]

**Solution:**

(a) Proof of symmetry:

\[(L_A u, v) = (-u''(x), v(x))\]
\[= -\int_0^1 u''(x)v(x) dx\]
\[= \int_0^1 u'(x)v'(x) dx - u'(x)v(x)|_0^1, \text{ integration by parts applied}\]
\[= \int_0^1 u'(x)v'(x) dx, \text{ boundary conditions applied}\]
\[= -\int_0^1 u(x)v''(x) dx + u(x)v'(x)|_0^1, \text{ integration by parts applied}\]
\[= -\int_0^1 u(x)\frac{d}{dx}(u)(x) dx, \text{ boundary conditions applied}\]
\[= (u(x), -v''(x)) = (u, L_A v)\]
Proof of all eigenvalues being positive: Assume \((\lambda, v)\) is an eigenpair of \(L_A\).

Then
\[
(L_A v, v) = (\lambda v, v) = \lambda \int_0^1 v^2(x) \, dx
\]  

(2)

Note that an eigenvector must be a nonzero vector, and thus \(\int_0^1 v^2(x) \, dx > 0\).

We can rewrite the left-hand-side as
\[
(L_A v, v) = -\int_0^1 v''(x)v(x) \, dx
\]
\[
= \int_0^1 [v'(x)]^2 \, dx - v'(x)v(x)|_0^1, \text{ integration by parts applied}
\]
\[
= \int_0^1 [v'(x)]^2 \, dx, \text{ boundary conditions applied}
\]

Equation (2) then becomes
\[
\int_0^1 [v'(x)]^2 \, dx = \lambda \int_0^1 v^2(x) \, dx.
\]  

(3)

Dividing both sides by \(\int_0^1 v^2(x) \, dx > 0\) yields
\[
\frac{\lambda}{\int_0^1 v^2(x) \, dx} = 0.
\]  

(4)

Now, assume that 0 is an eigenvalue, i.e. there exists \(v \neq 0\) such that \(L_A v = 0v = 0\).

Note that the general solution for \(\frac{d^2u}{dx^2} = 0\) is linear function \(v(x) = C_1x + C_2\). Plugging in the boundary conditions gives: \(v'(0) = C_1 = 0\) and \(\int_0^1 C_2 \, dx = C_2 = 0\). Thus, \(v(x) \equiv 0\). We have a contradiction.

Hence, zero cannot be an eigenvalue and all eigenvalues are positive.

(b) No. \(f(x)\) is not in the space spanned by the eigenfunctions.

(c) We first find a function \(p(x)\) of the format \(p(x) = C_1x^2 + C_2x + C_3\) satisfying the inhomogeneous boundary conditions.

\[
p'(0) = C_2 = a
\]
\[
p'(1) = 2C_1 + C_2 = b \implies C_1 = \frac{1}{2}(b - a)
\]
\[
\int_0^1 p(x) \, dx = \int_0^1 \frac{1}{2}(b - a)x^2 + ax + C_3 \, dx = 0 \implies C_3 = -\frac{1}{6}b - \frac{1}{3}a
\]

Thus, \(p(x) = \frac{1}{2}(b - a)x^2 + ax + \left(-\frac{1}{6}b - \frac{1}{3}a\right)\).

The general solution to the Neumann boundary value problem can then be expressed as \(u(x) = w(x) + p(x)\), where \(w(x)\) solve the homogeneous boundary value problem defined as
\[
-\frac{d^2w}{dx^2} = \frac{d^2p}{dx^2} = f(x) + (b - a) \quad 0 < x < 1
\]
\[
w'(0) = u'(0) - p'(0) = a - a = 0
\]
\[
w'(1) = u'(1) - p'(1) = b - b = 0
\]
\[
\int_0^1 w(x) \, dx = \int_0^1 u(x) \, dx - \int_0^1 p(x) \, dx = 0 - 0 = 0.
\]
Define $\tilde{f}(x)$ as $\tilde{f}(x) := f(x) + (b - a)$, we can solve

$$\begin{align*}
-d^2w & = \tilde{f}(x) \quad 0 < x < 1 \\
w'(0) &= w'(1) = 0 \\
\int_0^1 w(x) dx &= 0
\end{align*}$$  \hspace{1cm} (6)

via the spectral method:

$$w(x) = \sum_{j=1}^{\infty} \beta_j \phi_j(x), \hspace{1cm} (7)$$

where

$$\beta_j = \frac{1}{\lambda_j} \left( \phi_j, \tilde{f} \right).$$

Hence, the general solution to the Neumann boundary value problem would be

$$u(x) = w(x) + p(x) = \frac{1}{2} (b - a)x^2 + ax - \frac{1}{6} b - \frac{1}{3} a + \sum_{j=1}^{\infty} \beta_j \phi_j(x). \hspace{1cm} (8)$$

(d) With $a = 1$ and $b = 2$, we have $p(x) = \frac{1}{2} x^2 + x - \frac{2}{3}$ and $\tilde{f}(x) = f(x) + 1 = \cos(2\pi x) - 1 + 1 = \cos(2\pi x)$.

