Instructions:

1. Time limit: **3 hours**

2. Materials: You are allowed to use any reference material provided to you by the instructor. This exam was written assuming you would not have access to a calculator; you may not use a non-programmable, non-graphing scientific calculator. If you do so, do not give your final answers in decimal notation. You may not use any other materials, books, devices, or computer programs.

3. Answer format: Please answer the questions throughly (but succinctly!) and justify all your answers; start your answer to each problem on a new sheet of paper. For each part of each problem: you must box the final answer you wish to have scored. Please staple your entire submission together before submitting.

4. Partial credit: Please check your work! Partial credit can be awarded at the discretion of the instructor but the availability of partial credit should not be assumed. Writing neatly and clearly is strongly encouraged.

5. Print your name on the line below:

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

7. Staple this page to the front of your exam.
1. [X points] Consider the fourth-order linear operator $L : C_H^4[0, 1] \to C[0, 1]$,

$$Lu = u_{xxxx} - u_{xx},$$

where $C_H^4[0, 1]$ consists of all $C^4[0, 1]$ functions $v$ with hinged boundary conditions

$$v(0) = v''(0) = v(1) = v''(1) = 0.$$ 

It can be shown that $N(L) = 0$ (the nullspace of $L$ is trivial) in $C_H^4[0, 1]$. We use the standard $L^2$ inner product

$$(f, g) = \int_0^1 f(x)g(x) \, dx.$$ 

(a) The eigenvectors of $L$ are $\psi_k(x) = \sqrt{2}\sin(k\pi x)$, for $k = 1, 2, 3, \ldots$. Find the corresponding eigenvalues $\lambda_k$ of $L$.

(b) Compute the spectral method solution to the following homogeneous problem:

$$Lu = x$$

$$u(0) = 0 \quad u(1) = 0$$

$$u'(0) = 0 \quad u'(1) = 0$$

(c) Compute the spectral method solution to the following inhomogeneous problem:

$$Lu = x$$

$$u(0) = 1 \quad u(1) = 4$$

$$u'(0) = 3 \quad u'(1) = 3$$

2. [Z points]

(a) [2 points] Consider the boundary value problem

$$-\frac{\partial^2 u}{\partial x^2} = f(x)$$

$$u'(0) = 0$$

$$u'(1) = 0.$$ 

If we define $L_N u = -\frac{\partial^2 u}{\partial x^2}$ and the space $C_N^2[0, 1]

$$C_N^2[0, 1] = \{ u \in C^2[0, 1], u'(0) = u'(1) = 0 \},$$

this can be written as an operator equation

$$L_N u = f, \quad L : C_N^2[0, 1] \to C[0, 1].$$

Recall from class that an orthonormal set of eigenvectors for $L_N$ are $\psi_n(x) = \sqrt{2}\cos(n\pi x)$ for $n = 0, 1, 2, \ldots$ and the corresponding eigenvalues are $\lambda_n = n^2\pi^2$

Q: Are solutions to (1) unique? Why or why not? Fully justify your answer.
(b) [10 points] Consider a slightly different operator, by imposing an extra condition on the vector subspace $C_N^2[0,1]$, defined by

$$L_A u = -\frac{\partial^2 u}{\partial x^2}, \quad L_A : C_A^2[0,1] \to C[0,1].$$

where $C_A^2[0,1]$ contains functions in $C_N^2[0,1]$ with zero average

$$C_A^2[0,1] = \left\{ u \in C^2[0,1], u'(0) = u'(1) = 0, \int_0^1 u(x)dx = 0 \right\}.$$

Consider the boundary value problem

$$L_A u = f$$

Note that if $f(x)$ is in the range of $L_A$ then, by definition, there exists $u(x) \in C_A^2[0,1]$ such that $L_A u = f$. This implies that the average value of $f(x)$ on the interval $[0,1]$ must be zero since

$$\int_0^1 f(x) = \int_0^1 -\frac{\partial^2}{\partial x^2} u(x) = -(u'(1) - u'(0)) = 0$$

Where the fundamental theorem of calculus has been used in addition to the Neumann boundary conditions for $u(x) \in C_A^2[0,1]$. Since the boundary conditions have not changed, the eigenvectors of $L_A$ are precisely those of $L_N$ satisfying the additional condition $\int_0^1 \psi(x)dx = 0$.

