1. [20 points: 5 points each]

Let the inner product \((\cdot, \cdot) : C[0, 1] \times C[0, 1] \to \mathbb{R}\) be defined by

\[
(v, w) = \int_0^1 v(x)w(x) \, dx.
\]

Consider the linear operator \(L : C^2_m[0, 1] \to C[0, 1]\) defined by

\[
Lu = -u''
\]

where

\[
C^2_m[0, 1] = \{u \in C^2[0, 1] : u'(0) = u(1) = 0\}.
\]

(a) Is \(L\) symmetric?

(b) What is the null space of \(L\)?

(c) Show that \((Lu, u) \geq 0\) for all \(u \in C^2_m[0, 1]\) and explain why this and the answer to part (b) mean that \(\lambda > 0\) for all eigenvalues \(\lambda\) of \(L\).

(d) Find the eigenvalues and eigenfunctions of \(L\).

---

**Solution.**

(a) [5 points] Yes, \(L\) is symmetric.

Let \(u, v \in C^2_m[0, 1]\). Integrating by parts twice, we have

\[
(Lu, v) = \int_0^1 -u''(x)v(x) \, dx
\]

\[
= -[u'(x)v(x)]_0^1 + \int_0^1 u'(x)v'(x) \, dx
\]

\[
= -[u'(x)v(x)]_0^1 + [u(x)v'(x)]_0^1 - \int_0^1 u(x)v''(x) \, dx.
\]

Since \(u, v \in C^2_m[0, 1]\) we have \(u'(0) = 0\) and \(v(1) = 0\), and hence the first term in square brackets must be zero. Again using the fact that \(u, v \in C^2_m[0, 1]\) we have \(v'(0) = 0\) and \(u(1) = 0\), and hence the second term in square brackets is also zero. It follows that

\[
(Lu, v) = \int_0^1 u(x)(-v''(x)) \, dx = (u, Lv)
\]

for all \(u, v \in C^2_m[0, 1]\).
(b) [5 points] The general solution to the differential equation
\[-u''(x) = 0\]
has the form
\[u(x) = A + Bx\]
for constants $A$ and $B$. In order for $u$ to be in $C^2_m[0, 1]$, we must have $u'(0) = 0$ and so since $u'(x) = B$, we must have $B = 0$. Now $u \in C^2_m[0, 1]$ also requires $u(1) = 0$, and since $u(1) = A$, we conclude that $A = 0$ too, meaning that $u(x) = A + Bx = 0$ for all $x \in [0, 1]$. Thus, the only element of the null space is the zero function, that is, $N(L) = \{0\}$.

(c) [7 points] Let $u \in C^2_m[0, 1]$. Using the first integration by parts from part (a), we have
\[
(Lu, u) = -[u'(x)u(x)]_0^1 + \int_0^1 u'(x)u'(x) \, dx
\]
\[= \int_0^1 (u'(x))^2 \, dx.
\]
Thus, $(Lu, u)$ is the integral of a nonnegative function, so it is nonnegative. Consequently, $(Lu, u) \geq 0$ for all $u \in C^2_m[0, 1]$.
This statement implies that all eigenvalues of $L$ are non-negative, since if $\lambda$ is an eigenfunction of $L$ then, since $L$ is a symmetric linear operator, $\lambda \in \mathbb{R}$ and there exist nonzero $u \in C^2_m[0, 1]$ which are such that $Lu = \lambda u$ and hence
\[
\lambda(u, u) = (\lambda u, u) = (Lu, u) \geq 0,
\]
and so, since we know that $(u, u) > 0$ for all nonzero $u \in C^2_m[0, 1]$ due to the positivity-definiteness of the inner product, we have that
\[
\lambda = \frac{(Lu, u)}{(u, u)} \geq 0.
\]
If zero was an eigenvalue of $L$, then there would exist nonzero $u \in C^2_m[0, 1]$ which were such that $Lu = 0$. However, we showed in part (b) that there were no nonzero $u \in C^2_m[0, 1]$ which satisfied this and so zero cannot be an eigenvalue of $L$ and hence we can say that $\lambda > 0$ for all eigenvalues $\lambda$ of $L$.