The coefficients in the expansion are

$$\beta_j = \frac{1}{\lambda_j} \left( \phi_j, \tilde{f} \right) = \frac{1}{j\pi^2} \int_0^1 \cos(j\pi x) \cos(2\pi x) \, dx \int_0^1 \cos^2(j\pi x) \, dx$$

$$= \begin{cases} \frac{1}{2\pi^2}, & \text{if } j = 2 \\ 0, & \text{otherwise}. \end{cases}$$

Hence, the general solution to the Neumann boundary value problem is

$$u(x) = \frac{1}{2} x^2 + x - \frac{2}{3} + \frac{1}{2\pi^2} \cos(2\pi x). \hspace{1cm} (9)$$
3. [18 points]

Consider the following BVP with inhomogeneous boundary conditions:

\[-((1 + x^2)u')' = f(x), \quad 0 < x < 1,\]
\[u(0) = a,\]
\[u(1) = b\]

where \(a, b \in \mathbb{R}\) are constants.

(a) [5 points] The solution \(u(x)\) can be represented as \(u(x) = s(x) + g(x)\) where \(s(x) \in C^2_D[0,1]\) and \(g(x)\) is the lift function guaranteeing \(u(x)\) satisfies the boundary conditions. Derive the variational (weak) problem for \(s(x)\).

(b) [2 points] Let \(f(x) = x, \ a = 1\) and \(b = 2\). Then \(g(x) = 1 + x\) is an appropriate choice of lift function.

By approximating \(s(x)\) by \(s(x) \sim \sum_{j=1}^{n-1} \alpha_j \phi_j(x)\) where \(\{\phi_j\}_{j=1}^{n-1}\) denotes the basis for the discrete approximation space \(V_{n-1}\), write down the entries for the linear system whose solutions are \(\alpha_j\).

(c) [11 points] Let \([0,1]\) have a uniform mesh such that \(x_j = jh\) where \(h = 1/n\). For \(n = 3\), find \(u(x)\) when the basis for \(V_{n-1}\) is the hat functions.

Recall the hat function \(\phi_i(x)\) is defined as

\[
\phi_i(x) = \begin{cases} 
\frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{for } x_{i-1} < x \leq x_i \\
\frac{x_i-x}{x_{i+1}-x_i} & \text{for } x_i < x < x_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

Additional information: \[\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\]

Solution:

(a) The lift function \(g(x)\) will satisfy the inhomogeneous boundary conditions, and \(s(x)\) will solve the following homogeneous boundary value problem:

\[-((1 + x^2)s')' = -((1 + x^2)u')' + ((1 + x^2)g')' = f(x) + ((1 + x^2)g')', \quad 0 < x < 1,\]
\[s(0) = 0,\]
\[s(1) = 0.\]

Multiplying both sides by a test function \(v(x) \in C^2_D[0,1]\) and integrating w.r.t. \(x\) yields

\[
\int_0^1 -((1 + x^2)s'(x))'v(x) \, dx = \int_0^1 f(x)v(x) \, dx + \int_0^1 ((1 + x^2)g'(x))'v(x) \, dx
\]

\[
\int_0^1 (1 + x^2)s'(x)v'(x) \, dx - (1 + x^2)v(x)s'(x)|_0^1 = \int_0^1 f(x)v(x) \, dx
\]

\[
-\int_0^1 (1 + x^2)g'(x)v'(x) \, dx + (1 + x^2)v(x)g'(x)|_0^1
\]

\[
\int_0^1 (1 + x^2)s'(x)v'(x) \, dx = \int_0^1 f(x)v(x) \, dx - \int_0^1 (1 + x^2)g'(x)v'(x) \, dx
\]
Therefore the weak formulation is: find \( s \in C^2_D[0,1] \) such that

\[
\int_0^1 (1+x^2)s'(x)v'(x) \, dx = \int_0^1 f(x)v(x) \, dx - \int_0^1 (1+x^2)g'(x)v'(x) \, dx,
\]
for any \( v \in C^2_D[0,1] \).

(b) Plugging \( s_h(x) = \sum_{i=1}^{n-1} \alpha_i \phi_i(x) \) and \( v_h = \phi_j(x) \) into the weak formulation yields

\[
\sum_{i=1}^{n-1} \int_0^1 \alpha_i (1+x^2)\phi_i'(x)\phi_j'(x) \, dx = \int_0^1 f(x)\phi_j(x) \, dx - \int_0^1 (1+x^2)g'(x)\phi_j'(x) \, dx
\]

Then the linear systems will look like:

\[
A = (A_{i,j})_{i,j=1}^{n-1} = \left( \int_0^1 (1+x^2)\phi_i'(x)\phi_j'(x) \, dx \right)_{i,j=1}^{n-1},
\]

\[
\alpha = (\alpha_j)_{j=1}^{n-1},
\]

\[
b = (b_j)_{j=1}^{n-1} = \left( \int_0^1 f(x)\phi_j(x) \, dx - \int_0^1 (1+x^2)g'(x)\phi_j'(x) \, dx \right)_{j=1}^{n-1} = \left( \int_0^1 x\phi_j(x) \, dx - \int_0^1 (1+x^2)\phi_j'(x) \, dx \right)_{j=1}^{n-1}
\]

(c) We first evaluate the entries of \( A \):

We know if \(|i - j| \geq 2\), the support of \( \phi_i \) and \( \phi_j \) will not overlap, thus \( A_{i,j} = 0 \) for any \(|i - j| \geq 2\).

