Worksheet #4: Inner products

(1) Let $V = \mathbb{R}^3$ with inner product $(x, y) = x \cdot y$. Rewrite

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

as an orthogonal basis. Then normalize the basis.

Solution:

We use the Gram-Schmidt method. Denote

$$b_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Step 1: Take the first given vector

$$\hat{b}_1 = b_1.$$

Step 2: Since $\hat{b}_2$ is supposed to be orthogonal to $\hat{b}_1$, we eliminate the $\hat{b}_1$ component from $\hat{b}_2$. Let

$$\hat{b}_2 = b_2 - \text{proj}_{\hat{b}_1} b_2$$

$$= b_2 - \frac{(b_2, \hat{b}_1)}{\|\hat{b}_1\|} \hat{b}_1$$

$$= \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}.$$

Step 3: Since $\hat{b}_3$ is supposed to be orthogonal to $\hat{b}_1$ and $\hat{b}_2$, we eliminate the $\hat{b}_1$ and $\hat{b}_2$ component from $\hat{b}_3$. Let

$$\hat{b}_3 = b_3 - \text{proj}_{\hat{b}_1} b_3 - \text{proj}_{\hat{b}_2} b_3$$

$$= b_3 - \frac{(b_3, \hat{b}_1)}{\|\hat{b}_1\|^2} \hat{b}_1 - \frac{(b_3, \hat{b}_2)}{\|\hat{b}_2\|^2} \hat{b}_2$$

$$= \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}.$$

Step 4: Normalize the three vectors.

$$e_1 = \frac{\hat{b}_1}{\|\hat{b}_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix},$$

$$e_2 = \frac{\hat{b}_2}{\|\hat{b}_2\|} = \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{2/3} \\ 1/\sqrt{6} \end{bmatrix},$$

$$e_3 = \frac{\hat{b}_3}{\|\hat{b}_3\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$
(2) Consider the set of functions \( f_n(x) = \sin(n2\pi x) \), \( g_m(x) = \cos(m\pi x) \) for \( n, m = 1, 2, \ldots \) defined on the interval \([0, 1]\).

(a) Show that \( \{f_m\}_{m=1}^{\infty} \) is a mutually orthogonal set of functions on \([0, 1]\).

\textbf{Solution:}

For any \( m_1 \neq m_2 \in \{1, 2, \ldots \} \), we show that

\[
(f_{m_1}, f_{m_2}) = \int_0^1 \sin(m_1 2\pi x) \sin(m_2 2\pi x) \, dx
\]

\[
= \int_0^1 \frac{1}{2} (\cos((m_1 - m_2)2\pi x) - \cos((m_1 + m_2)2\pi x)) \, dx
\]

\[
= 0.
\]

In the second step, we used the sum and difference formula for \( \cos \).
(b) Show that \( \sin(2\pi nx) \) and \( \cos(2\pi mx) \) are orthogonal on \([0, 1]\).

**Solution:** We show that

\[
\langle \sin(2\pi nx), \cos(2\pi mx) \rangle = \int_0^1 \sin(2\pi nx) \cos(2\pi mx) \, dx
\]

\[
= \int_0^1 \frac{1}{2} (\sin(2\pi(n + m)x) + \sin(2\pi(n - m)x))
\]

\[
= 0
\]

We used the sum and difference formula for \( \sin \).
(c) Suppose the function \( u(x) \) on \([0, 1]\) can be expressed as

\[
u(x) = \sum_{i=1}^{\infty} \alpha_i f_i(x) + \sum_{j=1}^{\infty} \beta_j g_j(x).
\]

Find an expression for the coefficients \( \alpha_i \) and \( \beta_j, \ i, j = 1, 2, \ldots \).

NOTE: In the homework you will show that \( \{g_m\}_{m=1}^{\infty} \) is a mutually orthogonal set of functions on \([0, 1]\). Here you can use that result.

**Solution:**

It is clear by now that \( \{f_n\}_{n=1}^{\infty}, \{g_m\}_{m=1}^{\infty} \) is a mutually orthogonal set. We take any two functions from that set. If they both belong to \( \{f_n\}_{n=1}^{\infty} \), then the pair is orthogonal by (a). If they both belong to \( \{g_m\}_{m=1}^{\infty} \), the pair orthogonal by homework. If one belongs to \( \{f_n\}_{n=1}^{\infty} \) and the other belongs to \( \{g_m\}_{m=1}^{\infty} \), then the pair is orthogonal by (b).

