Homework 3 · Solutions

Posted Friday 23, September 2016. Due 5pm Friday 7, October 2016.

• A note on proofs (unless otherwise stated by the problem): When asked to prove a statement this means you are to show that the requisite properties (discussed in class) for the item referenced by the statement hold true. When asked to disprove something this means you are to come up with an example that shows the proposed premise is false.

• Unless explicitly stated otherwise in the problem you are free to use MATLAB as you see fit; including for those problems that do not explicitly require it. Please submit any code that you utilize as a printout or, if it is short enough, reference your steps in your writeup directly. (example: I used matlab to compute the inverse to the matrix B and got...

Please write your name and instructor on your homework.

There is a total of 80 points distributed among the problems below

1. [15 points : 5 each]
   (a) Show that if we have an orthogonal set of vectors $\phi_1, \ldots, \phi_k$, then $\phi_1, \ldots, \phi_k$ are linearly independent as well, i.e.
   \[
   \sum_{i=1}^{k} \alpha_i \phi_i = 0
   \]
   is only true if $\alpha_1, \ldots, \alpha_k = 0$.
   (b) Let $V$ be an inner product space (i.e. $V$ a vector space with an inner product). Suppose $\{v_1, v_2, v_3\}$ is a basis for $V$, and we would like to construct a is possible to construct a new orthogonal basis $\{\phi_1, \phi_2, \phi_3\}$ through the following procedure:
   \[
   \begin{align*}
   \phi_1 &= v_1 \\
   \phi_2 &= v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)} \phi_1 \\
   \phi_3 &= v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)} \phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)} \phi_2 \\
   & \quad \vdots \\
   \phi_k &= v_k - \sum_{i=1}^{k-1} \frac{(\phi_i, v_k)}{(\phi_i, \phi_i)} \phi_i
   \end{align*}
   \]
   This is called the Gram-Schmidt procedure.
   We refer to nonzero vectors $u_1, u_2, \ldots, u_k \in V$ an orthogonal set if they are orthogonal to each other: i.e. if
   \[
   (u_i, u_j) = 0, \quad i \neq j.
   \]
   Assuming we have $v_1, v_2, v_3$ and we define $\phi_1, \phi_2, \phi_3$ under the above process, show that $\phi_1, \phi_2, \phi_3$ form an orthogonal set, i.e.
   \[
   (\phi_i, \phi_j) = 0, \quad \text{if } 1 \leq i \neq j \leq 3.
   \]
   (c) Since we can define an inner product $\langle \cdot, \cdot \rangle$ on the function space $C[-1,1]$ as
   \[
   (u,v) = \int_{-1}^{1} u(x)v(x) \, dx,
   \]
   we can also use the Gram-Schmidt procedure to create orthogonal sets of functions. Using the Gram-Schmidt procedure above, compute the orthogonal vectors $\{\phi_1, \phi_2, \phi_3\}$ given starting vectors $\{v_1, v_2, v_3\} = \{1, x, x^2\}$. 
Solution.

(a) We are going to show that $(\phi_i, \phi_j) = 0$ if $1 \leq i \neq j \leq 3$. To check that these formulas yield an orthogonal sequence, first compute $(\phi_1, \phi_2)$ by substituting the above formula for $\phi_2$

$$(\phi_1, \phi_2) = (\phi_1, v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)} \phi_1)$$

$$= (\phi_1, v_2) - (\phi_1, \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)} \phi_1)$$

$$= (\phi_1, v_2) - (\phi_1, \phi_1)$$

$$= (\phi_1, v_2) - (\phi_1, v_2)$$

$$= 0.$$ 

Then use the fact that $(\phi_1, \phi_2) = 0$, to compute $(\phi_1, \phi_3)$. By substituting again the formula for $\phi_3$

$$(\phi_1, \phi_3) = (\phi_1, v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)} \phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)} \phi_2)$$

$$= (\phi_1, v_3) - (\phi_1, \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)} \phi_1) - (\phi_1, \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)} \phi_2)$$

$$= (\phi_1, v_3) - (\phi_1, \phi_1) - (\phi_1, \phi_1)$$

$$= (\phi_1, v_3) - (\phi_1, v_3)$$

$$= 0.$$ 

Similarly, using the symmetry property of inner product $(\phi_i, \phi_j) = (\phi_j, \phi_i)$ for all $i, j$. We can show $(\phi_2, \phi_3) = 0$.

