Instructions:

1. Time limit: **3 hours**
2. There are four questions worth a total of 100 points.

3. Materials: You are allowed to use any reference material *provided to you* by the instructor. This exam was written assuming you would not have access to a calculator; *you may not use* a non-programmable, non-graphing scientific calculator. If you do so, **do not** give your final answers in decimal notation. You may **not use** any other materials, books, devices, or computer programs.

4. Answer format: Please answer the questions thoroughly (but succinctly!) and justify all your answers; start your answer to each problem on a new sheet of paper. For each part of each problem: **you must box the final answer you wish to have scored.** Please **staple** your entire submission together before submitting.

5. Partial credit: Please **check your work!** Partial credit *can* be awarded at the discretion of the instructor but the availability of partial credit *should not be assumed.* Writing *neatly and clearly* is strongly encouraged.

6. Print **your name** on the line below:

6. Print your instructors name on the line below:

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

6. Staple this page to the front of your exam.
1. [24 points] Consider the fourth-order linear operator $L : C^4_H[0, 1] \rightarrow C[0, 1]$,

$$Lu = uxxxx - uxx,$$

where $C^4_H[0, 1]$ consists of all $C^4[0, 1]$ functions $v$ with hinged boundary conditions

$$v(0) = v''(0) = v(1) = v''(1) = 0.$$ 

It can be shown that $N(L) = 0$ (the nullspace of $L$ is trivial) in $C^4_H[0, 1]$. Here we use the inner product

$$(f, g) = \int_0^1 f(x)g(x) \, dx.$$ 

(a) 5 pts — Show that the eigenfunctions of $L$ are $\psi_k(x) = \sqrt{2} \sin(k\pi x)$, for $k = 1, 2, 3, \ldots$.

Find the corresponding eigenvalues $\lambda_k$ of $L$.

(b) 14 pts — Compute the spectral method solution to the following homogeneous problem:

$$Lu = x$$

$$u(0) = 0 \quad u(1) = 0$$

$$u''(0) = 0 \quad u''(1) = 0$$

(c) 5 pts — Compute the spectral method solution to the following inhomogeneous problem:

$$Lu = x$$

$$u(0) = 1 \quad u(1) = 4$$

$$u''(0) = 0 \quad u''(1) = 0$$

Hint: Look for a polynomial in the null space of $L$ in $C^4_M[0, 1]$ where $C^4_M[0, 1]$ consists of functions $v$ in $C^4[0, 1]$ with the following boundary conditions

$$v(0) = a \quad v(1) = b \quad v''(0) = v''(1) = 0.$$ 

2. [21 points]

(a) 5 points — Consider the Neumann boundary value problem

$$-\frac{\partial^2 u}{\partial x^2} = f(x)$$

$$u'(0) = 0$$

$$u'(1) = 0.$$ 

If we define $L_Nu = -\frac{\partial^2 u}{\partial x^2}$ and the space $C^2_N[0, 1]$

$$C^2_N[0, 1] = \{ u \in C^2[0, 1], u'(0) = u'(1) = 0 \},$$

this can be written as an operator equation

$$L_Nu = f, \quad L : C^2_N[0, 1] \rightarrow C[0, 1].$$

Are solutions to (1) unique? Why or why not? Fully justify your answer.
(b) 16 pts — Consider a slightly different operator, by imposing an extra condition on the vector subspace $C^2_A[0,1]$, defined by

$$L_A u = -\frac{\partial^2 u}{\partial x^2}, \quad L_A : C^2_A[0,1] \rightarrow C[0,1].$$

where $C^2_A[0,1]$ contains functions in $C^2_N[0,1]$ with zero average

$$C^2_A[0,1] = \left\{ u \in C^2[0,1], u'(0) = u'(1) = 0, \int_0^1 u(x)dx = 0 \right\}.$$ 

Consider the boundary value problem

$$L_A u = f$$

Recall from class that an orthonormal set of eigenvectors for $L_N$ are $\psi_n(x) = \sqrt{2}\cos(n\pi x)$ for $n = 0, 1, 2, \ldots$ and the corresponding eigenvalues are $\lambda_n = n^2\pi^2$. Since the boundary conditions have not changed, the eigenvectors of $L_A$ are precisely those of $L_N$ satisfying the additional condition $\int_0^1 \psi(x) dx = 0$.

Use this information to solve the following boundary value problem using the spectral method

$$-\frac{\partial^2 u}{\partial x^2} = x - \frac{1}{2}$$

$$u'(0) = 0$$

$$u'(1) = 0$$

$$\int_0^1 u = 0$$

That is, solve the equation $L_A u = x - \frac{1}{2}$ for $u(x) \in C^2_A[0,1]$ using the spectral method.

