CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 2

Posted Friday 16, September 2016. Due 5pm Friday 23, September 2016.

A note on proofs (unless otherwise stated by the problem): When asked to prove a statement this means you are to show that the requisite properties (discussed in class) for the item referenced by the statement hold true. When asked to disprove something this means you are to come up with an example that shows the proposed premise is false.

Unless explicitly stated otherwise in the problem you are free to use MATLAB as you see fit; including for those problems that do not explicitly require it. Please submit any code that you utilize as a printout or, if it is short enough, reference your steps in your writeup directly. (example: I used matlab to compute the inverse to the matrix B and got .. etc)

A total of 55 points is distributed among the following problems

Please write your name and instructor on your homework.


1. [ 8 points: 2 each]
Recall that a function $f : \mathbb{V} \rightarrow \mathbb{W}$ that maps a vector space $\mathbb{V}$ to a vector space $\mathbb{W}$ is a linear operator provided (1) $f(u + v) = f(u) + f(v)$ for all $u, v$ in $\mathbb{V}$ and (2) $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{R}$ and $v \in \mathbb{V}$.
Demonstrate whether each of the following functions is a linear operator.
(Show that both properties hold, or give an example showing that one of the properties must fail.)

(a) $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x) = x^T x$.
(b) $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$, $f(X) = AX + XB$ for fixed matrices $A, B \in \mathbb{R}^{n \times n}$.
(c) $L : C^1[0,1] \rightarrow C[0,1]$, $Lu = \frac{du}{dx}$
(d) $L : C^2[0,1] \rightarrow C[0,1]$, $Lu = \frac{d^2u}{dx^2} - \sin(x)\frac{du}{dx} + \cos(x)u$.

2. [ 12 points: (a),(b),(d) 2 each, (c) 6 pts] Characterize the range (1 point) and the nullspace (1 point) of the following linear operators:

(a) The first derivative operator $D_x : C^1[0,1] \rightarrow C[0,1]$ given by $D_x(f) = \partial_x f$.
(b) The second derivative operator $D_{xx} : C^2[0,1] \rightarrow C[0,1]$ given by $D_{xx}(f) = \partial_{xx} f$.
(c) Let $C^2_{D}[0,1]$ denote the set of all functions in $u \in C^2[0,1]$ such that $u(0) = u(1) = 0$. Consider again the second derivative operator $D_{xx} : C^2_{D}[0,1] \rightarrow C[0,1]$ given by $D_{xx}(f) = \partial_{xx} f$. What is the null space (1 pts)? You will determine the range by considering the following steps:

(i) [1 point ] Show that the range of the first derivative operator $\partial_x : C^2_{D}[0,1] \rightarrow C^1[0,1]$ consists of functions in $C^1[0,1]$ having have average value zero (ie. $\int_0^1 u = 0$).
(ii) [1 point ] Show that for each $u \in C^1[0,1]$ with average value zero there exists $f \in C^2_{D}[0,1]$ with $\partial_x f = u$.
(iii) [1 point ] Show that for every $g \in C[0,1]$ there exists $u \in C^1[0,1]$, having average value zero (ie. $\int_0^1 u = 0$), such that $\partial_x u = g$.
(iv) [2 points ] Interpret the meaning of the above sub-steps by clearly stating what the range is for the operator $\partial_{xx} : C^2_{D}[0,1] \rightarrow C[0,1]$ and using parts (i)-(iii) to thoroughly justify your answer.
(d) The matrix $A : \mathbb{R}^3 \rightarrow \mathbb{R}^5$ given by

$$A = \begin{bmatrix}
5 & 0 & 1 & 4 & 2 \\
6 & 3 & 8 & -2 & 4 \\
8 & 4 & 9 & 4 & 2 \\
-1 & 3 & 5 & 6 & 4 \\
11 & 3 & 9 & 2 & 6 \\
\end{bmatrix}$$
3. [10 points: 2 each]
Determine whether or not each of the following mappings is an inner product on the real vector space \( V \). If not, show all the properties of the inner product that are violated.

(a) \((\cdot, \cdot) : V \times V \to \mathbb{R}\) defined by \((u, v) = \int_0^1 u(x)v'(x) \, dx\) where \( V = C^1[0, 1] \).

(b) \((\cdot, \cdot) : V \times V \to \mathbb{R}\) defined by \((u, v) = \int_0^1 |u(x)||v(x)| \, dx\) where \( V = C[0, 1] \).