$$(\phi_2, \phi_3) = (\phi_2, v_3 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)} \phi_2)$$

$$= (\phi_2, v_3) - (\phi_2, \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)} \phi_2)$$

$$= (\phi_2, v_3) - (\phi_2, \phi_2)$$

$$= (\phi_2, v_3) - (\phi_2, v_3)$$

$$= 0.$$ 

By symmetry we can conclude that $(\phi_2, \phi_3) = (\phi_3, \phi_2) = 0$ and $(\phi_1, \phi_3) = (\phi_3, \phi_1) = 0$. This completes the proof.

(b) Consider a linear relationship

$$\sum_{i=1}^{k} \alpha_i \phi_i = 0$$

which can be written

$$\alpha_1 \phi_1 + \alpha_2 \phi_2 + \cdots + \alpha_k \phi_k = 0.$$
If \( 1 \leq i \leq k \) then taking the inner product of \( \phi_i \) with both sides of the equation and using the properties of inner product (Definition 3.32, page 58),

\[
(\phi_i, \alpha_1 \phi_1 + \alpha_2 \phi_2 + \cdots + \alpha_k \phi_k) = (\phi_i, 0)
\]
\[
(\phi_i, \alpha_1 \phi_1) + (\phi_i, \alpha_2 \phi_2) + \cdots + (\phi_i, \alpha_k \phi_k) = 0
\]
\[
\alpha_1 (\phi_i, \phi_1) + \alpha_2 (\phi_i, \phi_2) + \cdots + \alpha_k (\phi_i, \phi_k) = 0
\]
\[
\alpha_i (\phi_i, \phi_i) = 0
\]

shows, since \( \phi_i \) is nonzero, that \( \alpha_i \) for \( i = 1, \cdots, k \) is zero.

(c) We want to construct the new orthogonal bases for \( V \) by Gram-Schmidt procedure given starting vectors \( \{v_1, v_2, v_3\} = \{1, x, x^2\} \). Following the procedure we set

\[ \phi_1 = v_1 = 1 \]

and

\[ \phi_2 = v_2 - \frac{(\phi_1, v_2)}{(\phi_1, \phi_1)} \phi_1. \]

We compute

\[ (\phi_1, v_2) = \int_{-1}^{1} x \, dx = \left[ \frac{x^2}{2} \right]_{-1}^{1} = 0 \]

and

\[ (\phi_1, \phi_1) = \int_{-1}^{1} 1 \, dx = 2. \]

Now we can compute

\[ \phi_2 = x - \frac{0}{2}(1) = x. \]

Finally for \( \phi_3 \),

\[ \phi_3 = v_3 - \frac{(\phi_1, v_3)}{(\phi_1, \phi_1)} \phi_1 - \frac{(\phi_2, v_3)}{(\phi_2, \phi_2)} \phi_2 \]

\[ (\phi_1, v_3) = \int_{-1}^{1} x^2 \, dx = \left[ \frac{x^3}{3} \right]_{-1}^{1} = \frac{2}{3} \]

and

\[ (\phi_2, v_3) = \int_{-1}^{1} x^3 \, dx = \left[ \frac{x^4}{4} \right]_{-1}^{1} = 0 \]

and

\[ (\phi_2, \phi_2) = \int_{-1}^{1} x^2 \, dx = \left[ \frac{x^3}{3} \right]_{-1}^{1} = \frac{2}{3} \]

Substituting these inner products into the equation for \( \phi_3 \), we get

\[ \phi_3 = x^2 - \frac{2}{3}(1) - \frac{0}{2/3} (x) = x^2 - \frac{1}{3}. \]

This yields \( \{\phi_1, \phi_2, \phi_3\} = \{1, x, x^2 - \frac{1}{3}\} \) as desired.
2. [25 points: (a) 4 points; (b)-(d) 7 points each]