Solve the following boundary value problem using the spectral method

$$\begin{align*}
-\frac{\partial^2}{\partial x^2} u &= x - \frac{1}{2} \\
u'(0) &= 0 \\
u'(1) &= 0 \\
\int_0^1 u &= 0
\end{align*}$$

That is, solve the equation $L_A u = x - \frac{1}{2}$ for $u(x) \in C_A^2[0,1]$ using the spectral method.

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**Solution.**

(a) No they are not unique. Constant functions are contained in $C_N^2[0,1]$ and they are in the nullspace of $L_N$. (i.e. solutions differing by a constant are still solutions)

(b) If $L_N \phi_j = \lambda \phi_j$, then $\phi_j'' + \lambda \phi_j = 0$, which implies that $\phi_j(x)$ should have the form

$$\phi_j(x) = A \sin(\sqrt{\lambda_j}x) + B \cos(\sqrt{\lambda_j}x)$$

Then,

$$\phi_j'(x) = A \sqrt{\lambda_j} \cos(\sqrt{\lambda_j}x) - B \sqrt{\lambda_j} \sin(\sqrt{\lambda_j}x).$$

The boundary condition $\phi_j'(0) = 0$ then implies that $A = 0$. Likewise, the boundary condition $\phi_j'(1) = 0$ implies that

$$\phi_j'(1) = B \sqrt{\lambda_j} \sin(\sqrt{\lambda_j}x) = 0$$
so that $\sqrt{\lambda_j} = j\pi$, and

$$\phi_0 = 1 \quad \lambda_0 = 0 \quad \phi_j(x) = \cos(j\pi x), \quad \lambda_j = (j\pi)^2 \quad j = 1, 2, \ldots$$

Orthonormalizing these eigenvectors gives

$$\tilde{\phi}_0 = 1 \quad \tilde{\phi}_j(x) = \sqrt{2} \cos(j\pi x), \quad j = 1, 2, \ldots$$

(c) Yes. The null space of $L_A$ is trivial and hence solutions, when they exist, will be unique. If $v(x) \neq 0$ is in the null space of $L_A$ then $v(x) = ax + b$ for some constants $a$ and $b$ and $v(x) \in C^2_A[0, 1]$. The boundary condition $u'(1) = 0$ implies $a = 0$ so that $v(x) = b$. The average condition $\int_0^1 v(x) = 0$ then gives $\int_0^1 b = b = 0$ so that $v(x) = 0$.

(d) The orthonormal eigenfunctions for $L_A$ and their corresponding eigenvalues of are identical to those derived in part (b) with the exception that $\tilde{\psi}_0 = 1$ is no longer an eigenfunction. I.e.

$$\tilde{\phi}_j(x) = \sqrt{2} \cos(j\pi x), \quad \lambda_j = (j\pi)^2, \quad j = 1, 2, \ldots$$

(e) Expanding $f(x) = \sum_{n=1}^\infty \beta_n \tilde{\psi}_n(x)$ and computing $\beta_n = (x - \frac{1}{2}, \tilde{\psi}_n)$ gives

$$\beta_n = \int_0^1 \sqrt{2} \cos(n\pi x) \left(x - \frac{1}{2}\right) = \frac{\sqrt{2}}{(n\pi)^2} (\cos(n\pi) - 1) = \frac{\sqrt{2}}{(n\pi)^2} ((-1)^n - 1)$$