To get the coefficient, we use the trick of inner product by basis. We inner product both sides by \( f_i(x) \) and

\[
(u(x), f_i(x)) = \alpha_i (f_i(x), f_i(x)).
\]

All other terms disappear because of orthogonality. Therefore,

\[
\alpha_i = 2 \int_0^1 u(x) f_i(x) \, dx,
\]

where the coefficient \( 2 = \left( \int_0^1 \sin^2(2\pi nx) \, dx \right)^{-1} \).

We inner product both sides by \( g_j \) to get \( \beta_j \),

\[
\beta_j = 2 \int_0^1 u(x) g_j(x) \, dx.
\]
(3) Consider the vectors \( w_1(x) = 1 \) and \( w_2(x) = x \) and \( W = \text{span}\{w_1, w_2\} \) the vector space of \( P_1([0, 1]) \).

Let’s compute the best approximation to the function \( f(x) = \sin(2\pi x) + e^x \) in \( W \) with respect to the \( L^2 \) inner product \( (f, g) = \int_0^1 f(x)g(x)dx \). We will be looking for \( m = x_1w_1 + x_2w_2 \) such that \( \|f - m\| \leq \|f - u\| \) for every \( u \in W \).

To find the coefficients \( x_1 \) and \( x_2 \), we need to solve \( Gx = b \) where \( G \) is the Gram matrix

\[
G = \begin{bmatrix}
(w_1, w_1) & (w_2, w_1) \\
(w_1, w_2) & (w_2, w_2)
\end{bmatrix}
\]

and

\[
b = \begin{bmatrix}
(f, w_1) \\
(f, w_2)
\end{bmatrix}.
\]

What is the relationship between this approximation and the one you found in problem 2?

**Solution:**

We solve \( x \) from the Grammian system:

\[
\begin{bmatrix}
1 & 1/2 \\
1/2 & 1/3
\end{bmatrix} x = \begin{bmatrix}
e - 1 \\
1 - \frac{1}{2\pi}
\end{bmatrix},
\]

and

\[
x = \begin{bmatrix}
-10 + 4e + \frac{3}{\pi} \\
\frac{6(1-3\pi+e\pi)}{\pi}
\end{bmatrix} \approx \begin{bmatrix}
1.828 \\
-0.220
\end{bmatrix}
\]

This approximation happens in a finite dimensional space and the one in problem 2 happens in an infinite dimensional space.
(4) Suppose that \( V \) is a vector space and \( W \subset V \) is a finite dimensional subspace. We know that given \( v \in V \) we can compute the closest approximation \( w \in W \) to \( v \) by solving the Gram system \( G \alpha = c \) which arises from applying the projection theorem. If we were to construct a sequence of finite dimensional vector spaces \( W_1 \subset W_2 \subset W_3 \ldots \subset V \) and compute a sequence of best approximations \( w_1 \in W_1, w_2 \in W_2, w_3 \in W_3 \ldots \) to \( v \in V \) it is reasonable to assume that \( w_2 \) would be a better approximation to \( v \) than \( w_1 \), \( w_3 \) would be a better approximation to \( v \) than \( w_2 \), etc. This problem is designed to show you, visually, that this is indeed what occurs. Define the inner product \((u, v)\) to be
\[
(u, v) = \int_0^1 u(x)v(x) \, dx
\]
and let the norm \( \|v(x)\| \) be defined by
\[
\|v\| = \sqrt{(v, v)}.
\]
Let \( N \) be a positive integer and let \( \phi_1, \ldots, \phi_N \in C[0, 1] \) be such that \( \{\phi_1, \ldots, \phi_N\} \) is orthonormal with respect to the inner product \((\cdot, \cdot)\). We wish to approximate a continuous function \( f(x) \) with \( f_N(x) \)
\[
f_N(x) = \sum_{n=1}^{N} \alpha_n \phi_n(x)
\]
where
\[
\phi_n(x) = \sqrt{2} \sin(n\pi x), \quad n = 1, 2, \ldots
\]
and where \( \alpha_n = (f, \phi_n) \). (Note that \( f_N \) is the best approximation to \( f \) from \( \text{span} \{\phi_1, \ldots, \phi_N\} \) with respect to the norm \( \| \cdot \| \).)