(d) [8 points] The eigenvalues of $L$ are the real numbers $\lambda > 0$ for which there exist nonzero $u \in C^2_m[0, 1]$ which are such that $Lu = \lambda u$. When $\lambda > 0$, the general solution to the equivalent differential equation
\[-u''(x) = \lambda u(x)\]
has the form
\[u(x) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x)\]
where $A$ and $B$ are constants. Since
\[u'(x) = A\sqrt{\lambda} \cos(\sqrt{\lambda}x) - B\sqrt{\lambda} \sin(\sqrt{\lambda}x)\]
and thus
\[u'(0) = A\sqrt{\lambda},\]
the boundary condition $u'(0) = 0$ implies that $A = 0$. On the other hand, the boundary condition $u(1) = 0$ implies that
\[u(1) = B \cos(\sqrt{\lambda}) = 0,\]
which can be achieved with nonzero $B$ provided that $\sqrt{\lambda} = (n - 1/2)\pi$ for positive integers $n$. We thus have that $L$ has eigenvalues

$$\lambda_n = (n - 1/2)^2\pi^2$$

with corresponding eigenfunctions

$$u_n(x) = B_n \cos(\sqrt{\lambda_n}x) = B_n \cos((n - 1/2)\pi x)$$

for nonzero constants $B_n$, for $n = 1, 2, 3, \ldots$. 
2. [25 points: 5 points each]
If $(\cdot, \cdot)$ is an inner product on the vector space $V$ then the inner product property

- $(v, v) \geq 0$, and
- $(v, v) = 0$ implies that $v = 0$

is called the positive definite property. If $L$ is a symmetric linear operator on the inner product space $(V, (\cdot, \cdot))$ then we say that $L$ is positive definite if it is true that

- $(Lv, v) \geq 0$
- $(Lv, v) = 0$ implies that $v = 0$

(a) Consider the boundary value problem

$$-\frac{\partial^2 u}{\partial x^2} = f(x)$$

$$u'(0) = 0$$

$$u'(1) = 0.$$ 

If we define $Lu = -\frac{\partial^2 u}{\partial x^2}$ and the space $C^2_N[0, 1]$

$$C^2_N[0, 1] = \{u \in C^2[0, 1], u'(0) = u'(1) = 0\},$$

this can be written as an operator equation

$$Lu = f, \quad L : C^2_N[0, 1] \rightarrow C[0, 1].$$

or more compactly as

$$L_N u = f$$

Where the operator $L_N$ is understood to be the $L$ operator considered with the Neumann boundary conditions. Explain why $L$ is not positive-definite.

(b) Derive (do not just show) that the eigenfunctions $\phi_j(x)$ and corresponding eigenvalues $\lambda_j$ of the above operator equation are

$$\phi_j(x) = \cos(j\pi x), \quad \lambda_j = (j\pi)^2.$$ 

Specify the values of $j$ for which these formulas hold. Describe what problems arise with the use of the spectral method for the above problem.

(c) Earlier in the semester, we discussed that a solution only exists to the above problem if $\int_0^1 f(x)dx = 0$. Let $u(x)$ be a spectral method solution to $L_N u = f$ for some source function $f(x)$. Use the fact that $\Phi_0 = 1$ is an eigenvector of $L_N$, with zero eigenvalue, to show that

$$\int_0^1 f(x)dx = 0$$

(d) The above problem is also non-unique: for any solution $u(x)$, $u(x) + C$ is also a solution for constant $C$. One way to make the solution unique is to add a condition where the average of $u(x)$ is zero:

$$\int_0^1 u(x)dx = 0.$$

To this end, we can redefine our operator equation

$$L_A u = -\frac{\partial^2 u}{\partial x^2}, \quad L_A u = f, \quad L_A : C^2_A[0, 1] \rightarrow C[0, 1].$$
where \( C^2_A[0,1] \) contains functions in \( C^2_N[0,1] \) with zero average

\[
C^2_A[0,1] = \left\{ u \in C^2[0,1], u'(0) = u'(1) = 0, \int_0^1 u(x)dx = 0 \right\}.
\]

Show that \( L_A \) is positive definite.

(e) Determine eigenfunctions and eigenvalues for the operator \( L_A \), and give an expression for the spectral method solution for the above operator equation. Use these to give the exact spectral method solution to the equation

\[
-\frac{\partial^2 u}{\partial x^2} = 1 + \cos(\pi x)
\]

\( u'(0) = u'(1) = 0 \).

Solution.

(a) Note that for any constant \( C \neq 0 \), \( \frac{\partial^2 C}{\partial x^2} = 0 \). Additionally, \( C \in C^2_N([0,1]) \). Thus, we can conclude that \( LC = 0 \). As a result,

\[
(LC, C) = (0, C) = 0.
\]

This shows \( L \) is not positive definite, because otherwise \( (Lu, u) = 0 \) would imply \( u = 0 \).

