Homework 5

1. [24 points: 5 points (a), 3 points (b), 8 points (c) and (d)]
Let \( k(x) \) and \( p(x) \) be two positive-valued continuous functions on \([0, 1]\).

(a) Derive the weak form of the differential equation

\[ -\frac{d}{dx} \left( k(x) \frac{du}{dx} \right) + p(x)u = f(x), \quad 0 < x < 1, \]

subject to the boundary conditions

\[ u(0) = u(1) = 0. \]

Write the full weak problem statement.

(b) Verify that the bilinear form \( a(u, v) \) that you found in part (a) is an inner product.

(c) Let \( p(x) = 1 \), \( k(x) = \epsilon \), and let the source function \( f(x) = 1 \). Construct the finite element system \( Ax = b \), using the approximation space \( V_N \) given by the piecewise linear hat functions: for \( n \geq 1 \),

\[ h = \frac{1}{N + 1}, \quad x_k = kh \text{ for } k = 0, \ldots, N + 1: \]

\[ \phi_k(x) = \begin{cases} \frac{(x - x_{k-1})}{h}, & x \in [x_{k-1}, x_k); \\ \frac{(x_{k+1} - x)}{h}, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases} \]

Hint: For this specific choice of \( p(x) \) and \( k(x) \), it may be easier to show that you can express

\[ A = \epsilon K + M, \]

where \( K_{ij} = \int_0^1 \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \) and the form of \( M \) is that determined in Homework 3, problem 5c (the pledged problem).

(d) This specific equation corresponds to the simplest steady-state reaction-diffusion equation, where \( u(x) \) is the concentration of some solvent, and the choices of \( p(x) \) model local chemical reactions that may occur due to that solvent (multiple chemicals interacting may be modeled using systems of reaction-diffusion equations).

Use MATLAB to solve the above system with \( N = 32 \) and \( \epsilon = .1, .25, 1 \) and plot your results. In a separate figure, plot the results for \( \epsilon = .1, .01, .001 \). What do you observe about the solution as \( \epsilon \) decreases?
2. [26 points: 8 points (a) and (d); 5 points (b) and (c)]
A classical problem in quantum mechanics models a particle moving in an infinite square well, subject to an infinite potential at a point. The result is a Schrödinger operator posed on $C_0^2[0, 1]$ of the form

$$Lu = -u'' + \delta_{1/2}u,$$

where

$$\delta_{1/2}(x) = \delta(x - 1/2) = \begin{cases} 1 & \text{if } x = 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Note, for any function $g \in C[0, 1]$,

$$\int_0^1 \delta_{1/2}(x)g(x) \, dx = g(1/2).$$

The equivalent weak problem is: Find $u \in C_0^2[0, 1]$ such that for all $w \in V = C_0^2[0, 1]$

$$a(u, w) = (f, w)$$

where

$$a(u, w) = \int_0^1 \left( u'(x)w'(x) + \delta_{1/2}(x)u(x)w(x) \right) \, dx.$$

and $(f, w)$ is simply the $L^2$ inner product of $f$ and $w$.

Let $V_N = \text{span}\{\phi_1, \ldots, \phi_N\} \subset C_0^2[0, 1]$ where

$$\phi_k(x) = \sqrt{2} \sin(k\pi x), \quad k = 1, \ldots, N$$

denote our finite dimensional subspace.

(a) Compute a general formula for $a(\phi_j, \phi_k)$ for $1 \leq j \leq N$ and $1 \leq k \leq N$.

(b) Write out (by hand) the stiffness matrix for $N = 5$.

(c) Write down a general formula for the $k^{th}$ entry in the load vector, $(f, \phi_k)$, when $f(x) = 1$.

(d) Plot your approximate solutions to $-u''(x) + \delta_{1/2}(x)u(x) = 1$ for $N = 5, 10, 20, 35, 100$ in five separate plots (one for each value of $N$). Explain what you see as you increase $N$. Note: To see some of the differences between higher values of $N$, you may need to zoom in on the plots.

3. [10 points: 6 points (a), 4 points (b)]
Consider the following boundary value problem

$$-(k(x)u')' + q(x)u = f(x), \quad 0 < x < \ell,$$

subject to the boundary conditions

$$u(0) - k(0)u'(0) = \beta, \quad u'(\ell) = 0$$

for some constant $\beta$. Assume that $q$ and $f$ are continuous on $[0, \ell]$ and $k \in C^1[0, \ell]$.

(a) Derive the weak form of the above problem. Write the variational problem statement.

(b) Show that the weak form from part (a) is equivalent to the strong form of the boundary value problem.
4. [10 points: 5 points each] **This problem is pledged!**

You may not discuss this with anyone but your instructor. You may not consult any source other than approved textbooks, CAAM 336 lecture notes or your in-class notes to help you with the problem.

Let $k(x)$ and $p(x)$ be two positive-valued continuous functions on $[0, 1]$.

(a) Derive the weak form of the differential equation

$$- \frac{d}{dx}(k(x) \frac{du}{dx}) + p(x)u = f(x), \quad 0 < x < 1,$$

subject to the boundary conditions

$$u(0) = \frac{du}{dx}(1) = 0.$$

Write the variational problem statement.

(b) Show that the bilinear form $a(u, v)$ from the weak formulation in part (a) is an inner product for $u, v \in V$. 