(a) Assume that \( f_N \to f \) as \( N \to \infty \). Show that, since \( \phi_1, \ldots, \phi_N \) are orthonormal,
\[
\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2.
\]

(Hint: \( \|f - f_N\|^2 = (f - f_N, f - f_N) \))

Solution:

\[
\|f - f_N\|^2 = (f - f_N, f - f_N)
\]
\[
= (f, f) - 2(f_N, f) + (f_N, f_N)
\]
\[
= (f, f) - 2(f_N, f - f_N + f_N) + (f_N, f_N)
\]
\[
= (f, f) - 2(f_N, f - f_N) - 2(f_N, f_N) + (f_N, f_N)
\]
\[
= (f, f) - (f_N, f_N)
\]
\[
= \|f\|^2 - \left( \sum_{i=1}^{N} \alpha_i \phi_i, \sum_{j=1}^{N} \alpha_j \phi_j \right)
\]
\[
= \|f\|^2 - \sum_{i=1}^{N} \alpha_i \left( \phi_i, \sum_{j=1}^{N} \alpha_j \phi_j \right)
\]
\[
= \|f\|^2 - \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j (\phi_i, \phi_j) = \|f\|^2 - \sum_{i=1}^{N} \alpha_i^2.
\]
(b) Let \( f(x) = x(1-x) \). Solving the Gram system for the coefficients \( \alpha_n \) of the best approximation \( f_N \) to \( f(x) \) gives

\[
\alpha_n = \frac{2\sqrt{2}}{n^3\pi^3} (1 - (-1)^n).
\]

Plot the true function \( f(x) \) and compare it to \( f_N(x) \) for \( N = 5 \). On a separate figure, plot the norm of the error \( \|f - f_N\| \) using the above formula for \( N = 1, 2, \ldots, 100 \) on a log-log scale by using `loglog` in MATLAB.

**Solution:**

```matlab
%% Plot f versus f_N
x=linspace(0,1,100);
f=@(x) x.*(1-x);
f_N=zeros(size(x));
for i=1:5
    f_N = f_N + 2*sqrt(2)/(i^3* pi^3)*(1 -(-1)^i)*sqrt(2)*
    sin(i*pi*x);
end
plot(x,f(x),x,f_N), legend('f(x)', 'f_5(x)');

%% Loglog plot of error
N = 1:100;
S = (2*sqrt(2)./(N.^3* pi^3).*(1-(-1).^N)).^2;  
S = sqrt(1/30 - cumsum(S));  
figure, loglog(N,S)
```

![Plot of f(x) and f_5(x)](image1)

![Loglog plot of error](image2)
(c) For $f(x) = 1$ solving the Gram system gives that the coefficients of the best approximation satisfy

$$\alpha_n = 2\sqrt{2}/(n\pi)$$

for odd $n$, and $\alpha_n = 0$ for even $n$. Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 100$. On a separate figure, plot the norm of the error $\|f - f_N\|$ using the above formula for $N = 1, 2, \ldots, 100$ on a log-log scale by using \texttt{loglog} in MATLAB.

\textit{You may have noticed that the rate at which the coefficients $\alpha_n \to 0$ determines how fast the error decreases — this is not coincidental!}

\textbf{Solution:}

```matlab
%% Plot f versus f_N
x=linspace(0,1,100);
f = @(x) ones(size(x));
f_N = zeros(size(x));
for i = 1:2:100
    f_N = f_N + 2*sqrt(2)/(i*pi)*sqrt(2)*sin(i*pi*x);
end
plot(x,f(x),x,f_N), legend('f(x)', 'f_{100}(x)');

%% Loglog plot of error
N = 1:100;
S = (2*sqrt(2)./(N*pi)).^2;
S(2:2:100) = 0;
S = sqrt(1 - cumsum(S));
figure, loglog(N,S)
```

![Plot of f versus f_N](image1)

![Loglog plot of error](image2)