(b) If \( L\phi_j = \lambda \phi_j \), then \( -\phi''_j = \lambda \phi_j \), which implies that \( \phi_j(x) \) should have the form

\[
\phi_j(x) = A \sin(\sqrt{\lambda_j}x) + B \cos(\sqrt{\lambda_j}x).
\]

Then,

\[
\phi'_j(x) = A \sqrt{\lambda_j} \cos(\sqrt{\lambda_j}x) - B \sqrt{\lambda_j} \sin(\sqrt{\lambda_j}x).
\]

The boundary condition \( \phi'_j(0) = 0 \) then implies that \( A = 0 \). Likewise, the boundary condition \( \phi'_j(1) = 0 \) implies that

\[
\phi'_j(1) = B \sqrt{\lambda_j} \sin(\sqrt{\lambda_j}x) = 0
\]

so that \( \sqrt{\lambda_j} = j\pi \), and

\[
\phi_j(x) = \cos(j\pi x), \quad \lambda_j = j\pi^2.
\]

The above formula holds for \( j = 0, 1, 2, \ldots \), since for \( j = 0 \), \( \phi_j = 1 \), which is an eigenfunction of \( L \) with eigenvalue \( \lambda_0 = 0 \). This causes problems with the spectral method — the spectral method gives the solution

\[
u(x) = \sum_{j=0}^{\infty} \frac{(f, \phi_j)}{\lambda_j(\phi_j, \phi_j)} \phi_j(x).
\]

When \( j = 0 \), \( \lambda_j = 0 \), and we end up dividing by zero.

(c) Suppose \( u \) solves \( L_Nu = f \) and expand the functions \( u \), and \( f \) in terms of the (normalized) eigenfunctions \( \phi_j = \sqrt{2}\cos(j\pi x) \) to get

\[
\sum_{j=0}^{\infty} \alpha_j \phi_j(x) = \sum_{j=0}^{\infty} \beta_j \phi_j(x)
\]

Where \( \beta_k = (f, \phi_k) \). Using these expansions in the equation \( L_Nu = f \), and the fact that the \( \phi_j \) are eigenvalues of \( L_N \), gives the equality

\[
\lambda_k \alpha_k(\phi_k, \phi_k) = \beta_k.
\]

Picking the index \( k = 0 \) gives the eigenvector \( \phi_0(x) = 1 \) with coresponding eigenvalue \( \lambda_0 = (0 \star \pi)^2 = 0 \). Using this in the above expression gives

\[
0 = 0 \star \alpha_0 = \beta_0 = (f, \phi_0) = \sum_{j=0}^{\infty} \beta_j \phi_j(x)
\]

so that the result follows
(d) If we redefine $L_A : C_A^2[0, 1] \to C[0, 1]$, then we can show it is positive definite. First note that, by integration by parts, we get

$$(L_A u, u) = \int_0^1 L_A uu = \int_0^1 -u''(x)u(x) = [-u'(x)u(x)]_0^1 + \int_0^1 u'(x)^2 = \int_0^1 u'(x)^2 \geq 0.$$ 

Thus, we only have to show now that $(L_A u, u) = 0$ implies $u = 0$.

$$(L_A u, u) = 0 \Rightarrow \int_0^1 u'(x)^2 = 0$$

which implies $u'(x) = 0$, so that $u(x) = C$ for some constant $C$. However, since we require $u \in C_A^2[0, 1]$, $\int_0^1 u = \int_0^1 C = C = 0$, implying that for $(L_A u, u) = 0$, $u = 0$.

(e) The eigenfunctions and eigenvalues of $L_A$ are identical to those derived in part (b); the only difference is that $j = 1, 2, \ldots$ instead of $j = 0, 1, \ldots$

$$\phi_j(x) = \cos(j\pi x), \quad \lambda_j = (j\pi)^2, \quad j = 1, 2, \ldots.$$ 

Using the fact that

$$\int_0^1 (1 + \cos(\pi x)) \cos(j\pi x) = \int_0^1 \cos(j\pi x) + \int_0^1 \cos(\pi x) \cos(j\pi x) = \begin{cases} 1/2, & j = 1 \\ 0, & j \neq 1 \end{cases}$$

we have that representing $f(x)$ using eigenfunctions gives

$$f(x) = \sum_{j=1}^{\infty} \frac{(f, \phi_j)}{(\phi_j, \phi_j)} \phi_j(x) = \cos(\pi x).$$

Notice that using the eigenfunctions of $L_A$ does not exactly represent $f(x)$ — we remove 1 from $f(x)$, essentially removing the zero average part of $f(x)$. From part (c), this guarantees the existence of a solution.