- The \( i = j \) case:

\[
A_{i,i} = \int_0^1 (1+x^2)\left(\phi_i'(x)\right)^2 \, dx
\]

\[
= \frac{1}{h^2} \left\{ \int_{x_{i-1}}^{x_i} (1+x^2) \, dx + \int_{x_i}^{x_{i+1}} (1+x^2) \, dx \right\}
\]

\[
= \frac{1}{h^2} \left\{ \int_{x_{i-1}}^{x_i} 1 \, dx + \int_{x_i}^{x_{i+1}} 1 \, dx + \int_{x_{i-1}}^{x_i} x^2 \, dx + \int_{x_i}^{x_{i+1}} x^2 \, dx \right\}
\]

\[
= \frac{1}{h^2} \left\{ \int_{x_{i-1}}^{x_{i+1}} 1 \, dx + \int_{x_{i-1}}^{x_{i+1}} x^2 \, dx \right\}
\]

\[
= \frac{2}{h} + \frac{2x_i^2}{h} + \frac{2}{3} h = \frac{2}{h} + \left( 2l^2 + \frac{2}{3} \right) h
\]

For \( n = 3 \): \( A_{1,1} = \frac{62}{9} \) and \( A_{2,2} = \frac{80}{9} \).
– The $|i-j|=1$ case:

$$A_{i,i+1} = \int_{0}^{1} (1 + x^2) \phi_i'(x) \phi_{i+1}'(x) \, dx$$

$$= -\frac{1}{h^2} \int_{x_i}^{x_{i+1}} (1 + x^2) \, dx$$

$$= -\frac{1}{h^2} \left\{ x + \frac{1}{3} x^3 \right\}_{x_i}^{x_{i+1}} = -\frac{1}{h} - \frac{1}{3h^2} (x_{i+1}^3 - x_i^3) = -\frac{1}{h} \left( i^2 + i + \frac{1}{3} \right) h$$

$$A_{i,i-1} = \int_{0}^{1} (1 + x^2) \phi_i'(x) \phi_{i-1}'(x) \, dx$$

$$= -\frac{1}{h^2} \int_{x_{i-1}}^{x_i} (1 + x^2) \, dx$$

$$= -\frac{1}{h^2} \left\{ x + \frac{1}{3} x^3 \right\}_{x_{i-1}}^{x_i} = -\frac{1}{h} - \frac{1}{3h^2} (x_i^3 - x_{i-1}^3) = -\frac{1}{h} \left( i^2 - i + \frac{1}{3} \right) h$$

For $n=3$: $A_{1,2} = -3 - \frac{7}{9} = -\frac{34}{9}$ and $A_{2,1} = -3 - \frac{7}{9} = -\frac{34}{9}$.

We then evaluate the entries of $b$:

$$b_j = \int_{0}^{1} x \phi_j(x) \, dx - \int_{0}^{1} (1 + x^2) \phi_j'(x) \, dx$$

$$= \frac{1}{h} \left\{ \int_{x_{j-1}}^{x_j} x(x-jx_{j-1}) - (1 + x^2) \, dx + \int_{x_j}^{x_{j+1}} x(x_{j+1}-x) + (1 + x^2) \, dx \right\}$$

$$= \frac{1}{h} \left\{ \int_{x_j}^{x_{j+1}} (1 + x^2) \, dx - \int_{x_{j-1}}^{x_j} (1 + x^2) \, dx + \int_{x_{j-1}}^{x_j} x(x-x_{j-1}) \, dx + \int_{x_j}^{x_{j+1}} x(x_{j+1}-x) \, dx \right\}$$

$$= \frac{1}{h} \left\{ 2x_j h^2 + x_j h^2 \right\} = 3j h^2$$

For $n=3$: $b_1 = \frac{1}{3}$ and $b_2 = \frac{2}{3}$.

Thus

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 62/9 & -34/9 \\ -34/9 & 80/9 \end{bmatrix}^{-1} \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 37/317 \\ 79/634 \end{bmatrix}$$

Thus

$$s(x) \approx \alpha_1 \phi_1(x) + \alpha_2 \phi_2(x)$$

and

$$u(x) = s(x) + g(x) \approx \alpha_1 \phi_1(x) + \alpha_2 \phi_2(x) + 1 + x$$