Expanding $u(x)$ in terms of the eigenvectors, applying $L_A$, and using orthonormality gives

$$\lambda_m \alpha_m = \beta_m$$

So that

$$\alpha_m = \frac{\beta_m}{(m\pi)^2} = \frac{\sqrt{2}((-1)^n - 1)}{(n\pi)^4}$$

(Graders: the students may have put a 2 instead of a $\sqrt{2}$. If they did, and their answer to the overall problem is correct, please don’t take off anything. If their constant is incorrect and their answer to the overall problem is only off by that constant please count off only 1/2 a point). Therefore the solution is given by

$$\sum_{n=1}^\infty \frac{\sqrt{2}((-1)^n - 1)}{(n\pi)^4} \tilde{\psi}_n(x) = \sum_{n=1}^\infty \frac{2((-1)^n - 1)}{(n\pi)^4} \cos(n\pi x)$$

3. [W points] In class we discussed that one of the advantages of the finite element method over the spectral method was an ability to handle variable diffusivity functions $k(x)$. The remainder of this question considers the time-independent model problem:

$$-\frac{\partial}{\partial x} (k(x) \frac{\partial}{\partial x} u) = f$$

$$u(0) = u(1) = 0$$

Assume that $k(x) > 0$ on $[0, 1]$. A weak formulation for problem (3) can be expressed as

Find $u(x) \in V_0$ such that for every $v \in V_0$

$$a(u, v) = \int_0^1 f(x)v(x)$$

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Where the space \( V_0 \) is defined by:
\[
V_0 = \left\{ f \in L^2[0,1] \mid \frac{\partial f}{\partial x} \in L^2[0,1], f(0) = f(1) = 0 \right\}
\]
and \( L^2[0,1] \) is the set of all functions with \( \int_0^1 |f|^2 \, dx < \infty \). If \( V_h \) is any finite dimensional sub-vector space of \( V_0 \) a discrete weak problem can be formulated as:

Find \( u_h(x) \in V_h \) such that for every \( v_h \in V_h \)
\[
a(u_h, v_h) = \int_0^1 f(x)v_h(x)
\]

Let \( x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4, x_4 = 1 \) be a uniform partition of \([0,1]\), having three internal points, with mesh size \( h = 1/4 \). Let \( V_3 \) be the space of linear hat functions \( \{\psi_1, \psi_2, \psi_3\} \) satisfying \( \psi_i(x_i) = 1 \) and \( \psi_i(x_j) = 0 \) when \( i \neq j \). The general formula for the \( i^{th} \) hat function is:
\[
\psi_i(x) = \begin{cases} 
\frac{x - x_{i-1}}{h} & x \in [x_{i-1}, x_i] \\
\frac{x_{i+1} - x}{h} & x \in [x_i, x_{i+1}] \\
0 & \text{otherwise}
\end{cases}
\]

**Q:** Write the matrix system \( A\alpha = F \) that arises from the finite element method, using the hat function space \( V_3 \), from the boundary value problem:
\[
-\frac{\partial}{\partial x} \left( (1 + x) \frac{\partial u}{\partial x} \right) = 1
\]
\[
u(0) = 1, u(1) = 3
\]

**Solution.** GRADERS: The notation for different student answers may be different; ensure that they are equivalent. Keep in mind that the bilinear form \( a(u, v) \) is symmetric so it doesn’t matter what order the basis functions appear. They may have also explicitly written out what \( a(.,.) \) was; this is perfectly fine.

(a) i. \( a(u, v) = \int_0^1 k(x) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \)

ii. \( A_{ij} = a(\phi_j, \phi_i) \) and \( F_i = \int_0^1 f(x)\phi_i(x) \, dx = (f, \phi_i) \)

(b) GRADERS: the important part of this problem is that the students should notice or derive that the finite element method and the spectral method 4-basis-vector approximation will yield the same solution. They may do this in various ways; the following should receive full credit (1) if they give the general formula (5) or individually list the coefficients \( b_1, b_2, b_3 \) and \( b_4 \) correctly (2) formula (6) or (3) a verbal explanation that they are the same augmented by giving the form of the stiffness matrix and right-hand side vectors appearing in (4). The finite element method applied to the given weak problem yields a matrix equation of the form
\[
A\vec{b} = F
\]  
(4)