The spectral method then gives

$$u(x) = \frac{\cos(\pi x)}{\pi^2}.$$
3. [20 points: 4 points (b), 7 points (a),(c),(d)]
Define the inner product \((u,v)\) to be
\[
(u,v) = \int_0^1 u(x)v(x) \, dx
\]
and let the norm \(\|v(x)\|\) be defined by
\[
\|v\| = \sqrt{(v,v)}.
\]
Let \(N\) be a positive integer and let \(\phi_1, \ldots, \phi_N \in C[0,1]\) be such that \(\{\phi_1, \ldots, \phi_N\}\) is orthonormal with respect to the inner product \((.,.)\). We wish to approximate a continuous function \(f(x)\) with \(f_N(x)\)
\[
f_N(x) = \sum_{n=1}^{N} \alpha_n \phi_n(x)
\]
where
\[
\phi_n(x) = \sqrt{2} \sin(n\pi x), \quad n = 1, 2, \ldots.
\]
and where \(\alpha_n = (f,\phi_n)\). (Note that \(f_N\) is the best approximation to \(g\) from span \(\{\phi_1, \ldots, \phi_N\}\) with respect to the norm \(\|\cdot\|\).)

(a) Assume that \(f_N \to f\) as \(N \to \infty\). Show that, since \(\phi_1, \ldots, \phi_N\) are orthonormal,
\[
\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2.
\]

(b) The best approximation to \(f(x) = x(1-x)\) has coefficients \(\alpha_n\) which satisfy
\[
\alpha_n = \frac{2\sqrt{2}}{n^3 \pi^3} (1 - (-1)^n).
\]
Plot the true function \(f(x)\) and compare it to \(f_N(x)\) for \(N = 5\). On a separate figure, plot the norm of the error \(\|f - f_N\|\) using the above formula for \(N = 1, 2, \ldots, 100\) on a log-log scale by using \texttt{loglog} in MATLAB.

(c) For \(f(x) = 1\) (which does not satisfy the same boundary conditions as \(\phi_n(x)!\)), we computed in class that
\[
c_n = \frac{2\sqrt{2}}{(n\pi)}
\]
for odd \(n\), and \(c_n = 0\) for even \(n\). Plot the true function \(f(x)\) and compare it to \(f_N(x)\) for \(N = 100\). On a separate figure, plot the norm of the error \(\|f - f_N\|\) using the above formula for \(N = 1, 2, \ldots, 100\) on a log-log scale by using \texttt{loglog} in MATLAB.

You may have noticed that the rate at which the coefficients \(\alpha_n \to 0\) determines how fast the error decreases — this is not coincidental!

(d) For \(f(x) = 1\), the equation \(Lu = f\) has the exact solution \(u(x) = x(1-x)/2\). Given the result of part (c), the same argument used in part (a) tells us that
\[
\|u - u_N\|^2 = \|u\|^2 - \sum_{n=1}^{N} \frac{c_n^2}{\lambda_n^2}.
\]
(You do not need to show this explicitly.) Use this formula to produce a \texttt{loglog} plot of the error \(\|u - u_N\|\) for \(N = 1, \ldots, 100\) on the same plot you made in part (b). (Be aware that the error may appear to flatline around \(10^{-8}\): this is a consequence of the computer’s floating point arithmetic, and is not a concern of ours here. To learn more about this phenomenon, take CAAM 453!)
(a) [10 points] We have that
\[ \|f - f_N\|^2 = (f - f_N, f - f_N) \]
\[ = \left( f - \sum_{n=1}^{N} \alpha_n \psi_n, f - \sum_{m=1}^{N} \alpha_m \psi_m \right) \]
\[ = \left( f - \sum_{n=1}^{N} \alpha_n \psi_n, f \right) - \sum_{m=1}^{N} \alpha_m \left( f - \sum_{n=1}^{N} \alpha_n \psi_n, \psi_m \right) \]
\[ = (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{m=1}^{N} \alpha_m \sum_{n=1}^{N} \alpha_n (\psi_n, \psi_m) \]
\[ = (f, f) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n^2 \]
\[ = (f, f) - \sum_{n=1}^{N} \alpha_n^2 \]
where at each equal sign we have used: (1) the definition of the norm \( \| \cdot \| \); (2) the definition of \( g_N \); (3) linearity of the inner product in the second argument; (4) linearity of the inner product in the first argument; (5) the fact that \( (\psi_n, \psi_m) = 0 \) if \( n \neq m \), for \( m, n = 1, 2, \ldots, N \), since \( \{\psi_1, \ldots, \psi_N\} \) is orthonormal with respect to the inner product \( \langle \cdot, \cdot \rangle \); (6) the fact that \( (\psi_n, \psi_n) = 1 \), for \( n = 1, 2, \ldots, N \), since \( \{\psi_1, \ldots, \psi_N\} \) is orthonormal with respect to the inner product \( \langle \cdot, \cdot \rangle \); (7) the fact that \( (f, \psi_n) = (\psi_n, f) = \alpha_n \); (8) algebra; (9) the definition of the norm \( \| \cdot \| \).

