Homework 2 · Solutions

A total of 68 points is distributed among the following problems

Please write your name and instructor on your homework.

1. [12 points: 3 each]
   (a) Show that \((1 + x)^n = 1 + nx + o(x)\) as \(x \to 0\).
   (b) Show that \(x \sin \sqrt{x} = O(x^{3/2})\) as \(x \to 0^+\).
   (c) Show that \(e^{-t} = o\left(\frac{1}{t^2}\right)\) as \(t \to \infty\).
   (d) Show that \(\int_0^\varepsilon e^{-x^2} dx = O(\varepsilon)\) as \(\varepsilon \to 0\).

Solution.

(a) Recall the binomial series \((1 + x)^n = 1 + nx + \sum_{k=3}^{n} C^n_k x^k\). Therefore, \((1 + x)^n - 1 - nx = \frac{n(n-1)}{2!} x^2 + \sum_{k=3}^{n} C^n_k x^k\) and \(\lim_{x \to 0} \frac{(1+x)^n-1-nx}{x} = \lim_{x \to 0} \frac{n(n-1)}{2!} x + \sum_{k=3}^{n} C^n_k x^{k-1} = 0\).

(b) Approach 1: Notice that by mean value theorem:
\[
|\frac{x \sin \sqrt{x}}{x^{3/2}}| = \left| \frac{\sin \sqrt{x} - \sin 0}{\sqrt{x} - 0} \right| \leq 1,
\]
where \(K \in [0, \sqrt{x}]\).

Approach 2: We can write the numerator in terms of the Taylor series of \(\sin(\sqrt{x})\) so that
\[
\left| \frac{x \sin \sqrt{x}}{x^{3/2}} \right| = \left| \frac{\sin \sqrt{x} - \frac{\sqrt{x}}{3!} + \frac{1}{5!} + \cdots}{\sqrt{x}} \right| = \left| 1 - \frac{x}{3!} + \frac{x^2}{5!} + \cdots \right| \leq 1
\]
for \(x \leq 1\).

(c) For “little o” convergence, we need to show that \(\lim_{t \to \infty} \frac{e^{-1}}{t^3} = 0\).

Dropping the absolute value since the argument is nonnegative for all \(t\), we can use L’Hospital’s rule twice to get
\[
\frac{t^2}{e^t} = \lim_{t \to \infty} \frac{2t}{e^t} = \lim_{t \to \infty} \frac{2}{e^t} = 0.
\]

(d) For “Big O” convergence, we need to show that there exists a constant \(M\) such that
\[
\left| \frac{\int_0^\varepsilon e^{-x^2} dx}{\varepsilon} \right| \leq M.
\]

There are a couple of ways to approach this problem.

Approach 1: Notice that \(e^{-x^2} \leq 1\) for all \(x \in [0, \varepsilon]\). Then
\[
\int_0^\varepsilon e^{-x^2} dx \leq \int_0^\varepsilon 1 dx = \varepsilon.
\]
Hence,
\[ \left| \int_0^\varepsilon e^{-x^2} dx \right| \leq \varepsilon = 1, \]
and the “Big O” relationship is satisfied.

**Approach 2:** The Maclaurin series for \( e^{-x^2} \) is
\[ e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} + \ldots \]
Plugging this into the integral, we see that
\[ \int_0^\varepsilon e^{-x^2} dx = \left( x - \frac{x^3}{3} + \frac{x^5}{2!5} + \ldots \right) \bigg|_0^\varepsilon, \]
which implies that
\[ \left| \int_0^\varepsilon e^{-x^2} dx \right| = \left| \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{2!5} + \ldots \right| \leq 1. \]
Once again, we see that the “Big O” relationship is satisfied.

2. [8 points: 2 each]
Demonstrate whether each of the following functions is a linear operator.
(Show that both properties hold, or give an example showing that one of the properties must fail.)

(a) \( f : \mathbb{R}^2 \to \mathbb{R}, f(x) = x^T x \).

(b) \( f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}, f(X) = AX + XB \) for fixed matrices \( A, B \in \mathbb{R}^{n \times n} \).

(c) \( L : C^1[0,1] \to C[0,1], Lu = \frac{du}{dx} \).

