A total of 50 points is distributed among the following problems

Please write your name and instructor on your homework.

1. [6 points]
In class we saw examples of both first order and second order finite difference approximations to the first derivative at a mesh point $x_i$ (the forward and central difference approximations, respectively). Using Taylor series expansions, show that the following finite difference expression is a third order accurate approximation of $\partial_x u$ at the mesh point $x_i$.

$$
\frac{2u(x_{i+1}) + 3u(x_i) - 6u(x_{i-1}) + u(x_{i-2})}{6h}
$$

(1)

2. [8 points: 2 each]
Recall that a function $f : V \to W$ that maps a vector space $V$ to a vector space $W$ is a linear operator provided (1) $f(u + v) = f(u) + f(v)$ for all $u, v$ in $V$, and (2) $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{R}$ and $v \in V$.

Demonstrate whether each of the following functions is a linear operator.

(a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x^T x$.

(b) $f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$, $f(X) = AX + XB$ for fixed matrices $A, B \in \mathbb{R}^{n \times n}$.

(c) $L : C^1[0,1] \to C[0,1]$, $Lu = \frac{du}{dx}$.

(d) $L : C^2[0,1] \to C[0,1]$, $Lu = \frac{d^2u}{dx^2} - \sin(x)\frac{du}{dx} + \cos(x)u$.

3. [12 points: (a),(b),(d) 2 each, (c) 6 pts] Characterize the range (1 point) and the nullspace (1 point) of the following linear operators:

(a) The first derivative operator $D_x : C^1[0,1] \to C[0,1]$ given by $D_x(f) = \partial_x f$.

(b) The second derivative operator $D_{xx} : C^2[0,1] \to C[0,1]$ given by $D_{xx}(f) = \partial_{xx} f$.

(c) Let $C^2_0[0,1]$ denote the set of all functions in $u \in C^2[0,1]$ such that $u(0) = u(1) = 0$. Consider again the second derivative operator $D_{xx} : C^2_0[0,1] \to C[0,1]$ given by $D_{xx}(f) = \partial_{xx} f$. What is the null space (1 pts)? You will determine the range by considering the following steps:

(i) [1 point] Show that the range of the first derivative operator $\partial_x : C^2_0[0,1] \to C^1[0,1]$ consists of functions in $C^1[0,1]$ having have average value zero (i.e., $\int_0^1 u = 0$).

(ii) [1 point] Show that for each $u \in C^1[0,1]$ with average value zero there exists $f \in C^2_0[0,1]$ with $\partial_x f = u$. 

• A note on proofs (unless otherwise stated by the problem): When asked to prove a statement this means you are to show that the requisite properties (discussed in class) for the item referenced by the statement hold true. When asked to disprove something this means you are to come up with an example that shows the proposed premise is false.

• Unless explicitly stated otherwise in the problem you are free to use MATLAB as you see fit; including for those problems that do not explicitly require it. Please submit any code that you utilize as a printout or, if it is short enough, reference your steps in your writeup directly. (example: I used matlab to compute the inverse to the matrix $B$ and got .. etc)
(iii) [1 point] Show that for every $g \in C[0,1]$ there exists $u \in C^1[0,1]$, having average value zero (i.e. $\int_0^1 u \, dx = 0$), such that $\partial_x u = g$.

(iv) [2 points] Interpret the meaning of the above sub-steps by clearly stating what the range is for the operator $\partial_{xx} : C^2_D[0,1] \to C[0,1]$ and using parts (i)-(iii) to thoroughly justify your answer.

(d) The matrix $A : \mathbb{R}^5 \to \mathbb{R}^5$ given by

$$A = \begin{bmatrix} 5 & 0 & 1 & 4 & 2 \\ 6 & 3 & 8 & -2 & 4 \\ 8 & 4 & 9 & 4 & 2 \\ -1 & 3 & 5 & 6 & 4 \\ 11 & 3 & 9 & 2 & 6 \end{bmatrix}$$

4. [24 points: 4 each]

(a) Demonstrate whether or not the set $S_1 = \{ \mathbf{x} \in \mathbb{R}^2 : x_2 = x_1^3 \}$ is a subspace of $\mathbb{R}^2$.

(b) Demonstrate whether or not the set $S_2 = \{ \mathbf{x} \in \mathbb{R}^3 : 3x_1 + 2x_2 + x_3 = 0 \}$ is a subspace of $\mathbb{R}^3$.

(c) Demonstrate whether or not the set $S_3 = \{ f \in C[0,1] : f(x) \geq 0 \text{ for all } x \in [0,1] \}$ is a subspace of $C[0,1]$.

(d) Demonstrate whether or not the set $S_4 = \{ f \in C[0,1] : \max_{x \in [0,1]} f(x) \leq 1 \}$ is a subspace of $C[0,1]$.

(e) Demonstrate whether or not the set $S_5 = \{ f \in C^2[0,1] : f(1) = 1 \}$ is a subspace of $C^2[0,1]$.

(f) Demonstrate whether or not the set $S_6 = \{ f \in C^2[0,1] : f(1) = 0 \}$ is a subspace of $C^2[0,1]$.