(b) [10 points] First calculate the norm of \( f \)
\[ \|f\|^2 = \int_0^1 (f(x))^2 \, dx = \int_0^1 x^2(1-x)^2 \, dx = \frac{1}{30} \]
Then
\[ \|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2 \]
where
\[ \alpha_n = \frac{2\sqrt{2}}{n^3 \pi^3} \left( 1 - (-1)^n \right) \]
and so
\[ \|f - f_N\|^2 = \frac{1}{30} - \sum_{n=1}^{N} \alpha_n^2 \]
The requested plots are shown below.
Figure 1: Comparison of the true function $f(x)$ and $f_N(x)$ for $N = 5$

Figure 2: Norm of the error for $N = 1, 2 \cdots, 100$ on a log-log scale
Figure 3: Plot for (c).

Figure 4: Plot for (d).
(c) The requested plot for (c) and (d) is shown below.
The code that produced the plot above for (c) and (d) is shown below.

\begin{verbatim}
n = [1:100]';
cn = (sqrt(2)/pi)*(1+(-1).^(n+1))./(n);
lamn = pi^2*n.^2;
normf2 = 1;
normu2 = 1/120;

figure(1), clf
loglog([1:length(cn)], sqrt(normf2-cumsum(cn.^2)),'r.-')
set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|f-f_N\|$','$\|u-u_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr1

figure(2)
loglog(n, sqrt(normu2-cumsum((cn./lamn).^2)),'b.-')
set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|u-u_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr2
\end{verbatim}
Let the symmetric linear operator \( L : C^2_M[0, 1] \to C[0, 1] \) be defined by
\[
Lv = -v''
\]
where \( C^2_M[0, 1] = \{ w \in C^2[0, 1] : w'(0) = w(1) = 0 \} \).

Let \( N \) be a positive integer and let \( f \in C[0, 1] \) be defined by
\[
f(x) = \begin{cases} 
1 - 2x & \text{if } x \in [0, \frac{1}{2}] ; \\
0 & \text{otherwise}.
\end{cases}
\]

(a) The operator \( L \) has eigenvalues \( \lambda_n \) with corresponding eigenfunctions
\[
\phi_n(x) = \sqrt{2} \cos \left( \frac{2n - 1}{2} \pi x \right)
\]
for \( n = 1, 2, \ldots \). We have that, for \( m, n = 1, 2, \ldots \),
\[
(\phi_m, \phi_n) = \begin{cases} 
1 & \text{if } m = n; \\
0 & \text{if } m \neq n.
\end{cases}
\]
Obtain a formula for the eigenvalues \( \lambda_n \) for \( n = 1, 2, \ldots \).

(b) Compute \( f_N \), the best approximation to \( f \) from span \{\phi_1, \ldots, \phi_N\} with respect to the norm \( \| \cdot \| \).
Plot \( f_N \) for \( N = 1, 2, 3, 4, 5, 6 \).