(d) \( L : C^2[0,1] \to C[0,1], Lu = \frac{d^2u}{dx^2} - e^x \cos(x) \frac{du}{dx} + \sin(x)u \).

**Solution.** Graders: One point is to be given if the student identified linear v.s. nonlinear correctly; the other point is for the technical details. For example, if a student correctly identifies an operator as linear, but the proof of linearity is incorrect then this is worth one point.

(a) This function is not a linear operator.
Suppose \( x \in \mathbb{R}^n \). Then
\[ f(\alpha x) = (\alpha x)^T (\alpha x) = \alpha^2 x^T x = \alpha^2 f(x), \]
and thus if \( \alpha \neq \pm 1 \), we have \( f(\alpha x) \neq \alpha f(x) \).

(b) This function is a linear operator.
Suppose \( X, Y \in \mathbb{R}^{n \times n} \). Then
\[ f(X + Y) = A(X + Y) + (X + Y)B = AX + XB + AY + YB = f(X) + f(Y), \]
and if \( \alpha \in \mathbb{R} \), then
\[ f(\alpha X) = A(\alpha X) + (\alpha X)B = \alpha (AX + XB) = \alpha f(X). \]
(c) This function is not a linear operator. Suppose that \( u(x) = x \). Then 
\[
Lu = u\frac{du}{dx} = x \cdot 1 = x,
\]
yet for any \( \alpha \in \mathbb{R} \) we have 
\[
L(\alpha u) = (\alpha u)\frac{d(\alpha u)}{dx} = (\alpha x) \cdot \alpha = \alpha^2 x,
\]
so if \( \alpha \neq \pm 1 \), we have \( L(\alpha u) \neq \alpha Lu \).

(d) This function is a linear operator. Suppose that \( u, v \in C^2[0,1] \). Then 
\[
L(u + v) = \frac{d^2(u + v)}{dx^2} - e^x \cos(x)\frac{du}{dx} + \sin(x)(u + v)
\]
\[
= \frac{d^2u}{dx^2} - e^x \cos(x)\frac{du}{dx} + \sin(x)u + \frac{d^2v}{dx^2} - e^x \cos(x)\frac{dv}{dx} + \sin(x)v
\]
\[
= Lu + Lv,
\]
and for any \( \alpha \in \mathbb{R} \),
\[
L(\alpha u) = \frac{d^2(\alpha u)}{dx^2} - e^x \cos(x)\frac{d(\alpha u)}{dx} + \sin(x)(\alpha u) = \alpha \left( \frac{d^2u}{dx^2} - e^x \cos(x)\frac{du}{dx} + \sin(x)u \right) = \alpha L(u).
\]

3. [12 points: (a),(b),(d) 2 each, (c) 6 pts]
Characterize the range (1 point) and the nullspace (1 point) of the following linear operators:

(a) The first derivative operator \( D_x : C^1[0,1] \to C[0,1] \) given by \( D_x(f) = \partial_x f \)

(b) The second derivative operator \( D_{xx} : C^2[0,1] \to C[0,1] \) given by \( D_{xx}(f) = \partial_{xx} f \)

(c) Consider the second derivative operator \( D_{xx} : C^2_D[0,1] \to C[0,1] \) given by \( D_{xx}(f) = \partial_{xx} f \). What is the null space (1 pts)? You will determine the range by considering the following steps:
   (i) [1 point] Show that the range of the first derivative operator \( \partial_x : C^1_D[0,1] \to C^1[0,1] \) consists of functions in \( C^1[0,1] \) having have average value zero (i.e. \( \int_0^1 u = 0 \)).
   (ii) [1 point] Show that for each \( u \in C^1[0,1] \) with average value zero there exists \( f \in C^2_D[0,1] \) with \( \partial_x f = u \).
   (iii) [1 point] Show that for every \( g \in C[0,1] \) there exists \( u \in C^1[0,1] \), having average value zero (i.e. \( \int_0^1 u = 0 \)), such that \( \partial_x u = g \).
   (iv) [2 points] Interpret the meaning of the above sub-steps by clearly stating what the range is for the operator \( \partial_{xx} : C^2_D[0,1] \to C[0,1] \) and using parts (i)-(iii) to thoroughly justify your answer.

