CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 5

Posted Friday 24, February, 2017. Due 5pm Friday 3, March 2017.

• A note on proofs (unless otherwise stated by the problem): When asked to prove a statement this means you are to show that the requisite properties (discussed in class) for the item referenced by the statement hold true. When asked to disprove something this means you are to come up with an example that shows the proposed premise is false.

• Unless explicitly stated otherwise in the problem you are free to use MATLAB as you see fit; including for those problems that do not explicitly require it. Please submit any code that you utilize as a printout or, if it is short enough, reference your steps in your writeup directly. (example: I used matlab to compute the inverse to the matrix B and got ... etc)

Please write your name and instructor on your homework.

There is a total of 80 points distributed among the problems below

1. [20 points: 10 each]
   All parts of this question should be done by hand.
   (a) Let
   \[ D = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad g = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]
   Use the spectral method to obtain the solution \( c \in \mathbb{R}^2 \) to
   \[ Dc = g. \]

   (b) Let
   \[ A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}. \]
   Use the spectral method to obtain the solution \( x \in \mathbb{R}^3 \) to
   \[ Ax = b. \]

2. [20 points: 5 points each]
   Let the inner product \( (\cdot, \cdot) : C[0,1] \times C[0,1] \to \mathbb{R} \) be defined by
   \[ (v, w) = \int_0^1 v(x)w(x) \, dx. \]
   Consider the linear operator \( L : C^2_m[0,1] \to C[0,1] \) defined by
   \[ Lu = -u'' \]
   where
   \[ C^2_m[0,1] = \{ u \in C^2[0,1] : u'(0) = u(1) = 0 \}. \]
(a) Is $L$ symmetric?

(b) What is the null space of $L$?

(c) Show that $(Lu, u) \geq 0$ for all $u \in C^2_{m_0}[0, 1]$ and explain why this and the answer to part (b) mean that $\lambda > 0$ for all eigenvalues $\lambda$ of $L$.

(d) Find the eigenvalues and eigenfunctions of $L$.

3. [20 points: 4 points (b), 7 points (a),(c),(d)]

Define the inner product $(u, v)$ to be

$$(u, v) = \int_0^1 u(x)v(x) \, dx$$

and let the norm $\|v(x)\|$ be defined by

$$\|v\| = \sqrt{(v, v)}.$$

Let $N$ be a positive integer and let $\phi_1, \ldots, \phi_N \in C[0, 1]$ be such that $\{\phi_1, \ldots, \phi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$. We wish to approximate a continuous function $f(x)$ with $f_N(x)$

$$f_N(x) = \sum_{n=1}^N \alpha_n \phi_n(x)$$

where

$$\phi_n(x) = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, \ldots$$

and where $\alpha_n = (f, \phi_n)$. (Note that $f_N$ is the best approximation to $g$ from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\|\cdot\|$.)

(a) Assume that $f_N \to f$ as $N \to \infty$. Show that, since $\phi_1, \ldots, \phi_N$ are orthonormal,

$$\|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^N \alpha_n^2.$$

(b) The best approximation to $f(x) = x(1-x)$ has coefficients $\alpha_n$ which satisfy

$$\alpha_n = \frac{2\sqrt{2}}{n^3\pi^3}(1 - (-1)^n).$$

Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 5$. On a separate figure, plot the norm of the error $\|f - f_N\|$ using the above formula for $N = 1, 2, \ldots, 100$ on a log-log scale by using loglog in MATLAB.

(c) For $f(x) = 1$ (which does not satisfy the same boundary conditions as $\phi_n(x)$!), we computed in class that

$$c_n = 2\sqrt{2}/(n\pi)$$

for odd $n$, and $c_n = 0$ for even $n$. Plot the true function $f(x)$ and compare it to $f_N(x)$ for $N = 100$. On a separate figure, plot the norm of the error $\|f - f_N\|$ using the above formula for $N = 1, 2, \ldots, 100$ on a log-log scale by using loglog in MATLAB.

You may have noticed that the rate at which the coefficients $\alpha_n \to 0$ determines how fast the error decreases --- this is not coincidental!
(d) For \( f(x) = 1 \), the equation \( Lu = f \) has the exact solution \( u(x) = x(1 - x)/2 \). Given the result of part (c), the same argument used in part (a) tells us that

\[
\|u - u_N\|^2 = \|u\|^2 - \sum_{n=1}^{N} \frac{c_n^2}{\lambda_n^2}.
\]

(You do not need to show this explicitly.) Use this formula to produce a \( \log \log \) plot of the error \( \|u - u_N\| \) for \( N = 1, \ldots, 100 \) on the same plot you made in part (b). (Be aware that the error may appear to flatline around \( 10^{-8} \); this is a consequence of the computer’s floating point arithmetic, and is not a concern of ours here. To learn more about this phenomenon, take CAAM 453!)

4. [15 points: 5 points each]

Let the symmetric linear operator \( L : C^2_M[0, 1] \to C[0, 1] \) be defined by

\[
Lv = -v''
\]

where

\[
C^2_M[0, 1] = \{ w \in C^2[0, 1] : w'(0) = w(1) = 0 \}.
\]

Let \( N \) be a positive integer and let \( f \in C[0, 1] \) be defined by

\[
f(x) = \begin{cases} 
1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right]; \\
0 & \text{otherwise}.
\end{cases}
\]

(a) The operator \( L \) has eigenvalues \( \lambda_n \) with corresponding eigenfunctions

\[
\phi_n(x) = \sqrt{2} \cos \left( \frac{2n - 1}{2} \pi x \right)
\]

for \( n = 1, 2, \ldots \). We have that, for \( m, n = 1, 2, \ldots \),

\[
(\phi_m, \phi_n) = \begin{cases} 
1 & \text{if } m = n; \\
0 & \text{if } m \neq n.
\end{cases}
\]

Obtain a formula for the eigenvalues \( \lambda_n \) for \( n = 1, 2, \ldots \).

(b) Compute \( f_N \), the best approximation to \( f \) from \( \text{span} \{ \phi_1, \ldots, \phi_N \} \) with respect to the norm \( \| \cdot \| \). Plot \( f_N \) for \( N = 1, 2, 3, 4, 5, 6 \).

(c) Use the spectral method to obtain a series solution to the problem of finding \( \tilde{u} \in C^2[0, 1] \) such that

\[
-\tilde{u}''(x) = f(x), \quad 0 < x < 1
\]

and

\[
\tilde{u}'(0) = \tilde{u}(1) = 0.
\]