CAAM 336 · DIFFERENTIAL EQUATIONS
Homework 5 · Solutions

Posted Friday 24, February, 2017. Due 5pm Friday 3, March 2017.

- A note on proofs (unless otherwise stated by the problem): When asked to prove a statement this means you are to show that the requisite properties (discussed in class) for the item referenced by the statement hold true. When asked to disprove something this means you are to come up with an example that shows the proposed premise is false.

- Unless explicitly stated otherwise in the problem you are free to use MATLAB as you see fit; including for those problems that do not explicitly require it. Please submit any code that you utilize as a printout or, if it is short enough, reference your steps in your writeup directly. (example: I used matlab to compute the inverse to the matrix B and got ...

Please write your name and instructor on your homework.
There is a total of 80 points distributed among the problems below.

1. [20 points: 10 each]
All parts of this question should be done by hand.

(a) Let
\[ D = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad g = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]
Use the spectral method to obtain the solution \( c \in \mathbb{R}^2 \) to
\[ Dc = g. \]

(b) Let
\[ A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}. \]
Use the spectral method to obtain the solution \( x \in \mathbb{R}^3 \) to
\[ Ax = b. \]

Solution.

(a) [14 points] Since,
\[ \lambda I - D = \begin{bmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{bmatrix} \]
we have that
\[ \det (\lambda I - D) = (\lambda - 4)^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5) \]
and so
\[ \det (\lambda I - D) = 0 \]
when \( \lambda = 3 \) or \( \lambda = 5 \). Hence, the eigenvalues of \( D \) are
\[ \lambda_1 = 3 \]
and
\[ \lambda_2 = 5. \]
Moreover,
\[
(\lambda_1 \mathbf{I} - \mathbf{D}) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -f_1 - f_2 \\ -f_1 - f_2 \end{bmatrix}
\]
and so to make this vector zero we need to set \( f_2 = -f_1 \). Hence, any vector of the form
\[
\begin{bmatrix} f_1 \\ -f_1 \end{bmatrix}
\]
where \( f_1 \) is a nonzero constant is an eigenvector of \( \mathbf{D} \) corresponding to the eigenvalue \( \lambda_1 \). Let us choose
\[
v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]
Furthermore,
\[
(\lambda_2 \mathbf{I} - \mathbf{D}) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_1 - d_2 \\ -d_1 + d_2 \end{bmatrix}
\]
and so to make this vector zero we need to set \( d_2 = d_1 \). Hence, any vector of the form
\[
\begin{bmatrix} d_1 \\ d_1 \end{bmatrix}
\]
where \( d_1 \) is a nonzero constant is an eigenvector of \( \mathbf{D} \) corresponding to the eigenvalue \( \lambda_2 \). Let us choose
\[
v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]
Since \( \mathbf{D} = \mathbf{D}^T \), \( \mathbf{D} v_1 = \lambda_1 v_1 \), \( \mathbf{D} v_2 = \lambda_2 v_2 \) and \( \lambda_1 \neq \lambda_2 \), \( v_1 \cdot v_2 = 0 \). Now,
\[
\mathbf{g} \cdot v_1 = 2 - 3 = -1,
\]
\[
v_1 \cdot v_1 = 1^2 + (-1)^2 = 1 + 1 = 2,
\]
\[
\mathbf{g} \cdot v_2 = 2 + 3 = 5,
\]
and
\[
v_2 \cdot v_2 = 1^2 + 1^2 = 1 + 1 = 2.
\]
The spectral method then yields that
\[
c = \frac{1}{\lambda_1} \frac{\mathbf{g} \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{1}{\lambda_2} \frac{\mathbf{g} \cdot v_2}{v_2 \cdot v_2} v_2
\]
\[
\begin{aligned}
&= \frac{1}{3} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-1}{3} + \frac{1}{5} \\ \frac{2}{3} + \frac{2}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{15} \\ \frac{16}{15} \end{bmatrix}.
\end{aligned}
\]
(b) [14 points] For this matrix $A$ we have

$$
\lambda I - A = \begin{bmatrix}
\lambda - 3 & 0 & 0 \\
0 & \lambda & 1 \\
0 & 1 & \lambda
\end{bmatrix},
$$

and hence the characteristic polynomial is

$$
det(\lambda I - A) = (\lambda - 3)(\lambda^2 - 1) = (\lambda - 3)(\lambda - 1)(\lambda + 1).
$$