(d) The matrix \( A : \mathbb{R}^5 \to \mathbb{R}^5 \) given by
\[
A = \begin{bmatrix}
5 & 0 & 1 & 4 & 2 \\
6 & 3 & 8 & -2 & 4 \\
8 & 4 & 9 & 4 & 2 \\
-1 & 3 & 5 & 6 & 4 \\
11 & 3 & 9 & 2 & 6
\end{bmatrix}
\]

Solution.
(a) The range of $D_x$ is, from the fundamental theorem of calculus, all of $C[0,1]$. To see this let $g \in C[0,1]$ and define $G(x) = \int_0^x g(t) \, dt$. Then $D_x(G) = g$. The Nullspace of $D$ is the set of constant functions.

(b) The range of $D_{xx}$ is, again, all of $C[0,1]$ by applying the fundamental theorem of Calculus twice. The Null space is the set of all linear functions of the form $ax + b$ for $a, b$ real numbers.

(c) The null space (1 point) contains only the zero vector; i.e. the function $u(x) = 0$. To see this note that the the null space of the general second derivative is all lines $ax + b$ (as in part (b)) however the homogeneous Dirichlet boundary conditions require that $a = b = 0$. We now present the solutions to each sub-part:

i. Consider $\partial_x : C^2_D[0,1] \to C^1[0,1]$ and let $f \in \text{Range}(\partial_x)$. Then there exists $u \in C^2_D[0,1]$ with $\partial_x u = f$. Integrating both sides of this equation, applying the fundamental theorem of calculus, and the homogeneous Dirichlet conditions gives $\int_0^x f = \int_0^1 \partial_x u = u(1) - u(0) = 0$ so that if $f \in \text{Range}(\partial_x)$ then $f$ has average value zero, as desired.

ii. Let $u \in C^1[0,1]$ with average value zero. We note that given any integrable function $h(x)$ on $[0,1]$ the function $\hat{h}(x) = h(x) - \int_0^1 h(s) \, ds$ is equal to $h(x)$ plus a constant and has average value zero. Consider the function $f(x) = \int_0^x u(t) \, dt$. Then $f \in C^2[0,1]$ and we will be done if we can show that $f(1) = f(0) = 0$. One is easy; we have that $f(0) = \int_0^0 u(t) \, dt = 0$ and we also have that $f(1) = \int_0^1 u(t) \, dt$ but by hypothesis $u$ has average value zero which means that $f(1) = 0$. So, indeed, $f(x) \in C^2[0,1]$ and $\partial_x f = u$.

iii. Let $g \in C[0,1]$. We want to find a function $u \in C^1[0,1]$ having average value zero such that $\partial_x u = g$. Stating this another way: we want to find an antiderivative of $g$ that has average value zero. The fundamental theorem of calculus tells us that $G(x) = \int_0^x g(t) \, dt$ is an antiderivative of $g(x)$ and thus, as we saw in the previous part, $\hat{G}(x) = G(x) - \int_0^1 G(s) \, ds$ has average value zero and satisfies $\partial_x \hat{G} = g$. Thus, $u(x) = \hat{G}(x)$ has average value zero and is mapped by $\partial_x$ to $g$, since $g \in C[0,1]$ was arbitrary this gives the result.

iv. Putting the previous items together gives that, for $\partial_{xx} : C^2_D[0,1] \to C[0,1]$, the set $\text{Range}(\partial_{xx}) = C[0,1]$ since given $g \in C[0,1]$ we can find $u \in C^1$ with average value zero such that $\partial_x u = g$ and we can find $f \in C^2_D[0,1]$ with $\partial_x f = u$ so that $\partial_{xx} f = g$ as needed.

(d) Graders: Please mark this problem with full points if it is clear that the student ascertained the general idea that the Range is a four-dimensional thing and the Null Space is a one-dimensional thing. The matrix $A : \mathbb{R}^5 \to \mathbb{R}^5$ has row-reduced echelon form, which can be easily computed using MATLAB, given by

$$
\text{rref}(A) = \begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{4}{\pi} \\
0 & 1 & 0 & 0 & -\frac{2}{\pi} \\
0 & 0 & 1 & 0 & \frac{16}{7} \\
0 & 0 & 0 & 1 & \frac{5}{7} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