Let \( \phi_1 \in C[-1, 1], \phi_2 \in C[-1, 1], \phi_3 \in C[-1, 1], \) and \( f \in C[-1, 1] \) be defined by

\[
\phi_1(x) = 1, \quad \phi_2(x) = x, \quad \phi_3(x) = 3x^2 - 1,
\]

and

\[
f(x) = e^x,
\]

for all \( x \in [-1, 1] \). Let the inner product \((\cdot, \cdot) : C[-1, 1] \times C[-1, 1] \rightarrow \mathbb{R}\) be defined by

\[
(u, v) = \int_{-1}^{1} u(x)v(x) \, dx.
\]

Let the norm \( \| \cdot \| : C[-1, 1] \rightarrow \mathbb{R} \) be defined by

\[
\|u\| = \sqrt{(u, u)}.
\]

Note that \( \{ \phi_1, \phi_2, \phi_3 \} \) is orthogonal with respect to the inner product \((\cdot, \cdot)\), which is defined on \([-1, 1]\).

(a) Construct the best approximation \( f_1 \) to \( f \) from \( \text{span}\{\phi_1\} \) with respect to the norm \( \| \cdot \| \).

(b) Construct the best approximation \( f_2 \) to \( f \) from \( \text{span}\{\phi_1, \phi_2\} \) with respect to the norm \( \| \cdot \| \).

(c) Construct the best approximation \( f_3 \) to \( f \) from \( \text{span}\{\phi_1, \phi_2, \phi_3\} \) with respect to \( \| \cdot \| \).

(d) Produce a plot that superimposes your best approximations from parts (a), (b), and (c) on top of a plot of \( f(x) \).

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**Solution.**

(a) [4 points] The best approximation to \( f(x) = e^x \) from \( \text{span}\{\phi_1\} \) with respect to the norm \( \| \cdot \| \) is

\[
f_1(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)} \phi_1(x).
\]

We compute

\[
(\phi_1, \phi_1) = \int_{-1}^{1} 1^2 \, dx = |x|_1^1 = 1 - (-1) = 2
\]

and

\[
(f, \phi_1) = \int_{-1}^{1} e^x \, dx = [e^x]_1^{-1} = e^1 - e^{-1} = e - \frac{1}{e}
\]

and hence

\[
f_1(x) = \frac{1}{2} \left( e - \frac{1}{e} \right).
\]

(b) [7 points] Since \( \phi_1 \) and \( \phi_2 \) are orthogonal with respect to the inner product \((\cdot, \cdot)\), i.e., \((\phi_1, \phi_2) = 0\), the best approximation to \( f(x) = e^x \) from \( \text{span}\{\phi_1, \phi_2\} \) with respect to the norm \( \| \cdot \| \) is

\[
f_2(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)} \phi_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)} \phi_2(x) = f_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)} \phi_2(x).
\]
Noting that
\[
(\phi_2, \phi_2) = \int_{-1}^{1} x^2 \, dx = \left[ \frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{3} - \left( -\frac{1}{3} \right) = \frac{2}{3}
\]
and
\[
(f, \phi_2) = \int_{-1}^{1} x e^x \, dx = [x e^x]_{-1}^{1} - \int_{-1}^{1} e^x \, dx = e^1 - (e^{-1}) - (f, \phi_1) = e + \frac{1}{e} - e + \frac{1}{e} = \frac{2}{e}
\]
we can compute that
\[
f_2(x) = f_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)} \phi_2(x) = \frac{1}{2} \left( e - \frac{1}{e} \right) + \frac{3}{e} x.
\]

(c) [7 points] Since,
\[
(\phi_1, \phi_2) = (\phi_1, \phi_3) = (\phi_2, \phi_3) = 0,
\]
the best approximation to \( f(x) = e^x \) from \( \text{span}\{\phi_1, \phi_2, \phi_3\} \) with respect to the norm \( \| \cdot \| \) is
\[
f_3(x) = \frac{(f, \phi_1)}{(\phi_1, \phi_1)} \phi_1(x) + \frac{(f, \phi_2)}{(\phi_2, \phi_2)} \phi_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)} \phi_3(x) = f_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)} \phi_3(x).
\]

