CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 2

Posted Monday, September 11. Due Thursday, September 28, by 5pm.

- A note on proofs (unless otherwise stated by the problem): When asked to prove a statement this means you are to show that the requisite properties (discussed in class) for the item referenced by the statement hold true. When asked to disprove something this means you are to come up with an example that shows the proposed premise is false.
- Unless explicitly stated otherwise in the problem you are free to use MATLAB as you see fit; including for those problems that do not explicitly require it. Please submit any code that you utilize as a printout or, if it is short enough, reference your steps in your writeup directly. (example: I used MATLAB to compute the inverse to the matrix B and got .. etc)

A total of 68 points is distributed among the following problems

Please write your name and instructor on your homework.

1. [12 points: 3 each]
   (a) Show that \((1 + x)^n = 1 + nx + o(x)\) as \(x \to 0\).
   (b) Show that \(x \sin \sqrt{x} = O(x^{3/2})\) as \(x \to 0\).
   (c) Show that \(e^{-t} = o\left(\frac{1}{t}\right)\) as \(t \to \infty\).
   (d) Show that \(\int_0^\varepsilon e^{-x^2}dx = O(\varepsilon)\) as \(\varepsilon \to 0\).

2. [8 points: 2 each]
   Demonstrate whether each of the following functions is a linear operator.
   (Show that both properties hold, or give an example showing that one of the properties must fail.)
   (a) \(f : \mathbb{R}^2 \to \mathbb{R}, f(x) = x^T x\).
   (b) \(f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}, f(X) = AX + XB\) for fixed matrices \(A, B \in \mathbb{R}^{n \times n}\).
   (c) \(L : C^1[0, 1] \to C[0, 1], Lu = \frac{du}{dx}\).
   (d) \(L : C^2[0, 1] \to C[0, 1], Lu = \frac{d^2u}{dx^2} - e^{\pi} \cos(x) \frac{du}{dx} + \sin(x) u\).

3. [12 points: (a),(b),(d) 2 each, (c) 6 pts]
   Characterize the range (1 point) and the nullspace (1 point) of the following linear operators:
   (a) The first derivative operator \(D_x : C^1[0, 1] \to C[0, 1]\) given by \(D_x(f) = \frac{\partial x f}{\partial x}\).
   (b) The second derivative operator \(D_{xx} : C^2[0, 1] \to C[0, 1]\) given by \(D_{xx}(f) = \frac{\partial x^2 f}{\partial x^2}\).
   (c) Consider the second derivative operator \(D_{xx} : C^2_D[0, 1] \to C[0, 1]\) given by \(D_{xx}(f) = \frac{\partial x^2 f}{\partial x^2}\). What is the null space (1 pts)? You will determine the range by considering the following steps:
      (i) [1 point] Show that the range of the first derivative operator \(\partial_x : C^2_D[0, 1] \to C^1[0, 1]\) consists of functions in \(C^1[0, 1]\) having average value zero (ie. \(\int_0^1 u = 0\)).
      (ii) [1 point] Show that for each \(u \in C^1[0, 1]\) with average value zero there exists \(f \in C^2_D[0, 1]\) with \(\partial_x f = u\).
      (iii) [1 point] Show that for every \(g \in C[0, 1]\) there exists \(u \in C^1[0, 1]\), having average value zero (ie. \(\int_0^1 u = 0\)), such that \(\partial_x u = g\).
      (iv) [2 points] Interpret the meaning of the above sub-steps by clearly stating what the range is for the operator \(\partial_{xx} : C^2_D[0, 1] \to C[0, 1]\) and using parts (i)-(iii) to thoroughly justify your answer.
(d) The matrix $A : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ given by

$$A = \begin{bmatrix}
5 & 0 & 1 & 4 & 2 \\
6 & 3 & 8 & -2 & 4 \\
8 & 4 & 9 & 4 & 2 \\
-1 & 3 & 5 & 6 & 4 \\
11 & 3 & 9 & 2 & 6
\end{bmatrix}$$

4. [16 points: 4 each]

(a) Demonstrate whether or not the set $S_1 = \{ x \in \mathbb{R}^2 : x_2 = x_1^3 \}$ is a subspace of $\mathbb{R}^2$.

(b) Demonstrate whether or not the set $S_2 = \{ x \in \mathbb{R}^3 : 3x_1 + 2x_2 + x_3 = 0 \}$ is a subspace of $\mathbb{R}^3$.

(c) Demonstrate whether or not the set $S_3 = \{ f \in C^2[0,1] : f(1) = 1 \}$ is a subspace of $C^2[0,1]$.

(d) Demonstrate whether or not the set $S_4 = \{ f \in C^2[0,1] : f(1) = 0 \}$ is a subspace of $C^2[0,1]$.

5. [12 points: 4 each]

Determine whether or not each of the following mappings is an inner product on the real vector space $V$. If not, show all the properties of the inner product that are violated.

(a) $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ defined by $(u, v) = \int_0^1 |u(x)||v(x)| \, dx$ where $V = C[0, 1]$.

(b) $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ defined by $(u, v) = \int_0^1 u(x)v(x)e^{-x} \, dx$ where $V = C[0, 1]$.

(c) $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ defined by $(u, v) = \int_0^1 (u(x) + v(x)) \, dx$ where $V = C[0, 1]$.

6. [8 points: 4 each]

This problem is pledged! You may not discuss this with anyone but your instructor. You may not consult any source other than approved textbooks, CAAM 336 lecture notes or your in-class notes to help you with the problem.

(a) Show that the functions $\{ \cos(n \pi x) \mid n = 1, 2, \ldots \}$ are mutually orthogonal in $C[0,1]$ with respect to $L^2$ inner product.

(b) What is the length of $f_n(x) = \cos(n \pi x)$ with respect to the $L^2$ inner product on $C[0,1]$?