The reduced Row Echelon form for $A$ shows that an output vector $y$ of $Ax = y$ will have only its first four coordinates determined by the components of $x$. Therefore the range is a four-dimensional subvector space of $\mathbb{R}^5$ and is therefore ‘like’ $\mathbb{R}^4$ (the mathematical word for ‘like’ in this context is isomorphic). Since there is one row of zeros in the reduced row echelon form for $A$ the Null Space is a one-dimensional subspace of $\mathbb{R}^5$ and is therefore ‘like’ $\mathbb{R}^1$. To see this more rigorously we can write down the equations. Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. Let $\hat{A}$ denote the reduced row-echelon form of $A$. Then from linear algebra we know that the Range and Null Space of $\hat{A}$ and that of $A$ are the same. Then we look at the equations resulting from the multiplication
A \times x = y and we see the following:

\begin{align*}
y_1 &= x_1 - \frac{4}{3}x_5 \\
y_2 &= x_2 - \frac{29}{3}x_5 \\
y_3 &= x_3 + \frac{16}{3}x_5 \\
y_4 &= x_4 + \frac{5}{6}x_5
\end{align*}

Which clearly shows that the range is only four dimensional (as we only get four equations from the matrix). To see that the Null Space is one dimensional we re-write the above equations and we see that, for a fixed \(y\), a vector \(x\) mapping to it (via multiplication by \(A\)) has the form

\[
x = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \frac{4}{3} \\ -\frac{16}{3} \\ 5 \\ 1 \end{bmatrix}
\]

where \(x_5\) can be any number. Let \(\tilde{x} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ 0 \end{bmatrix}\) and let \(z = \begin{bmatrix} \frac{4}{3} \\ -\frac{16}{3} \\ 5 \\ 1 \end{bmatrix}\).

Then the discussion above means that \(\tilde{A}\tilde{x} = A(\tilde{x} + z) = y\) so that subtracting these two quantities and using that \(\tilde{A}\) is linear gives \(\tilde{A}z = 0\) and therefore \(A(\alpha z) = 0\) for all real numbers \(\alpha\) as well.

This means that the null space consists of any multiple of the single vector, \(z\). I.E. it is a vector subspace of \(\mathbb{R}^5\) which has only one dimension and therefore ‘looks like’ a copy of \(\mathbb{R}^1\).

4. [16 points: 4 each]

(a) Demonstrate whether or not the set \(S_1 = \{x \in \mathbb{R}^2 : x_2 = x_1^3\}\) is a subspace of \(\mathbb{R}^2\).

(b) Demonstrate whether or not the set \(S_2 = \{x \in \mathbb{R}^3 : 3x_1 + 2x_2 + x_3 = 0\}\) is a subspace of \(\mathbb{R}^3\).

(c) Demonstrate whether or not the set \(S_3 = \{f \in C^2[0,1] : f(1) = 1\}\) is a subspace of \(C^2[0,1]\).

(d) Demonstrate whether or not the set \(S_4 = \{f \in C^2[0,1] : f(1) = 0\}\) is a subspace of \(C^2[0,1]\).

Solution.

(a) [4 points] The set \(S_1\) is not a subspace of \(\mathbb{R}^2\). The vector \(x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\) is in the set \(S_1\), yet \(2x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\) is not, since \(2 \neq 2^3 = 8\). Consequently, the set \(S_1\) is not a subspace of \(\mathbb{R}^2\).

(b) [4 points] The set \(S_2\) is a subspace of \(\mathbb{R}^3\). The set \(S_2\) is a subset of \(\mathbb{R}^3\) and \(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\) is a member of the set \(S_2\). Now, suppose \(x\) and \(y\) are members of the set \(S_2\). Then \(3x_1 + 2x_2 + x_3 = 0\) and \(3y_1 + 2y_2 + y_3 = 0\). Adding these two equations gives

\[3(x_1 + y_1) + 2(x_2 + y_2) + (x_3 + y_3) = 0,\]

and hence \(x + y\) is also in the set \(S_2\). Multiplying \(3x_1 + 2x_2 + x_3 = 0\) by an arbitrary constant \(\alpha \in \mathbb{R}\) gives

\[3(\alpha x_1) + 2(\alpha x_2) + \alpha x_3 = 0,\]

and hence \(\alpha x\) is also in the set \(S_2\). Consequently, the set \(S_2\) is a subspace of \(\mathbb{R}^3\).
(c) [4 points] The set $S_3$ is not a subspace of $C^2[0,1]$.