(c) \((\cdot, \cdot) : V \times V \to \mathbb{R}\) defined by \((u, v) = \int_0^1 u(x)v(x)e^{-x} \, dx\) where \( V = C[0, 1] \).

(d) \((\cdot, \cdot) : V \times V \to \mathbb{R}\) defined by \((u, v) = \int_0^1 (u(x) + v(x)) \, dx\) where \( V = C[0, 1] \).

(e) \((\cdot, \cdot) : V \times V \to \mathbb{R}\) defined by \((u, v) = \int_{-1}^1 xu(x)v(x) \, dx\) where \( V = C[-1, 1] \).

4. [8 points: 4 each] Recall the \( L^2 \) inner product we discussed in class given by

\[
(f, g)_{L^2} = \int_a^b f(x)g(x) \, dx
\]  

(a) Recall: two vectors are orthogonal with respect to an inner product if \((v, w) = 0\). Show that the functions \(\{\cos(n\pi x) \mid n = 1, 2, \ldots\}\) are mutually orthogonal in \( C[0, 1] \) (i.e. that \(\cos(n\pi x)\) and \(\cos(m\pi x)\) are orthogonal when \( n \neq m \)).

(b) What is the length of \( f_n(x) = \cos(n\pi x) \) with respect to the \( L^2 \) inner product on \( C[0, 1] \) ?

5. [5 points] Suppose \( V \) is a vector space with an associated inner product. The angle \( \angle(u, v) \) between \( u \) and \( v \in V \) is defined via the equation

\[
(u, v) = \|u\|\|v\| \cos \angle(u, v).
\]

Let \( V = C[0, 1] \) and \((u, v) = \int_0^1 u(x)v(x) \, dx\). Compute \( \cos \angle(x^n, x^m) \) between \( u(x) = x^n \) and \( v(x) = x^m \) for non-negative integers \( m \) and \( n \). What happens to \( \angle(x^n, x^{n+1}) \) as \( n \to \infty \)?

6. [12 points: 4 each]
Suppose \( N \geq 1 \) is an integer and define \( h = 1/(N + 1) \) and \( x_j = jh \) for \( j = 0, 1, \ldots, N + 1 \). Consider the \( N \) hat functions \( \phi_k \in C[0, 1] \), defined as

\[
\phi_k(x) = \begin{cases} 
\frac{x - x_{k-1}}{h} & \text{if } x \in [x_{k-1}, x_k); \\
\frac{x_{k+1} - x}{h} & \text{if } x \in [x_k, x_{k+1}); \\
0 & \text{otherwise}; 
\end{cases}
\]

for \( k = 1, \ldots, N \). Let the inner product \((\cdot, \cdot) : C[0, 1] \times C[0, 1] \to \mathbb{R}\) be defined by

\[
(u, v) = \int_0^1 u(x)v(x) \, dx
\]

and let the norm \( \| \cdot \| : C[0, 1] \to \mathbb{R} \) be defined by

\[
\|u\| = \sqrt{(u, u)}.
\]
(a) For \( j = 1, \ldots, N \), what is \( \phi_j(x_k) \) for \( k = 0, 1, \ldots, N+1 \)? Simplify your answer as much as possible.

(b) Show that \( \{\phi_1, \ldots, \phi_N\} \) is linearly independent by showing that if \( c_k \in \mathbb{R} \) and \( \sum_{k=1}^{N} c_k \phi_k(x) = 0 \) for all \( x \in [0, 1] \) then \( c_k = 0 \) for \( k = 1, \ldots, N \).

(c) Compute by hand the inner products \( (\phi_j, \phi_k) \) for \( j, k = 1, \ldots, N \). Your final answers should be simplified as much as possible and in your formulas \( h \) should be left as \( h \) and not be replaced with \( 1/(N + 1) \). You must clearly state which values of \( j \) and \( k \) each formula you obtain is valid for. An acceptable way to present the final answer would be:

For \( j, k = 1, \ldots, N \),

\[
(\phi_j, \phi_k) = \begin{cases} 
? & \text{if } k = j, \\
? & \text{if } |j - k| = 1, \\
? & \text{otherwise.}
\end{cases}
\]

with the question marks replaced with the correct values. Hint:

\[
\int_{x_{j-1}}^{x_j} \left( \frac{x - x_{j-1}}{h} \right)^2 \, dx = \frac{1}{h^2} \int_{x_{j-1} - x_{j-1}}^{x_j - x_{j-1}} (s + x_{j-1} - x_{j-1})^2 \, ds = \frac{1}{h^2} \int_{0}^{h} s^2 \, ds
\]

where \( s = x - x_{j-1} \).