Worksheet #2: Differential equations, and finite differences

Order

(1) Is \( \tan \epsilon = o(\epsilon) \) as \( \epsilon \to 0? \)

Solution:

No.

\[
\lim_{\epsilon \to 0} \frac{\tan \epsilon}{\epsilon} = \lim_{\epsilon \to 0} \frac{\sin \epsilon}{\epsilon \cos \epsilon} = \lim_{\epsilon \to 0} \frac{\cos \epsilon}{\cos \epsilon - \epsilon \sin \epsilon} = \frac{1}{1} \neq 0 \implies \tan \epsilon \neq o(\epsilon).
\]

(2) Is \( \tan \epsilon = O(\epsilon) \) as \( \epsilon \to 0? \)

Solution:

Yes. We know the limit is 1. Thus, we can choose any number \( M > 1 \) such that

\[
\forall \epsilon \in [0, c) : \frac{\tan \epsilon}{\epsilon} < M,
\]

which implies

\[
\tan \epsilon = O(\epsilon).
\]

Finite differences

Suppose \( N \geq 1 \) is an integer and define \( h = 1/(N + 1) \) and \( x_j = ih \) for \( i = 0, \ldots, N + 1 \). We can approximate the differential equation

\[
-u''(x) = f(x), \quad 0 < x < 1,
\]

with homogeneous Dirichlet boundary conditions \( u(0) = u(1) = 0 \) by the matrix equation

\[
-\frac{1}{h^2} \begin{bmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
& 1 & -2 & \ddots & \\
& & \ddots & 1 & \\
& & & 1 & -2
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{N-1} \\
u_N
\end{bmatrix} = \begin{bmatrix}
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_{N-1}) \\
f(x_N)
\end{bmatrix},
\]

where \( u_i \approx u(x_i) \). (Entries of the matrix that are not specified are zero.)

(1) Explain what adjustments to the right hand side of the matrix equation are necessary to accommodate the inhomogeneous Dirichlet boundary conditions

\[
u(0) = 1, \quad u(1) = 2.
\]

Solution:

The finite difference at the \( x_1 \) is

\[
-\frac{1}{h^2}(u_0 - 2u_1 + u_2) = f(x_1).
\]

Now \( u_0 \) changes from 0 to 1, the above equation changes to

\[
-\frac{1}{h^2}(-2u_1 + u_2) = f(x_1) + \frac{1}{h^2}.
\]

Similarly the finite different at \( x_N \) changes to

\[
-\frac{1}{h^2}(u_{N-1} - 2u_N) = f(x_N) + \frac{2}{h^2}.
\]
Therefore the right hand side changes to
\[
\begin{bmatrix}
  f(x_1) + \frac{1}{h^2} \\
f(x_2) \\
  \vdots \\
f(x_{N-1}) \\
f(x_N) + \frac{2}{h^2}
\end{bmatrix}
\]
Suppose that we have
\[-u''(x) = (2\pi)^2 \sin(2\pi x), \quad 0 < x < 1,\]
\[u(0) = 1\]
\[u(1) = 2.\]

Since this differential equation is linear, we can split up the solution into
\[u(x) = u_1(x) + u_2(x),\]
where \(u_1(x)\) satisfies
\[-u_1''(x) = 0, \quad 0 < x < 1,\]
\[u_1(0) = 1\]
\[u_1(1) = 2\]
and \(u_2(x)\) satisfies the equation
\[-u_2''(x) = (2\pi)^2 \sin(2\pi x), \quad 0 < x < 1,\]
\[u_2(0) = 0\]
\[u_2(1) = 0.\]

Show that \(u(x) = u_1(x) + u_2(x)\) satisfies the original differential equation, and determine \(u_1(x), u_2(x)\) and the exact solution \(u(x)\).