Toward this end, compute
\[
(\phi_3, \phi_3) = \int_{-1}^{1} (3x^2 - 1)^2 \, dx
\]
\[
= \int_{-1}^{1} 9x^4 - 6x^2 + 1 \, dx
\]
\[
= \int_{-1}^{1} 9x^4 \, dx - 6(\phi_2, \phi_2) + (\phi_1, \phi_1)
\]
\[
= \left[ \frac{9x^5}{5} \right]_{-1}^{1} - \frac{6\cdot 2}{3} + 2
\]
\[
= \frac{9}{5} - \left( -\frac{9}{5} \right) - \frac{12}{3} + 2
\]
\[
= \frac{18}{5} - \frac{12}{3} + 2
\]
\[
= \frac{54}{15} - \frac{60}{15} + \frac{30}{15}
\]
\[
= \frac{24}{15}
\]
\[
= \frac{8}{5}
\]
and
\[
(f, \phi_3) = \int_{-1}^{1} (3x^2 - 1)e^x \, dx
\]
\[
= \int_{-1}^{1} 3x^2 e^x \, dx - (f, \phi_1)
\]
\[
= \left[3x^2 e^x\right]_{-1}^{1} - \int_{-1}^{1} 6xe^x \, dx - \left( e - \frac{1}{e} \right)
\]
\[
= 3e^1 - 3e^{-1} - 6(f, \phi_2) - \left( e - \frac{1}{e} \right)
\]
\[
= 2e - \frac{2}{e} - \frac{12}{e}
\]
\[
= 2e - \frac{14}{e}
\]

thus giving
\[
f_3(x) = f_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)} \phi_3(x) = \frac{1}{2} \left( e - \frac{1}{e} \right) + \frac{3}{e} x + \frac{5}{4} \left( e - \frac{7}{e} \right) (3x^2 - 1).
\]

(d) [7 points] The following plot compares the best approximations to \( f(x) \).

The code use to produce it is below.

```matlab
clear
clc
figure(1)
clf
x=linspace(-1,1,1000);
f=exp(x);
f1=(exp(1)-exp(-1))/2+x-x;
f2=f1+3*exp(-1)*x;
f3=f2+5*(exp(1)-7*exp(-1))*(3*x.^2-1)/4;
plot(x,f,'-k')
hold on
plot(x,f1,'-r')
plot(x,f2,'-b')
```

```latex
\[
(f, \phi_3) = \int_{-1}^{1} (3x^2 - 1)e^x \, dx
\]
\[
= \int_{-1}^{1} 3x^2 e^x \, dx - (f, \phi_1)
\]
\[
= \left[3x^2 e^x\right]_{-1}^{1} - \int_{-1}^{1} 6xe^x \, dx - \left( e - \frac{1}{e} \right)
\]
\[
= 3e^1 - 3e^{-1} - 6(f, \phi_2) - \left( e - \frac{1}{e} \right)
\]
\[
= 2e - \frac{2}{e} - \frac{12}{e}
\]
\[
= 2e - \frac{14}{e}
\]

thus giving
\[
f_3(x) = f_2(x) + \frac{(f, \phi_3)}{(\phi_3, \phi_3)} \phi_3(x) = \frac{1}{2} \left( e - \frac{1}{e} \right) + \frac{3}{e} x + \frac{5}{4} \left( e - \frac{7}{e} \right) (3x^2 - 1).
\]

(d) [7 points] The following plot compares the best approximations to \( f(x) \).

The code use to produce it is below.