The eigenvalues of $A$ are the roots of the characteristic polynomial, which we label

$$
\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 3.
$$

To compute the eigenvectors associated with the eigenvalue $\lambda_1 = -1$, we seek $u = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$
(\lambda_1 I - A)u = \begin{bmatrix}
-4 & 0 & 0 \\
0 & -1 & 1 \\
0 & 1 & -1
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix} = \begin{bmatrix}
-4u_1 \\
-u_2 + u_3 \\
u_2 - u_3
\end{bmatrix}.
$$

To make this vector zero we need to set $u_1 = 0$ and $u_3 = u_2$. Thus any vector of the form

$$
\begin{bmatrix}
0 \\
u_2 \\
u_2
\end{bmatrix}, \quad u_2 \neq 0
$$

is an eigenvector associated with the eigenvalue $\lambda_1 = -1$.

To compute the eigenvectors associated with the eigenvalue $\lambda_2 = 1$ we now seek $u = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$
(\lambda_2 I - A)u = \begin{bmatrix}
-2 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix} = \begin{bmatrix}
-2u_1 \\
u_2 + u_3 \\
u_3 + u_2
\end{bmatrix}.
$$

To make this vector zero we need to set $u_1 = 0$ and $u_3 = -u_2$. Thus any vector of the form

$$
\begin{bmatrix}
0 \\
u_2 \\
-u_2
\end{bmatrix}, \quad u_2 \neq 0
$$

is an eigenvector associated with the eigenvalue $\lambda_2 = 1$.

To compute the eigenvectors associated with the eigenvalue $\lambda_3 = 3$ we now seek $u = (u_1, u_2, u_3)^T$ that makes the following vector zero:

$$
(\lambda_3 I - A)u = \begin{bmatrix}
0 & 0 & 0 \\
0 & 3 & 1 \\
0 & 1 & 3
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix} = \begin{bmatrix}
0 \\
3u_2 + u_3 \\
u_2 + 3u_3
\end{bmatrix}.
$$

To make the second component zero we need $u_2 = -u_3/3$, while to make the third component zero we need $u_3 = -u_2/3$. The only way to accomplish both is to set $u_2 = u_3 = 0$. Thus any vector of the form

$$
\begin{bmatrix}
u_1 \\
0 \\
0
\end{bmatrix}, \quad u_1 \neq 0
$$

is an eigenvector associated with the eigenvalue $\lambda_3 = 3$. 
We choose the eigenvectors

\[ u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \]

\[ u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \]

and

\[ u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \]

We can compute that

\[ u_1^T u_2 = u_2^T u_1 = 0 \cdot 0 + (1/\sqrt{2}) \cdot (1/\sqrt{2}) + (1/\sqrt{2}) \cdot (-1/\sqrt{2}) = 0, \]

\[ u_1^T u_3 = u_3^T u_1 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (1/\sqrt{2}) \cdot 0 = 0, \]

and

\[ u_2^T u_3 = u_3^T u_2 = 0 \cdot 1 + (1/\sqrt{2}) \cdot 0 + (-1/\sqrt{2}) \cdot 0 = 0. \]

Now, for \( j = 1, 2, 3 \), \( A u_j = \lambda_j u_j \) and \( u_j^T u_j = 1 \). Since \( A = A^T \), the spectral method then yields that

\[ x = \sum_{j=1}^{3} \frac{1}{\lambda_j} u_j^T b u_j = \sum_{j=1}^{3} \frac{u_j^T b}{\lambda_j} u_j. \]

We can compute that

\[ u_1^T b = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (1/\sqrt{2}) \cdot 3 = \sqrt{2}, \]

\[ u_2^T b = 0 \cdot 2 + (1/\sqrt{2}) \cdot (-1) + (-1/\sqrt{2}) \cdot 3 = -2\sqrt{2}, \]

and

\[ u_3^T b = 1 \cdot 2 + 0 \cdot (-1) + 0 \cdot 3 = 2, \]

and hence

\[ x = \sqrt{2} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{-2\sqrt{2}}{3} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -3/1 \end{bmatrix}. \]

We can multiply \( Ax \) out to verify that the desired \( b \) is obtained.