Where \( A \) is the \( 4 \times 4 \) matrix \( A_{ij} = a(\phi_j, \phi_i) \), \( \vec{b} \) is the column vector of unknown coefficients of the solution \( u_h(x) \) and \( F \) is the right-hand side vector \( F_i = \int_0^1 f(x)\phi_i(x) \, dx \). Given
the choice of basis functions for the problem we compute that
\[
A_{ij} = a(\phi_j, \phi_i) = \int_0^1 \frac{\partial \sqrt{2} \sin(j\pi x)}{\partial x} \frac{\partial \sqrt{2} \sin(i\pi x)}{\partial x} \, dx \\
= ij\pi^2 \int_0^1 \sqrt{2} \cos(j\pi x) \sqrt{2} \cos(i\pi x) \, dx \\
= \begin{cases} \\
\kappa(j\pi)^2 & j = i \\
0 & j \neq i \\
\end{cases}
\]

Where the last equation follows since the vectors \( \phi_n = \sqrt{2} \cos(n\pi x) \) for \( n = 1, 2, \ldots \) defines an orthonormal set of vectors (as discussed throughout the semester). Therefore the matrix \( A \) is diagonal and given by
\[
A = \begin{bmatrix} \\
\kappa\pi^2 & 0 & 0 & 0 \\
0 & \kappa4\pi^2 & 0 & 0 \\
0 & 0 & \kappa9\pi^2 & 0 \\
0 & 0 & 0 & \kappa16\pi^2 \\
\end{bmatrix}
\]

Since we have been asked to compare the final finite element solution to the solution we would get by applying the spectral method now is a good time to note that the \( j^{th} \) diagonal entry of the matrix \( A \) is exactly the corresponding eigenvalue for the eigenvector \( \phi_j = \sqrt{2} \sin(j\pi x) \) of the operator \( L = -\frac{\partial^2}{\partial x^2} \). The right-hand side vector is given by the formula
\[
F_i = \int_0^1 f(x) \sqrt{2} \sin(j\pi x) \, dx = (f, \phi_j)
\]

Since the matrix \( A \) is diagonal we can directly solve equation (4) for the coefficient vector \( \vec{b} \) to get
\[
b_j = \frac{(f, \phi_j)}{\kappa j^2\pi^2} \tag{5}
\]

So that
\[
u_h(x) = \sum_{j=1}^4 \frac{(f, \phi_j)}{\kappa j^2\pi^2} \phi_j(x) \tag{6}
\]

Which is exactly the same as the approximate spectral method solution in the case of using only the first four basis vectors; e.g.
\[
u_s(x) = \sum_{j=1}^4 \frac{(f, \phi_j)}{\kappa \lambda_j} \phi_j(x)
\]

4. [T points] Consider the equation Euler Bernoulli beam equation,
\[
\frac{\partial^2}{\partial x^2} (k(x) \frac{\partial^2 u}{\partial x^2}) = f(x), \quad 0 < x < 1,
\]

Here \( k(x) > 0 \) and the boundary conditions are \( u(0) = u(1) = 0 \) With these boundary conditions, the eigenvalues and eigenvectors of this operator are difficult to compute, even
if \( k(x) = 1 \). As a result, we will consider the finite element approximation of this problem. Solving the beam equation with the finite element method leads to a weak problem of the form:

$$ \text{Find } u \in V_0^2 \text{ such that} $$

$$ a(u, v) = (f, v); \quad \text{for all } v \in V_0^2, $$

where

$$ V_0^2 = \left\{ u \in L^2[0, 1] \mid \frac{\partial u}{\partial x} \in L^2[0, 1] \text{ and } \frac{\partial^2 u}{\partial x^2} \in L^2[0, 1], \; u(0) = u(1) = u'(0) = u'(1) = 0 \right\} $$

(a) Thoroughly derive that:

$$ a(u, v) = \int_0^1 \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} \, dx $$

is the bilinear form appropriate to frame the weak formulation of the beam problem (one can show that \( a(\cdot, \cdot) \) is an inner product on \( V_0^2 \) but you do not need to show this).

(b) To produce a matrix problem of the form \( A\alpha = F \) from the weak problem we need to select a finite dimensional subspace of \( V_0^2 \). What problem prevents us from using the piecewise linear hat functions that we used in class?