```matlab
clear
clc
figure(1)
clf
x=linspace(-1,1,1000);
f=exp(x);
f1=(exp(1)-exp(-1))/2+x-x;
f2=f1+3*exp(-1)*x;
f3=f2+5*(exp(1)-7*exp(-1))*(3*x.^2-1)/4;
plot(x,f,'-k')
hold on
plot(x,f1,'-r')
plot(x,f2,'-b')
```
3. [20 points: 10 each]
All parts of this question should be done by hand.

(a) Let

\[
D = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad g = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.
\]

Use the spectral method to obtain the solution \( c \in \mathbb{R}^2 \) to

\[
Dc = g.
\]

(b) Let

\[
A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.
\]

Use the spectral method to obtain the solution \( x \in \mathbb{R}^3 \) to

\[
Ax = b.
\]

Solution.

(a) [14 points] Since,

\[
\lambda I - D = \begin{bmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{bmatrix}
\]

we have that

\[
\det (\lambda I - D) = (\lambda - 4)^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5)
\]

and so

\[
\det (\lambda I - D) = 0
\]

when \( \lambda = 3 \) or \( \lambda = 5 \). Hence, the eigenvalues of \( D \) are

\[
\lambda_1 = 3
\]

and

\[
\lambda_2 = 5.
\]

Moreover,

\[
(\lambda_1 I - D) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -f_1 - f_2 \\ -f_1 - f_2 \end{bmatrix}
\]

and so to make this vector zero we need to set \( f_2 = -f_1 \). Hence, any vector of the form

\[
\begin{bmatrix} f_1 \\ -f_1 \end{bmatrix}
\]

where \( f_1 \) is a nonzero constant is an eigenvector of \( D \) corresponding to the eigenvalue \( \lambda_1 \). Let us choose

\[
v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]
Furthermore, 
\[(\lambda_2 I - D) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_1 - d_2 \\ -d_1 + d_2 \end{bmatrix}\]
and so to make this vector zero we need to set \(d_2 = d_1\). Hence, any vector of the form \(\begin{bmatrix} d_1 \\ d_1 \end{bmatrix}\) where \(d_1\) is a nonzero constant is an eigenvector of \(D\) corresponding to the eigenvalue \(\lambda_2\). Let us choose \(v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\).

Since \(D = D^T\), \(Dv_1 = \lambda_1 v_1\), \(Dv_2 = \lambda_2 v_2\) and \(\lambda_1 \neq \lambda_2\), \(v_1 \cdot v_2 = 0\). Now,
\[g \cdot v_1 = 2 - 3 = -1,\]
\[v_1 \cdot v_1 = 1^2 + (-1)^2 = 1 + 1 = 2,\]
\[g \cdot v_2 = 2 + 3 = 5,\]
and
\[v_2 \cdot v_2 = 1^2 + 1^2 = 1 + 1 = 2.\]
The spectral method then yields that
\[c = \frac{1}{\lambda_1} \frac{g \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{1}{\lambda_2} \frac{g \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} + \frac{5}{6} \\ \frac{1}{6} + \frac{5}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.\]

(b) [14 points] For this matrix \(A\) we have
\[\lambda I - A = \begin{bmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix},\]
and hence the characteristic polynomial is
\[\det(\lambda I - A) = (\lambda - 3)(\lambda^2 - 1) = (\lambda - 3)(\lambda - 1)(\lambda + 1).\]
The eigenvalues of \(A\) are the roots of the characteristic polynomial, which we label \(\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 3.\)
To compute the eigenvectors associated with the eigenvalue \( \lambda_1 = -1 \), we seek \( u = (u_1, u_2, u_3)^T \) that makes the following vector zero:

\[
(\lambda_1 I - A)u = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -4u_1 \\ -u_2 + u_3 \\ u_2 - u_3 \end{bmatrix}.
\]

To make this vector zero we need to set \( u_1 = 0 \) and \( u_3 = u_2 \). Thus any vector of the form

\[
\begin{bmatrix} 0 \\ u_2 \\ u_2 \end{bmatrix}, \quad u_2 \neq 0
\]

is an eigenvector associated with the eigenvalue \( \lambda_1 = -1 \).

