Worksheet #1: Differential equations, modeling physics and finite differences

(1) Classify the differential equations: ODE, PDE, linear, non-linear, constant coefficient, variable coefficient, order, etc.
(a) \( x \frac{\partial u}{\partial x} - t \frac{\partial u}{\partial t} + u = 0 \)
(b) \( \cos(u) \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = \sqrt{x^2 + t^2} \)

(2) Let \( x_1(t), x_2(t) \) and \( x_3(t) \) denote the three unknown functions of \( t \). The following is a set of linear differential equations
\[
\begin{align*}
\frac{\partial x_1}{\partial t}(t) &= 2x_1(t) - 3x_3(t) \\
\frac{\partial x_2}{\partial t}(t) &= 2x_2(t) \\
\frac{\partial x_3}{\partial t}(t) &= -x_2(t) + 10x_3(t)
\end{align*}
\]
Rewrite the set of linear differential equations as a linear differential equation system. In other words, write the set of differential equations in matrix form.

(3) Solve the following steady state heat equation with non-homogeneous boundary condition.
\[
\frac{\partial^2 u}{\partial x^2} = 0 \\
u(0) = 4; \quad u(l) = 6
\]

(4) Consider the one sided finite difference approximation of \( \frac{\partial^2 u}{\partial x^2}(x_i) \) given by
\[
\frac{\partial^2 u}{\partial x^2}(x) \sim \frac{\alpha u(x) + \beta u(x + \Delta x) + \gamma u(x + 2\Delta x)}{(\Delta x)^2}.
\]
For what choice of constants \( \alpha, \beta, \) and \( \gamma \) is this scheme first order accurate?

(5) Prove that the following finite difference approximation is third order accurate.
\[
\left( \frac{\partial u}{\partial x} \right)(x) \sim \frac{2u(x + \Delta x) + 3u(x) - 6u(x - \Delta x) + u(x - 2\Delta x)}{6\Delta x}.
\]