CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 1

Posted Friday, September 2nd 2016. Due Friday, September 9th by 5pm.

A reminder from the course syllabus: Mathematically rigorous solutions are expected; strive for clarity and elegance. You may collaborate on the problems, but your write-up must be your own independent work. Transcribed solutions and copied MATLAB code are both unacceptable. You may not consult solutions from previous sections of this class.

Unless it is specified that a particular calculation must be performed ‘by hand,’ you are encouraged to use MATLAB’s Symbolic Math Toolbox (or Mathematica/Wolfram Alpha/Maple) to compute and simplify tedious integrals and derivatives on the problem sets. As always, you must document your calculations clearly.

A total of 83 points is distributed among the following problems

1. [12 pts (2 ea)]
   For each of the following equations, specify whether each is (a) an ODE or a PDE; (b) determine its order; (c) specify whether it is linear or nonlinear. For those that are linear, specify whether they are (d) homogeneous or inhomogeneous, and (e) whether they have constant or variable coefficients.

   \[
   \begin{align*}
   (1.1) \quad & \frac{dv}{dx} + \frac{2}{x}v = 0 \\
   (1.2) \quad & \frac{\partial v}{\partial t} - \frac{3}{x} \frac{\partial v}{\partial x} = x - t \\
   (1.3) \quad & \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left[ 2u \frac{\partial u}{\partial x} \right] = 0 \\
   (1.4) \quad & \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \\
   (1.5) \quad & \frac{d^2 y}{dx^2} - \mu (1 - y^2) \frac{dy}{dx} + y = 0 \\
   (1.6) \quad & \frac{d^2}{dx^2} \left[ \rho(x) \frac{d^2 u}{dx^2} \right] = \sin(x)
   \end{align*}
   \]

2. [21 pts (7 ea)]
   Consider the temperature function
   \[ u(x, t) = e^{-\kappa \theta^2 t/(\rho c)} \sin(\theta x) \]
   for constant \( \kappa, \rho, c, \) and \( \theta. \)
   (a) Show that this function \( u(x, t) \) is a solution of the homogeneous heat equation
   \[ \rho c \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad \text{for } 0 < x < \ell \text{ and all } t. \]
   (b) For which values of \( \theta \) will \( u \) satisfy homogeneous Dirichlet boundary conditions at \( x = 0 \) and \( x = \ell? \)
   (c) Suppose \( \kappa = 2.37 \text{ W/(cm K)} \), \( \rho = 2.70 \text{ g/cm}^3 \), and \( c = 0.897 \text{ J/(g K)} \) (approximate values for aluminum found on Wikipedia), and that the bar has length \( \ell = 10 \) cm. Let \( \theta \) be such that \( u(x, t) \) satisfies homogeneous Dirichlet boundary conditions as in part (b) and \( u(x, t) \geq 0 \) for all \( x \) and \( t. \)
   Use MATLAB to plot the solution \( u(x, t) \) for \( 0 \leq x \leq \ell \) and time \( 0 \leq t \leq 20 \) sec.
   You may choose to do this in one of the following ways: (1) Plot the solution for \( 0 \leq x \leq \ell \) at times \( t = 0, 4, 8, \ldots, 20 \) sec., superimposing all six plots on the same axis (helpful commands: \texttt{linspace}, \texttt{plot}, \texttt{hold on}); (2) Create a three-dimensional plot of the data using \texttt{surf}, \texttt{mesh}, or \texttt{waterfall}. In either case, be sure to produce an attractive, well-labeled plot.
3. [26 points: 8 each (a),(b), 10 for (c)]
   Recall the 1D steady-state heat equation with constant diffusivity over the interval [0, 1]
   \[-\frac{\partial^2 u}{\partial x^2} = f\]
   \[u(0) = u(1) = 0.\]
   Recall from class the finite difference approximation to this problem: given a set of points \(x_0, \ldots, x_{N+1}\),
   solved for the solution \(u(x_i)\) at each point by approximating \(\frac{\partial^2 u}{\partial x^2}\) with
   \[u''(x_i) \approx \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2},\quad i = 1, \ldots, N\]
   (where \(h\) is the spacing between points \(x_{i+1}\) and \(x_i\)) along with the conditions that
   \[u(x_0) = u(x_{N+1}) = 0.\]
   We will modify this finite difference approximation to accommodate instead the Neumann boundary condition of \(u'(1) = 0\) at \(x = 1\).
   (a) We would like to enforce that \(u'(x_{N+1}) = 0\), but if we approximate \(u'(x_{N+1})\) with a central difference
   \[u'(x_{N+1}) \approx \frac{u(x_{N+\frac{1}{2}}) - u(x_{N-\frac{1}{2}})}{h},\]
   we end up with an equation involving \(u(x_{N+\frac{1}{2}})\), which does not lie inside the interval [0, 1].
   Instead, we can define a backward difference approximation to the derivative
   \[u'(x_{N+1}) \approx \frac{u(x_{N+1}) - u(x_N)}{h} = 0\]
   and set this to zero instead. Write out the expression for \(u''(x)\) in terms of \(u(x_i)\) and use the backward difference approximation for \(u'(x_{N+1})\) to eliminate \(u(x_{N+1})\).
   (b) Determine the exact solution to \(-u''(x) = 1\) for \(u(0) = 0, u'(1) = 0\) (hint: the solution is a quadratic function).
   (c) Create a MATLAB script that constructs the matrix system \(Au = f\) resulting from the finite difference equations when \(f = 1\). Plot the computed solution values \(u(x_i)\) for \(i = 0, \ldots, N+1\) for \(N = 16, 32, 64, 128\). On a separate plot, compute the maximum error \(e_h\) for a given \(h\)
   \[e_h = \max_{0 \leq i \leq N+1} |u(x_i) - u_i|\]
   and plot \(\log(h)\) against \(\log(e_h)\). Does the error decrease faster or slower compared to the error computed in the finite difference example on the worksheet? Can you explain why?

4. [24 points: 4 each]
   (a) Demonstrate whether or not the set \(S_1 = \{x \in \mathbb{R}^2 : x_2 = x_1^3\}\) is a subspace of \(\mathbb{R}^2\).
   (b) Demonstrate whether or not the set \(S_2 = \{x \in \mathbb{R}^3 : 3x_1 + 2x_2 + x_3 = 0\}\) is a subspace of \(\mathbb{R}^3\).
   (c) Demonstrate whether or not the set \(S_3 = \{f \in C[0, 1] : f(x) \geq 0\text{ for all }x \in [0, 1]\}\) is a subspace of \(C[0, 1]\).
   (d) Demonstrate whether or not the set \(S_4 = \{f \in C[0, 1] : \max_{x \in [0, 1]} f(x) \leq 1\}\) is a subspace of \(C[0, 1]\).
   (e) Demonstrate whether or not the set \(S_5 = \{f \in C^2[0, 1] : f(1) = 1\}\) is a subspace of \(C^2[0, 1]\).
   (f) Demonstrate whether or not the set \(S_6 = \{f \in C^2[0, 1] : f(1) = 0\}\) is a subspace of \(C^2[0, 1]\).