Worksheet #9: Time stepping schemes and stability

(1) Consider applying both forward and backward Euler to the initial value problem
\[
\begin{align*}
y'(t) &= f(t, y(t)) \\
y(0) &= C \in \mathbb{R}.
\end{align*}
\]
Prove that these methods are first order accurate.

Solution: Need to show that the truncation error is \( O(h) \). For forward Euler, truncation error is given by
\[
T_k = \frac{y(t_{k+1}) - y(t_k)}{h} - f(t_k, y(t_k)).
\]
We first apply Taylor’s theorem to the term \( y(t_{k+1}) \), centered at \( t_k \), using the fact that \( t_{k+1} = t_k + h \);
\[
y(t_{k+1}) = y(t_k) + hy'(t_k) + \frac{h^2}{2}y''(\xi)
\]
for some \( \xi \in (t_k, t_{k+1}) \). Making use of the ODE we wish to solve, that is \( f(t_k, y(t_k)) = y'(t_k) \), and our expression for \( y(t_{k+1}) \), we have
\[
T_k = \left[ y(t_k) + hy'(t_k) + \frac{h^2}{2}y''(\xi) \right] - y(t_k) - \frac{h}{2}y''(\xi).
\]

The truncation error for backward Euler is given by
\[
T_k = \frac{y(t_{k+1}) - y(t_k)}{h} - f(t_{k+1}, y(t_{k+1})).
\]
We apply Taylor’s theorem to \( y(t_k) \) centered at \( t_{k+1} \), using \( t_k = t_{k+1} - h \);
\[
y(t_k) = y(t_{k+1}) - hy'(t_{k+1}) + \frac{h^2}{2}y''(\xi)
\]
for some \( \xi \in (t_k, t_{k+1}) \). Making use of the ODE we wish to solve, that is \( f(t_{k+1}, y(t_{k+1})) = y'(t_{k+1}) \), and our expression for \( y(t_k) \), we have
\[
T_k = \frac{y(t_{k+1}) - [y(t_{k+1}) - hy'(t_{k+1}) + \frac{h^2}{2}y''(\xi)]}{h} - y'(t_{k+1}) = \frac{h}{2}y''(\xi).
\]
(2) Consider the initial value problem
\[ y'(t) = f(t, y(t)) \]
\[ y(0) = C \in \mathbb{R}. \]

(a) Let \( \Delta t = t_{n+1} - t_n \) for all \( n \). Derive the time stepping scheme that results from approximating the integral of \( f(t, y(t)) \) over the interval \([t_n, t_{n+1}]\) with the area of the trapezoid.

(b) Apply the trapezoidal method you just found to “test problem.” In other words, apply the trapezoidal method to
\[ y'(t) = -\lambda y(t) \]
\[ y(0) = 1 \]
where \( \lambda > 0 \).

(c) Determine a criterion (if any is needed) for \( \Delta t \) which guarantees convergence of the method.

(d) Classify the stability of this method? (unconditionally stable or conditionally stable)

**Solution:**

(a) We derive a time stepping scheme by integrating our ODE and applying the trapezoidal rule on the integral over \( f(t, y(t)) \);
\[
\int_{t_n}^{t_{n+1}} f(t, y(t)) \, dt \approx \frac{\Delta t}{2} [f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))],
\]
\[
\int_{t_n}^{t_{n+1}} y'(t) \, dt = y(t_{n+1}) - y(t_n),
\]
thus
\[
y(t_{n+1}) = y(t_n) + \frac{\Delta t}{2} [f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))].
\]

(b) Applying the trapezoidal method gives,
\[
y_{n+1} = y_n + \frac{\Delta t}{2} [-\lambda y_n - \lambda y_{n+1}].
\]

After some algebra we get
\[
y_{n+1} = y_n \frac{1 - \frac{1}{2} \Delta t \lambda}{1 + \frac{1}{2} \Delta t \lambda}.
\]

(c) We require the following criteria to guarantee convergence,
\[
\frac{1 - \frac{1}{2} \Delta t \lambda}{1 + \frac{1}{2} \Delta t \lambda} < 1.
\]

Note that this inequality is automatically achieved since \( \lambda, \Delta t > 0 \).

(d) This method is unconditionally stable (also A-stable).