To compute the eigenvectors associated with the eigenvalue \( \lambda_2 = 1 \) we now seek \( u = (u_1, u_2, u_3)^T \) that makes the following vector zero:

\[
(\lambda_2 I - A)u = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2u_1 \\ u_2 + u_3 \\ u_3 + u_2 \end{bmatrix}.
\]

To make this vector zero we need to set \( u_1 = 0 \) and \( u_3 = -u_2 \). Thus any vector of the form

\[
\begin{bmatrix} 0 \\ u_2 \\ -u_2 \end{bmatrix}, \quad u_2 \neq 0
\]

is an eigenvector associated with the eigenvalue \( \lambda_2 = 1 \).

To compute the eigenvectors associated with the eigenvalue \( \lambda_3 = 3 \) we now seek \( u = (u_1, u_2, u_3)^T \) that makes the following vector zero:

\[
(\lambda_3 I - A)u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3u_2 + u_3 \\ u_2 + 3u_3 \end{bmatrix}.
\]

To make the second component zero we need \( u_2 = -u_3/3 \), while to make the third component zero we need \( u_3 = -u_2/3 \). The only way to accomplish both is to set \( u_2 = u_3 = 0 \). Thus any vector of the form

\[
\begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix}, \quad u_1 \neq 0
\]

is an eigenvector associated with the eigenvalue \( \lambda_3 = 3 \).

We choose the eigenvectors

\[
u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},
\]

\[
u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},
\]

and

\[
u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

We can compute that
\( \mathbf{u}_1^T \mathbf{u}_2 = \mathbf{u}_2^T \mathbf{u}_1 = 0 \cdot 0 + (1/\sqrt{2}) \cdot (1/\sqrt{2}) + (1/\sqrt{2}) \cdot (-1/\sqrt{2}) = 0, \)
\( \mathbf{u}_1^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_1 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (1/\sqrt{2}) \cdot 0 = 0, \)
and
\( \mathbf{u}_2^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_2 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (-1/\sqrt{2}) \cdot 0 = 0. \)

Now, for \( j = 1, 2, 3, \) \( \mathbf{A} \mathbf{u}_j = \lambda_j \mathbf{u}_j \) and \( \mathbf{u}_j^T \mathbf{u}_j = 1. \) Since \( \mathbf{A} = \mathbf{A}^T, \) the spectral method then yields that
\[
\mathbf{x} = \sum_{j=1}^{3} \frac{1}{\lambda_j} \mathbf{u}_j^T \mathbf{b} \mathbf{u}_j = \sum_{j=1}^{3} \frac{\mathbf{u}_j^T \mathbf{b}}{\lambda_j} \mathbf{u}_j.
\]

We can compute that
\[
\mathbf{u}_1^T \mathbf{b} = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (1/\sqrt{2}) \cdot 3 = \sqrt{2},
\]
\[
\mathbf{u}_2^T \mathbf{b} = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (-1/\sqrt{2}) \cdot 3 = -2\sqrt{2},
\]
and
\[
\mathbf{u}_3^T \mathbf{b} = 1 \cdot 2 + 0 \cdot (-1) + 0 \cdot 3 = 2,
\]
and hence
\[
\mathbf{x} = \frac{\sqrt{2}}{4} \left[ \begin{array}{c} 0 \\ \frac{\sqrt{2}}{\sqrt{2}} \\ \frac{\sqrt{2}}{1} \end{array} \right] + \frac{-2\sqrt{2}}{1} \left[ \begin{array}{c} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{-\sqrt{2}} \end{array} \right] + \frac{2}{3} \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] = \left[ \begin{array}{c} 2/3 \\ -3/3 \\ 1 \end{array} \right].
\]

We can multiply \( \mathbf{A} \mathbf{x} \) out to verify that the desired \( \mathbf{b} \) is obtained.

4. [20 points: 10 each]

You may find this problem challenging. Working with a group is suggested.

This problem deals with vectors and matrices defined in \( \mathbb{R}^n \) and therefore it is assumed that any reference to an inner product is the dot product and any reference to a norm is the euclidean norm (i.e. the inner product norm where the inner product is the dot product).