The function $z$ defined by $z(x) = 0$ for $x \in [0,1]$ is not in the set $S_3$ since $z(1) = 0$ and thus violates the requirement for membership in the set $S_3$. Consequently, the set $S_3$ is not a subspace of $C^2[0,1]$.

(d) [4 points] The set $S_4$ is a subspace of $C^2[0,1]$.

The set $S_4$ is a subset of $C^2[0,1]$ and the function $z$ defined by $z(x) = 0$ for $x \in [0,1]$ is in the set $S_4$. If $f$ and $g$ are in the set $S_4$, then $f(1) = g(1) = 0$, so

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$$

and hence $f + g$ is in the set $S_4$. Also, if $f$ is in the set $S_4$ and $\alpha \in \mathbb{R}$, then

$$(\alpha f)(1) = \alpha f(1) = \alpha \cdot 0 = 0$$

and hence $\alpha f$ is in the set $S_4$. Consequently, the set $S_4$ is a subspace of $C^2[0,1]$.

5. [12 points: 4 each]

Determine whether or not each of the following mappings is an inner product on the real vector space $V$. If not, show all the properties of the inner product that are violated.

(a) $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ defined by $(u, v) = \int_0^1 |u(x)||v(x)| \, dx$ where $V = C[0,1]$.

(b) $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ defined by $(u, v) = \int_0^1 u(x)v(x)e^{-x} \, dx$ where $V = C[0,1]$.

(c) $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ defined by $(u, v) = \int_0^1 (u(x) + v(x)) \, dx$ where $V = C[0,1]$.

Solution.

(a) [4 points] This mapping is not an inner product: it is not linear in the first argument.

If $u, v, w \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$ then

$$(\alpha u + \beta v, w) = \int_0^1 |\alpha u(x) + \beta v(x)||w(x)| \, dx$$

and

$$\alpha(u, w) + \beta(v, w) = \alpha \int_0^1 |u(x)||w(x)| \, dx + \beta \int_0^1 |v(x)||w(x)| \, dx.$$  

However, if $u(x) = 1$, $v(x) = 0$, $w(x) = 1$, $\alpha = -1$ and $\beta = 0$ then

$$\int_0^1 |\alpha u(x) + \beta v(x)||w(x)| \, dx = \int_0^1 |1||1| \, dx = \int_0^1 1 \, dx = 1$$

but

$$\int_0^1 |\alpha u(x) + \beta v(x)||w(x)| \, dx = - \int_0^1 |1||1| \, dx = - \int_0^1 1 \, dx = -1$$

and so the mapping is not linear in the first argument.

The mapping is symmetric, as

$$(u, v) = \int_0^1 |u(x)||v(x)| \, dx = \int_0^1 |v(x)||u(x)| \, dx = (v, u)$$
for all \( u, v \in C[0, 1] \).
Moreover, the mapping is positive definite as for all \( u \in C[0, 1] \)

\[
(u, u) = \int_0^1 |u(x)|^2 \, dx
\]

is the integral of a nonnegative function, and hence is nonnegative and \((u, u) = 0\) only if \( u = 0 \).

(b) [4 points] This mapping is an inner product.
The mapping is symmetric, as

\[
(u, v) = \int_0^1 u(x)v(x)e^{-x} \, dx = \int_0^1 v(x)u(x)e^{-x} \, dx = (v, u)
\]
for all \( u, v \in C[0, 1] \).
The mapping is also linear in the first argument since

\[
(\alpha u + \beta v, w) = \int_0^1 (\alpha u(x) + \beta v(x))w(x)e^{-x} \, dx
\]

\[
= \alpha \int_0^1 u(x)w(x)e^{-x} \, dx + \beta \int_0^1 v(x)w(x)e^{-x} \, dx
\]

\[
= \alpha (u, w) + \beta (v, w)
\]
for all \( u, v, w \in C[0, 1] \) and all \( \alpha, \beta \in \mathbb{R} \).
The function \( e^{-x} \) is positive valued for all \( x \in [0, 1] \), so we have that

\[
(u, u) = \int_0^1 (u(x))^2 e^{-x} \, dx
\]
is the integral of a nonnegative function, and hence is also nonnegative. If \((u, u) = 0\) then \((u(x))^2 e^{-x} = 0\) for all \( x \in [0, 1] \) and, since \( e^{-x} > 0 \) for all \( x \in [0, 1] \), this means that \( u(x) = 0 \) for all \( x \in [0, 1] \), i.e., \( u = 0 \). Hence, the mapping is positive definite.