**Solution:**

Since the second derivative of \(u_1\) is zero, we know \(u_1\) is a linear function: \(u_1(x) = C_1 x + C_2\) where \(C_1\) and \(C_2\) are two constants to be determined from the boundary conditions: \(C_1 = C_2 = 1.\)

To solve \(u_2\), we take the indefinite integral of the right hand side once: \(u_2'(x) = 2\pi \cos(2\pi x) + C_1\) and a second time: \(u_2(x) = \sin(2\pi x) + C_1 x + C_2\) where \(C_1\) and \(C_2\) are integration constants which can be determined from the two boundary conditions: \(C_1 = 0, C_2 = 0.\)

The exact solution is then \(u(x) = u_1(x) + u_2(x) = \sin(2\pi x) + x + 1.\)
(3) Compute and plot the approximate solutions for $N = 8, 16, 32, 64$, and compare it to the exact solution $u(x)$. On a separate plot, compute the maximum error $e_h$ for a given $h$

$$e_h = \max_{0 \leq i \leq N+1} |u(x_i) - u_i|$$

and plot $\log(h)$ against $\log(e_h)$ (in class, we showed this line should have slope 2 - you may wish to check this is true by also plotting $\log(h)$ against $2 \log(h)$ along with the error. Both the error and this line should have identical slopes).

**Solution:** The code to plot the approximate solution against the exact solution:

```matlab
N = 8;
h = 1/(N+1);
grid = linspace(0,1,N+2);
f = @(x) (2*pi)^2* sin(2*pi*x);
exact = @(x) sin(2*pi*x)+x+1;
RHS = f(grid(2:end-1))';
RHS(1) = RHS(1) + 1/h^2;
RHS(end) = RHS(end) + 2/h^2;
LHS = -full(gallery('tridiag',N,1,-2,1))/h^2;
sol = LHS\RHS;
plot(grid,[1;sol;2],grid,exact(grid)), legend('approximate','exact')
```
The code to plot the error graph:

```matlab
f = @(x) (2*pi)^2*sin(2*pi*x);
exact = @(x) sin(2*pi*x)+x+1;
error = zeros(1,4);
for i=1:4
    N = 2^(i+2);
    h(i) = 1/(N+1);
    grid = linspace(0,1,N+2);

    RHS = f(grid(2:end-1))';
    RHS(1) = RHS(1) + 1/h(i)^2;
    RHS(end) = RHS(end) + 2/h(i)^2;
    LHS = -full(gallery('tridiag',N,1,-2,1))/h(i)^2;
    sol = LHS\RHS;
    error(i) = max(abs(exact(grid)'-[1;sol;2]));
end
loglog(h,error,'x-',h,h.^2,'r--'), legend('h-log(e_h).','slope 2')
```
Vector spaces
A set $V$ is a set that satisfies the following properties

(i.) For all $u, v \in V$, $u + v = v + u$.
(ii.) For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.
(iii.) $0 \in V$ and satisfies $u + 0 = u$.
(iv.) For each $u \in V$, then there exist a vector $-u \in V$ such that $u + (-u) = 0$.
(v.) $\alpha(u + v) = \alpha u + \alpha v$ for all $u, v \in V$ and scalars $\alpha$.
(vi.) $(\alpha + \beta)u = \alpha u + \beta u$ for all $u \in V$ and scalars $\alpha, \beta$.
(vii.) $\alpha(\beta u) = (\alpha\beta)u$ for all $u \in V$ and scalars $\alpha, \beta$.
(viii.) $1u = u$ for all $u \in V$.

Determine if the following spaces are vector spaces. If the space is not a vector space, show what property it fails. If the space is a vector space, prove it.

(1) (TA led example) $\mathbb{R}^n$
Solution: Yes.

(2) $P^n = \{p(x)\text{such that } p(x)\text{is a polynomial of degree at least } n\}$
Solution: No, fails (i).

(3) The set of polynomials of degree $n > 0$
Solution: No, fails (iii).

(4) The set of real valued functions on $[0,1]$ i.e. $\{f : f(x) \to \mathbb{R} \text{for } x \in [0,1]\}$
Solution: Yes.

(5) $C^2[a, b]$ This is the set of functions $f(x)$ such that $f(x), f'(x), f''(x) \to \mathbb{R}$ are continuous.
Solution: Yes. To verify some function lies in $C^2[a, b]$, one needs to show
- the function itself is defined and continuous on $[a, b]$.
- its first derivative is defined and continuous on $[a, b]$.
- its second derivative is defined and continuous on $[a, b]$.

(6) $C^2_D[a, b]$ This is the set of functions $f(x)$ such that $f(x), f'(x), f''(x) \to \mathbb{R}$ are continuous and $f(a) = f(b) = 0$.
Solution: Yes.

(7) $C^2_D[a, b]$ This is the set of functions $f(x)$ such that $f(x), f'(x), f''(x) \to \mathbb{R}$ are continuous and $f(a) = f(b) = 1$.
Solution: No, fails (i).