(c) Use the spectral method to obtain a series solution to the problem of finding \( \tilde{u} \in C^2[0, 1] \) such that
\[
-\tilde{u}''(x) = f(x), \quad 0 < x < 1
\]
and
\[
\tilde{u}'(0) = \tilde{u}(1) = 0.
\]

**Solution.**

(a) [5 points] We can compute that, for \( n = 1, 2, \ldots \),
\[
\phi_n'(x) = -\sqrt{2} \left( \frac{2n - 1}{2} \right) \pi \sin \left( \frac{2n - 1}{2} \pi x \right).
\]
and
\[
\phi_n''(x) = -\sqrt{2} \left( \frac{2n - 1}{2} \right)^2 \pi^2 \cos \left( \frac{2n - 1}{2} \pi x \right).
\]
and so
\[
L\phi_n = -\phi_n'' = \left( \frac{2n - 1}{2} \right)^2 \pi^2 \phi_n.
\]
Hence,
\[
\lambda_n = \left( \frac{2n - 1}{2} \right)^2 \pi^2 = (2n - 1) \frac{\pi^2}{4} \quad \text{for } n = 1, 2, \ldots.
\]
(b) [5 points] Since $\{\phi_1, \ldots, \phi_N\}$ is orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$, the best approximation to $f$ from $\text{span} \{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\| \cdot \|$ is

$$f_N = \sum_{n=1}^{N} (f, \phi_n) \phi_n.$$ 

Now, for $n = 1, 2, \ldots$,

$$\begin{align*}
(f, \phi_n) &= \int_{0}^{1} f(x) \phi_n(x) \, dx \\
&= \int_{0}^{1/2} f(x) \phi_n(x) \, dx + \int_{1/2}^{1} f(x) \phi_n(x) \, dx \\
&= \int_{0}^{1/2} (1 - 2x) \sqrt{2} \cos \left( \frac{2n - 1}{2} \pi x \right) \, dx + \int_{1/2}^{1} 0 \, dx \\
&= \sqrt{2} \int_{0}^{1/2} (1 - 2x) \cos \left( \frac{2n - 1}{2} \pi x \right) \, dx + 0 \\
&= \sqrt{2} \left[ (1 - 2x) \frac{2}{(2n - 1) \pi} \sin \left( \frac{2n - 1}{2} \pi x \right) \right]_{0}^{1/2} - \int_{0}^{1/2} (-2) \frac{2}{(2n - 1) \pi} \sin \left( \frac{2n - 1}{2} \pi x \right) \, dx \\
&= \sqrt{2} \left( 0 - 0 + \frac{4}{(2n - 1) \pi} \int_{0}^{1/2} \sin \left( \frac{2n - 1}{2} \pi x \right) \, dx \right) \\
&= \sqrt{2} \frac{4}{(2n - 1) \pi} \left[ - \frac{2}{(2n - 1) \pi} \cos \left( \frac{2n - 1}{2} \pi x \right) \right]_{0}^{1/2} \\
&= \frac{4\sqrt{2}}{(2n - 1) \pi} \left( - \frac{2}{(2n - 1) \pi} \cos \left( \frac{2n - 1}{4} \pi \right) - \left( - \frac{2}{(2n - 1) \pi} \right) \right) \\
&= \frac{8\sqrt{2}}{(2n - 1)^2 \pi^2} \left( 1 - \cos \left( \frac{2n - 1}{4} \pi \right) \right). 
\end{align*}$$

Hence,

$$f_N(x) = \sum_{n=1}^{N} (f, \phi_n) \phi_n(x)$$

$$= \sum_{n=1}^{N} (f, \phi_n) \sqrt{2} \cos \left( \frac{2n - 1}{2} \pi x \right)$$

$$= \sum_{n=1}^{N} \frac{16}{(2n - 1)^2 \pi^2} \left( 1 - \cos \left( \frac{2n - 1}{4} \pi \right) \right) \cos \left( \frac{2n - 1}{2} \pi x \right).$$

The requested plot is below.
The above plot was produced using the following MATLAB code.

```
clear
clc
colors='rgbcmy';
x = linspace(0,1,1000);
figure(1)
clf
legendStr{1}=['$f(x)$'];
plot(x,-(x-1/2)+(x-1/2).*sign(x-1/2),'k-')
hold on
fk = zeros(size(x));
for k=1:6
    fk = fk + 16*(1-cos(((2*k-1)/4)*pi))./((2*k-1).^2*pi^2)*cos(((2*k-1)/2)*pi*x);
    plot(x,fk,colors(k))
    legendStr{k+1}=['$f_{' num2str(k) '} (x)$'];
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(1),'hw72c','epsc')
```

(c) [5 points] Now, \( \tilde{u} \) is the solution to \( L\tilde{u} = f \) and so the spectral method yields the series solution

\[
\tilde{u}(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{\lambda_n} \phi_n(x) = \sum_{n=1}^{\infty} \frac{64}{(2n-1)^4 \pi^4} \left( 1 - \cos \left( \frac{2n-1}{4} \pi \right) \right) \cos \left( \frac{2n-1}{2} \pi x \right).
\]