Consider \( n \) evenly spaced internal points \( x_k \) in the interval \( I = [0, 1] \). Label the internal points \( x_k \) with indices \( k = 1, 2, \ldots, n \) and let \( x_0 = 0 \) and \( x_{n+1} = 1. \) The internal points \( x_1, x_2, \ldots, x_n \) subdivide \([0, 1]\) into \( n + 1 \) intervals each of length \( h = 1/(n + 1) \). It therefore follows that for each \( k = 1, 2, \ldots, n \) the internal point \( x_k \) is the number \( x_k = kh. \)

For each \( j = 1, 2, \ldots, n \) define a vector \( \mathbf{s}^j \) whose \( j^{th} \) entry is the function \( f(x) = \sin(j \pi x) \) (\( f(x) \) is a sine wave with frequency \( j \)) sampled at the internal point \( x_k; \) that is \( \left[ \mathbf{s}^j \right]_k = \sin(j \pi x_k) = \sin(j \pi h k). \)

Now define the symmetric \( n \times n \) matrix \( \mathbf{L} \) whose entry in the \( i^{th} \) row and \( j^{th} \) column is determined by the formula
\[
\mathbf{L}_{ij} = \begin{cases} 
\frac{2}{h^2} & i = j \\
-\frac{1}{h^2} & |i - j| = 1 \\
0 & \text{otherwise}
\end{cases}
\]

(a) [10 points] Show that each of the vectors \( \mathbf{s}^j \) for \( j = 1, 2, \ldots, n \) is an eigenvector of \( \mathbf{L} \) and determine the corresponding eigenvalue \( \lambda_j. \) (Observe that, if this problem is done correctly, the eigenvalues \( \lambda_j \) for \( j = 1, 2, \ldots, n \) will satisfy \( \lambda_j \neq \lambda_i \) if \( i \neq j \).)

Hints

(1) You will need to use the sine summation trigonometric identity (2) You may find it easier to first look at rows 2, 3, \ldots, \( n - 1 \) of the matrix and then treat rows 1 and \( n \) separately. When looking at row 1 and row \( n \) keep in mind that \( \sin(0) = \sin(j \pi) = 0 \) for \( j = 1, 2, \ldots, n. \)
(b) [10 points] The vectors $s^j$ are orthogonal since they are eigenvectors of the symmetric matrix $L$ corresponding to different eigenvalues. Since there are $n$ of them the set $B = \{s^1, s^2, \ldots, s^n\}$ forms a basis of $\mathbb{R}^n$. Each of the vectors $s^j$ can be shown to have length

$$\|s^j\| = 1/\sqrt{2h}$$

Therefore the set $\tilde{B} = \{w_1, w_2, \ldots, w_n\} = \{\sqrt{2hs^1}, \sqrt{2hs^2}, \ldots, \sqrt{2hs^n}\}$ is an orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $L$ (Recall that dividing an eigenvector by its length does not change the associated eigenvalue). Let $b_{r,t}$, where $1 \leq r \leq t$ be the vector which is 1 in row $r$ and row $t$ but is zero in every other entry. Use the spectral method to find a general form for the solution of the problem

$$L\vec{v} = b_{r,t}$$

Solution.

(a) [10 points] To do this problem we will make use of the trigonometric identity of equation (1). Let $j = 1, 2, \ldots, n$ be fixed and consider the vector $s^j$. To say that $s^j$ is an eigenvector of $L$ with eigenvalue $\lambda_j$ means we need to show $Ls^j = \lambda_j s^j$

$$\sin(u) + \sin(v) = 2\sin \left( \frac{u + v}{2} \right) \cos \left( \frac{u - v}{2} \right)$$

(1)

We will show this entry-by-entry by looking at the $n$ equations that arise from the matrix-vector multiplication. As per the hint we consider the equations coming from the first row and last row of the matrix $L$ last. Consider $2 \leq k \leq n - 1$ the corresponding $k^j h$ row of the vector $Ls^j$ is given by the expression

$$\frac{1}{h^2} (2\sin(j\pi h k) - 2\sin(j\pi h (k + 1)) - 2\sin(j\pi h (k - 1)))$$

(2)

Using the trigonometric identity (1) on expression (2) gives that

$$\frac{1}{h^2} \left[ 2\sin(j\pi h k) - (\sin(j\pi h (k + 1)) + \sin(j\pi h (k - 1))) \right]$$

$$= \frac{1}{h^2} \left[ 2\sin(j\pi h k) - 2\sin(j\pi h k) \cos(j\pi h) \right]$$

$$= \frac{2(1 - \cos(j\pi h))}{h^2} \sin(j\pi h)$$

(3)

