Worksheet #2: Differential equations, and finite differences

Finite differences

Suppose \( N \geq 1 \) is an integer and define \( h = 1/(N+1) \) and \( x_j = ih \) for \( i = 0, \ldots, N+1 \). We can approximate the differential equation

\[-u''(x) = f(x), \quad 0 < x < 1,\]

with homogeneous Dirichlet boundary conditions \( u(0) = u(1) = 0 \) by the matrix equation

\[
\begin{array}{ccc}
-2 & 1 &  \\
1 & -2 & 1 \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
1 & -2 & \\
\end{array}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{N-1} \\
u_N
\end{bmatrix} =
\begin{bmatrix}
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_{N-1}) \\
f(x_N)
\end{bmatrix},
\]

where \( u_i \approx u(x_i) \). (Entries of the matrix that are not specified are zero.)

1. Explain what adjustments to the right hand side of the matrix equation are necessary to accommodate the inhomogeneous Dirichlet boundary conditions

\[ u(0) = 1, \quad u(1) = 2. \]

2. Suppose that we have

\[-u''(x) = (2\pi)^2 \sin(2\pi x), \quad 0 < x < 1,\]

\[ u(0) = 1 \]

\[ u(1) = 2. \]

Since this differential equation is linear, we can split up the solution into

\[ u(x) = u_1(x) + u_2(x), \]

where \( u_1(x) \) satisfies

\[-u_1''(x) = 0, \quad 0 < x < 1,\]

\[ u_1(0) = 1 \]

\[ u_1(1) = 2 \]

and \( u_2(x) \) satisfies the equation

\[-u_2''(x) = (2\pi)^2 \sin(2\pi x), \quad 0 < x < 1,\]

\[ u_2(0) = 0 \]

\[ u_2(1) = 0. \]

Show that \( u(x) = u_1(x) + u_2(x) \) satisfies the original differential equation, and determine \( u_1(x), u_2(x) \) and the exact solution \( u(x) \).

3. Compute and plot the approximate solutions for \( N = 8, 16, 32, 64, \) and compare it to the exact solution \( u(x) \). On a separate plot, compute the maximum error \( e_h \) for a given \( h \)

\[ e_h = \max_{0 \leq i \leq N+1} |u(x_i) - u_i| \]

and plot \( \log(h) \) against \( \log(e_h) \) (in class, we showed this line should have slope 2 - you may wish to check this is true by also plotting \( \log(h) \) against \( 2 \log(h) \) along with the error. Both the error and this line should have identical slopes).
Vector spaces
A set \( V \) is a set that satisfies the following properties

(i.) For all \( u, v \in V \), \( u + v = v + u \).
(ii.) For all \( u, v, w \in V \), \((u + v) + w = u + (v + w)\).
(iii.) \( 0 \in V \) and satisfies \( u + 0 = u \).
(iv.) For each \( u \in V \), then there exist a vector \( -u \in V \) such that \( u + (-u) = 0 \).
(v.) \( \alpha(u + v) = \alpha u + \alpha v \) for all \( u, v \in V \) and scalars \( \alpha \).
(vi.) \( (\alpha + \beta)u = \alpha u + \beta u \) for all \( u \in V \) and scalars \( \alpha, \beta \).
(vii.) \( \alpha(\beta u) = (\alpha\beta)u \) for all \( u \in V \) and scalars \( \alpha, \beta \).
(viii.) \( 1u = u \) for all \( u \in V \).

Determine if the following spaces are vector spaces. If the space is not a vector space, show what property it fails. If the space is a vector space, prove it.

1. (TA led example) \( \mathbb{R}^n \)
2. \( P^n = \{ p(x) \text{ such that } p(x) \text{ is a polynomial of degree at least } n \} \)
3. The set of polynomials of degree \( n \geq 0 \)
4. The set of real valued functions on \( [0,1] \) i.e. \( \{ f : f(x) \to \mathbb{R} \text{ for } x \in [0,1] \} \)
5. \( C^2[a,b] \) This is the set of functions \( f(x) \) such that \( f(x), f'(x), f''(x) \to \mathbb{R} \) are continuous.
6. \( C^2_D[a,b] \) This is the set of functions \( f(x) \) such that \( f(x), f'(x), f''(x) \to \mathbb{R} \) are continuous and \( f(a) = f(b) = 0 \).
7. \( \hat{C}^2[a,b] \) This is the set of functions \( f(x) \) such that \( f(x), f'(x), f''(x) \to \mathbb{R} \) are continuous and \( f(a) = f(b) = 0 \).