(c) We will use a very simple mesh and employ Hermite finite element functions to put forth a very basic approximation to the Euler beam equation boundary value problem. Consider the the interval \([0, 1]\), and take a mesh with one internal node; i.e. \( x_0 = 0, \; x_1 = \frac{1}{2}, \; x_3 = 1 \). Define the finite dimensional space \( V_h \subset V_0^2 \) to be the space spanned by the functions:

$$ \psi_1(x) = \begin{cases} 
4x^2(3 - 4x) & x \in [0, \frac{1}{2}] \\
4(4x^3 - 9x^2 + 6x - 1) & x \in [\frac{1}{2}, 1] 
\end{cases} $$

$$ \psi_2(x) = \begin{cases} 
4(2x^3 - x^2) & x \in [0, \frac{1}{2}] \\
4(2x^3 - 5x^2 + 4x - 1) & x \in [\frac{1}{2}, 1] 
\end{cases} $$

Taking \( f(x) = 1 \) in the boundary value problem, set up and solve the resulting discrete weak problem: Find \( u_h(x) \in V_h \) such that for all \( v_h(x) \in v_h \)

$$ a(u_h, v_h) = \int_0^1 v_h \, dx $$

You may use the fact that the inverse of a \( 2 \times 2 \) matrix is:

$$ \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} $$

Solution.
(a) For all \( v \in V \) multiply both side of BVP with the test function \( v \)

\[
(k(x)u''(x))''v = f(x)v, \quad \forall v \in V,
\]

Now integrate over 0 to 1

\[
\int_0^1 (k(x)u''(x))''v(x)dx = \int_0^1 f(x)v(x)dx \quad \forall v \in V
\]

Apply integration by parts the left hand side

\[
\int_0^1 (k(x)u''(x))''v(x)dx = \left[ (k(x)u''(x))'v(x) \right]_0^1 - \int_0^1 (k(x)u''(x))'v'(x)dx
\]

\[
= (k(1)u''(1))v(1) - (k(0)u''(0))v(0) - \int_0^1 (k(x)u''(x))'v'(x)dx
\]

\[
= - \int_0^1 (k(x)u''(x))'v'(x)dx \quad \text{by imposing} \quad v(0) = 0, v(1) = 0
\]

\[
= \left[ -(k(x)u''(x))v'(x) \right]_0^1 + \int_0^1 (k(x)u''(x))''v(x)dx \quad \text{by integration by parts}
\]

\[
= -(k(1)u''(1))v'(1) + (k(0)u''(0))v'(0) + \int_0^1 (k(x)u''(x))''v(x)dx
\]

\[
= \int_0^1 (k(x)u''(x))''v(x)dx \quad \text{by imposing} \quad v'(0) = 0, v'(1) = 0 \quad \forall v \in V
\]

Thus we get

\[
\int_0^1 (k(x)u''(x))''v(x)dx = \int_0^1 f(x)v(x)dx
\]

where

\[
a(u, v) = \int_0^1 (k(x)u''(x))''v(x)dx \quad \text{and} \quad (f, v) = \int_0^1 f(x)v(x)dx \quad \forall v \in V
\]

To show that the form \( a(u, v) \) in part is an inner product, we must verify the three basic properties:

- **Symmetry** is apparent by inspection:

\[
a(u, v) = \int_0^1 k(x)u''(x)v''(x)dx = \int_0^1 k(x)v''(x)u''(x)dx = a(v, u).
\]
• **Linearity** follows from the linearity of differentiation and integration:

\[
a(\alpha u + \beta v, w) = \int_0^1 k(x)(\alpha u + \beta v)(x)w(x) dx
\]

\[
= \int_0^1 k(x)(\alpha u(x) + \beta v)(x)w''(x) dx
\]

\[
= \alpha \int_0^1 k(x)u''(x)w(x) dx + \beta \int_0^1 k(x)v''(x)w''(x) dx
\]

\[
= \alpha a(u, w) + \beta a(v, w).
\]