Since the $k^j h$ row of the vector $s^j$ is precisely $\sin(j\pi h k)$ equation (3) shows that if $s^j$ is to be an eigenvector the corresponding eigenvalue will have to be

$$\lambda_j = \frac{2(1 - \cos(j\pi h))}{h^2}$$

(4)

We aren’t quite done yet as we still need to make sure that the relationship (3) holds for the first and last rows of the vector $Ls^j$. The corresponding equations are

$$\frac{1}{h^2} (2\sin(j\pi h) - 2\sin(j\pi 2h)) \quad \leftrightarrow \quad \text{first row equation}$$

$$\frac{1}{h^2} (2\sin(j\pi h) - 2\sin(j\pi (n - 1)h)) \quad \leftrightarrow \quad \text{last row equation}$$

(5)

At first glance it appears that we will need to use more trigonometric identities to handle the two special cases set forth in (5). However, if we use the fact that $\sin(0) = \sin(j\pi (1 - 1)h) = 0$ in the equation for the first row and $\sin(j\pi (n + 1)h) = \sin(j\pi (n + 1)\frac{1}{n+1} h) = \sin(j\pi) = 0$ in the last row then the expressions (5) have the same general form as in equation (2) and hence the same relationship set forth in (3) holds for the first and last rows. Therefore the eigenvalue $\lambda_j$ corresponding to $s^j$ is given by equation (4).
(b) [10 points] The basic idea of the spectral method (for matrices) is to find an orthonormal basis of eigenvectors (which we know exists as a result of the spectral theorem), expand the unknown solution vector and the right hand side vector in terms of this basis and then carry out the matrix multiplication on the left and compare coefficients. To do this we first write

\[
b_{r,t} = \sum_{m=1}^{n} \beta_m w_m
\]

and we write

\[
\vec{x} = \sum_{m=1}^{n} \alpha_m w_m
\]

The goal of this game is, of course, to find the unknown coefficients \(\alpha_i\). We can compute the coefficients \(\beta_i\) of \(b_{r,t}\) because we know what the vector \(b_{r,t}\) is. Namely the orthonormality of the basis vectors \(w_i\), and the fact that our inner product of choice for this problem is the dot product, give that

\[
\beta_m = (b_{r,t}, w_m) = b_{r,t} \cdot w_m
\]

Since \(b_{r,t}\) is zero in every row except the \(r^{th}\) and \(t^{th}\) we can once again use the trigonometric identity (1) to get

\[
\beta_m = b_{r,t} \cdot w_m = \sqrt{2h} (\sin(m\pi rh) + \sin(m\pi th)) = 2\sqrt{2h} \left( \sin\left(\frac{m\pi(r+t)h}{2}\right) \cos\left(\frac{m\pi(r-t)h}{2}\right) \right)
\]

Using the expansion of \(\vec{x}\) in terms of the orthonormal basis of eigenvectors gives that

\[
L\vec{x} = L \left( \sum_{m=1}^{n} \alpha_m w_m \right) = \sum_{m=1}^{n} \lambda_m \alpha_m w_m
\]

Putting everything together we see that \(L\vec{x} = b_{r,t}\) if and only if

\[
\sum_{m=1}^{n} \lambda_m \alpha_m w_m = \sum_{m=1}^{n} 2\sqrt{2h} \left( \sin\left(\frac{m\pi(r+t)h}{2}\right) \cos\left(\frac{m\pi(r-t)h}{2}\right) \right) w_m \tag{6}
\]

Let \(j = 1, 2, \ldots, n\) be a fixed number and take the inner product of both sides of (6) with the vector \(w_j\) and use the fact that the basis vectors are orthonormal to derive that

\[
\alpha_j = \frac{2\sqrt{2h}}{\lambda_j} \left( \sin\left(\frac{m\pi(r+t)h}{2}\right) \cos\left(\frac{m\pi(r-t)h}{2}\right) \right)
\]

Simplifying this expression (which is optional) can be done using equation (4) to get

\[
\alpha_j = \frac{h^2 \sqrt{2h}}{1 - \cos(j\pi h)} \left( \sin\left(\frac{m\pi(r+t)h}{2}\right) \cos\left(\frac{m\pi(r-t)h}{2}\right) \right)
\]

Now that we have determined the unknown coefficients \(\alpha_j\) of the vector \(\vec{x}\) we are finished. We know exactly what the orthonormal basis vectors are and we could therefore write down the explicit formula for \(\vec{x}\) if we so desired.