(c) [4 points] This mapping is not an inner product: it is not linear in the first argument and it is not positive definite.
If \( u, v, w \in C[0, 1] \) and \( \alpha, \beta \in \mathbb{R} \) then

\[
(\alpha u + \beta v, w) = \int_0^1 (\alpha u(x) + \beta v(x) + w(x)) \, dx
\]

and

\[
\alpha (u, w) + \beta (v, w) = \alpha \int_0^1 (u(x) + w(x)) \, dx + \beta \int_0^1 (v(x) + w(x)) \, dx.
\]
However, if \( u(x) = 1, v(x) = 0, w(x) = 1, \alpha = 2 \) and \( \beta = 0 \) then

\[
(\alpha u + \beta v, w) = \int_0^1 (2 + 1) \, dx = \int_0^1 3 \, dx = 3
\]

but

\[
\alpha (u, w) + \beta (v, w) = 2 \int_0^1 (1 + 1) \, dx = 2 \int_0^1 2 \, dx = 4
\]
and so \((\cdot, \cdot)\) is not linear in the first argument.
The mapping \((\cdot, \cdot)\) is also not positive definite. For example, if \(u(x) = -1\), then
\[
(u, u) = \int_{0}^{1} (u(x) + u(x)) \, dx = \int_{0}^{1} -2 \, dx = -2 < 0.
\]

The mapping is symmetric, as
\[
(u, v) = \int_{0}^{1} (u(x) + v(x)) \, dx = \int_{0}^{1} (v(x) + u(x)) \, dx = (v, u)
\]
for all \(u, v \in C[0, 1]\).

6. [8 points: 4 each]
This problem is pledged! You may not discuss this with anyone but your instructor. You may not consult any source other than approved textbooks, CAAM 336 lecture notes or your in-class notes to help you with the problem.

(a) Show that the functions \(\{\cos(n\pi x) \mid n = 1, 2, \ldots\}\) are mutually orthogonal in \(C[0, 1]\) with respect to \(L^2\) inner product.

(b) What is the length of \(f_n(x) = \cos(n\pi x)\) with respect to the \(L^2\) inner product on \(C[0, 1]\) ?

Solution.
(a) Consider \(f(x) = \cos(n\pi x)\) and \(g(x) = \cos(m\pi x)\) and consider \(I = (f, g) = \int_{0}^{1} f(x)g(x) \, dx\). Integrating by parts once and using the fact that \(\sin(n\pi) = \sin(0) = 0\) gives
\[
I = \int_{0}^{1} \cos(n\pi x)\cos(m\pi x) \, dx = \frac{m}{n} \int_{0}^{1} \sin(n\pi x)\sin(m\pi x) \, dx
\]
Using integration by parts once more, applied to the right hand term, produces
\[
I = \int_{0}^{1} \cos(n\pi x)\cos(m\pi x) \, dx = \frac{m^2}{n^2} \int_{0}^{1} \cos(n\pi x)\cos(m\pi x) \, dx = \frac{m^2}{n^2} I
\]
for \(m \neq n\) we therefore have that \(I\) is a number which is equal to a multiple of itself. The only way this can be true is if \(I = 0\) that is if \((f, g) = 0\).

(b) The length (or norm) defined by the \(L^2\) inner produce is, as we saw in class, \(\|f\|_{L^2} = \sqrt{(f, f)}\). Thus for fixed \(n\) we have \(f_n(x) = \cos(n\pi x)\) so that \(I = (f, f) = \int_{0}^{1} \cos^2(n\pi x) \, dx\). Using the trigonometric identity \(\cos^2(u) = \frac{1 + \cos(2u)}{2}\) to compute \(I\) we see that \(I = \frac{1}{2}\) so that \(\|f\|_{L^2} = \frac{1}{\sqrt{2}}\).