• **Positivity** requires that \(a(u, u) \geq 0\) and \(a(u, u) = 0\) only when \(u = 0\). Note that

\[
a(u, u) = \int_0^1 k(x)u''(x)u''(x) dx
\]

\[
= \int_0^1 k(x)(u''(x))^2 dx.
\]

Since \(k(x)\) is positive for all \(x \in [0, 1]\), integrand is non-negative, and hence \(a(u, u) \geq 0\). To have \(a(u, u) = 0\), we must have \(u''(x) = 0\) for all \(x \in [0, 1]\), which is only possible if \(u(x) = bx + c\) by BC \(b = c = 0\) then \(u(x) = 0\) for all \(x \in [0, 1]\), i.e., \(u = 0\).

(b) Let \(V_n = \text{span}\{\phi_1, \cdots, \phi_n\}\) is an n-dimensional subspace of \(C^4_2[0, 1]\). We would like to shot that

\[a(u_n, v) = (f, v), \quad \text{for all } v \in V_n\]

leads to a linear system.

The Galerkin solution can be defined by the linear combination of basis function \(u_n = \sum_{j=1}^{n} u_j \phi_j(x)\) for coefficients \(u_j\). Then

\[a(\sum_{j=1}^{n} u_j \phi_j(x), \phi_i(x)) = (f, \phi_i(x)), \quad \text{for } i = 1, \cdots, n\]

By the linearity of bilinear form

\[\sum_{j=1}^{n} a(\phi_j(x), \phi_i(x))u_j = (f, \phi_i(x)), \quad \text{for } i = 1, \cdots, n\]

This leads to a linear system of \(N\) equations with \(N\) unknown. If we define that system as \(Ku = f\) then each component of \(K\) and \(f\) will be

\[K_{ij} = a(\phi_j(x), \phi_i(x)) = \int_0^1 k(x)\phi_j''(x)\phi_i''(x) dx \quad \text{for } i, j = 1, \cdots, n\]

and

\[f_i = (f(x), \phi_i(x)) = \int_0^1 f(x)\phi_i(x) dx \quad \text{for } i = 1, \cdots, n\]

\(u\) is our solution vector with \(u = [u_1, u_2, \cdots, u_n]^T\).
(c) Now if we take for $\phi_1, \ldots, \phi_n$ the standard piecewise linear 'hat' functions as in Problem 2. We will get $\phi_i''(x) = 0$ for $i = 1, \ldots, n$. That leads our stiffness matrix $K = 0$. Therefore, this hat function is not suitable for Euler Bernoulli beam problem.

By the same idea, we want to define a hat function satisfy following property

$$\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j; \end{cases}$$

Also for

$$K_{ij} = \int_0^1 k(x)\phi_j''(x)\phi_i''(x) \, dx$$

we need second derivative of the hat function. For simplify, let pick a hat function second derivative is constant.

In that case , the new hat function

$$\phi_i(x) = \begin{cases} \left( \frac{x - x_{i-1}}{h} \right)^2 & \text{if } x \in [x_{i-1}, x_i]; \\ \left( \frac{x_{i+1} - x}{h} \right)^2 & \text{if } x \in [x_i, x_{i+1}); \\ 0 & \text{otherwise}; \end{cases}$$

With these choose we will get again tridiagonal matrix $K$ such that

$$K_{ii} = \int_0^1 k(x)\phi_i''(x)\phi_i''(x) \, dx = \int_{x_{i-1}}^{x_{i+1}} k(x)\phi_i''(x)\phi_i''(x) \, dx$$

$$K_{i(i+1)} = \int_0^1 k(x)\phi_i''(x)\phi_{i+1}''(x) \, dx = \int_{x_i}^{x_{i+1}} k(x)\phi_i''(x)\phi_{i+1}''(x) \, dx$$

By symmetry $K_{ij} = K_{ji}$.

Finally, for $|i - j| > 1$,

$$K_{ij} = \int_0^1 k(x)\phi_i''(x)\phi_j''(x) \, dx = 0$$