3. [26 points] Considers the time-independent variable diffusitivity model problem:

$$-\frac{\partial}{\partial x} \left( k(x) \frac{\partial u}{\partial x} \right) = f$$

Assume that $k(x) > 0$ on $[0,1]$. A weak formulation for problem (3) is

Find $u(x) \in V_0$ such that for every $v_h \in V_0$

$$a(u,v) = \int_0^1 f(x)v(x)$$

where the space $V_0$ is defined by:

$$V_0 = \left\{ f \in L^2[0,1] \mid \frac{\partial f}{\partial x} \in L^2[0,1] \right\}$$

and $L^2[0,1]$ is the set of all functions with $\int_0^1 |f|^2 \, dx < \infty$. If $V_n$ is any finite dimensional sub-vector space of $V_0$ a discrete weak problem can be formulated as:

Find $u_n(x) \in V_n$ such that for every $v_h \in V_n$

$$a(u_n,v_n) = \int_0^1 f(x)v_n(x)$$
Let $x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4, x_4 = 1$ be a uniform partition of $[0, 1]$, having three internal points, with mesh size $h = 1/4$. Let $V_3$ be the space $V_3 = \text{span} \{\psi_1, \psi_2, \psi_3\}$ where $\psi_i$ is the $i^{th}$ linear hat function; recall that $\psi_i(x)$ is linear on each interval and satisfies $\psi_i(x_i) = 1$ and $\psi_i(x_j) = 0$ when $i \neq j$. The general formula for the $i^{th}$ hat function is:

\[
\psi_i(x) = \begin{cases} 
\frac{x-x_{i-1}}{h} & x \in [x_{i-1}, x_i] \\
\frac{x_{i+1}-x}{h} & x \in [x_i, x_{i+1}] \\
0 & \text{otherwise}
\end{cases}
\]

(a) 20 pts — Write the linear system $A\alpha = F$ that arises from the finite element method, using the hat function space $V_3$, from the boundary value problem:

\[-\frac{\partial}{\partial x} \left((1 + x) \frac{\partial}{\partial x} u\right) = 1 \]

$u(0) = u(1) = 0$

(b) 6 pts — Write the linear system $A\alpha = F$ that arises from the finite element method, using the hat function space $V_3$, from the boundary value problem:

\[-\frac{\partial}{\partial x} \left((1 + x) \frac{\partial}{\partial x} u\right) = 1 \]

$u(0) = 1, u(1) = 3$

4. [29 points] Consider the equation Euler Bernoulli beam equation,

\[\frac{\partial^2}{\partial x^2}(k(x) \frac{\partial^2 u}{\partial x^2}) = f(x), \quad 0 < x < 1,\]

Here $k(x) > 0$ and the boundary conditions are $u(0) = u(1) = 0$ With these boundary conditions, the eigenvalues and eigenvectors of this operator are difficult to compute, even if $k(x) = 1$. Instead consider the finite element approximation of this problem which is associated with a weak problem of the form:

Find $u \in V_0^2$ such that

\[a(u, v) = (f, v); \quad \text{for all } v \in V_0^2,\]

where

\[V_0^2 = \left\{ u \in L^2[0, 1] | \frac{\partial u}{\partial x} \in L^2[0, 1] \text{ and } \frac{\partial^2 u}{\partial x^2} \in L^2[0, 1], \; u(0) = u(1) = u'(0) = u'(1) = 0 \right\}\]

(a) 10 pts — For any $v \in V_0^2$, derive that the bilinear form for the weak formulation of this problem is given by :

\[a(u, v) = \int_0^1 \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} dx.\]

Show all your work.

NOTE: $a(\cdot, \cdot)$ is an inner product on $V_0^2$. (You do not need to show this!)
(b) 4 pts — To produce a linear system $A\alpha = F$ of the weak problem, a finite dimensional subspace of $V_0^2$ is needed. Show that the piecewise hat functions are not elements in $V_0^2$.

(c) 15 pts — Consider the beam problem on the interval $[0,1]$. Take a simple mesh with one internal node; i.e. $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_3 = 1$. Define the finite dimensional space $V_h \subset V_0^2$ to be the space spanned by the functions:

$$\psi_1(x) = \begin{cases} 4x^2(3 - 4x) & x \in [0, \frac{1}{2}] \\ 4(4x^3 - 9x^2 + 6x - 1) & x \in [\frac{1}{2}, 1] \end{cases}$$

$$\psi_2(x) = \begin{cases} 4(2x^3 - x^2) & x \in [0, \frac{1}{2}] \\ 4(2x^3 - 5x^2 + 4x - 1) & x \in [\frac{1}{2}, 1] \end{cases}$$

Take $f(x) = 1$ in the boundary value problem. Find the linear linear system resulting from the discrete weak problem:

Find $u_h(x) \in V_h$ such that for all $v_h(x) \in v_h$

$$a(u_h, v_h) = \int_0^1 v_h \, dx.$$