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2. [20 points: 5 points each]

Let the inner product \((\cdot, \cdot) : C[0, 1] \times C[0, 1] \rightarrow \mathbb{R}\) be defined by

\[ (v, w) = \int_0^1 v(x)w(x) \, dx. \]

Consider the linear operator \( L : C_2^2[0, 1] \rightarrow C[0, 1] \) defined by

\[ Lu = -u'' \]

where

\[ C_2^2[0, 1] = \{ u \in C^2[0, 1] : u'(0) = u(1) = 0 \}. \]
(a) Is $L$ symmetric?

(b) What is the null space of $L$?

(c) Show that $(Lu, u) \geq 0$ for all $u \in C^2_m[0,1]$ and explain why this and the answer to part (b) mean that $\lambda > 0$ for all eigenvalues $\lambda$ of $L$.

(d) Find the eigenvalues and eigenfunctions of $L$.

Solution.

(a) [5 points] Yes, $L$ is symmetric.
Let $u, v \in C^2_m[0,1]$. Integrating by parts twice, we have

$$(Lu, v) = \int_0^1 -u''(x)v(x) \, dx$$

$$= -[u'(x)v(x)]_0^1 + \int_0^1 u'(x)v'(x) \, dx$$

$$= -[u'(x)v(x)]_0^1 + [u(x)v'(x)]_0^1 - \int_0^1 u(x)v''(x) \, dx.$$ 

Since $u, v \in C^2_m[0,1]$ we have $u'(0) = 0$ and $v(1) = 0$, and hence the first term in square brackets must be zero. Again using the fact that $u, v \in C^2_m[0,1]$ we have $v'(0) = 0$ and $u(1) = 0$, and hence the second term in square brackets is also zero. It follows that

$$(Lu, v) = \int_0^1 u(x)(-v''(x)) \, dx = (u, Lv)$$

for all $u, v \in C^2_m[0,1]$.

(b) [5 points] The general solution to the differential equation

$$-u''(x) = 0$$

has the form

$$u(x) = A + Bx$$

for constants $A$ and $B$. In order for $u$ to be in $C^2_m[0,1]$, we must have $u'(0) = 0$ and so since $u'(x) = B$, we must have $B = 0$. Now $u \in C^2_m[0,1]$ also requires $u(1) = 0$, and since $u(1) = A$, we conclude that $A = 0$ too, meaning that $u(x) = A + Bx = 0$ for all $x \in [0,1]$. Thus, the only element of the null space is the zero function, that is, $N(L) = \{0\}$.

(c) [7 points] Let $u \in C^2_m[0,1]$. Using the first integration by parts from part (a), we have

$$(Lu, u) = -[u'(x)u(x)]_0^1 + \int_0^1 u'(x)u'(x) \, dx$$

$$= \int_0^1 (u'(x))^2 \, dx.$$ 

Thus, $(Lu, u)$ is the integral of a nonnegative function, so it is nonnegative. Consequently, $(Lu, u) \geq 0$ for all $u \in C^2_m[0,1]$. 
This statement implies that all eigenvalues of $L$ are non-negative, since if $\lambda$ is an eigenfunction of $L$ then, since $L$ is a symmetric linear operator, $\lambda \in \mathbb{R}$ and there exist nonzero $u \in C^2_m[0,1]$ which are such that $Lu = \lambda u$ and hence

$$\lambda(u,u) = \langle \lambda u, u \rangle = \langle Lu, u \rangle \geq 0,$$

and so, since we know that $(u,u) > 0$ for all nonzero $u \in C^2_m[0,1]$ due to the positivity-definiteness of the inner product, we have that

$$\lambda = \frac{\langle Lu, u \rangle}{(u,u)} \geq 0.$$

If zero was an eigenvalue of $L$, then there would exist nonzero $u \in C^2_m[0,1]$ which were such that $Lu = 0$. However, we showed in part (b) that there were no nonzero $u \in C^2_m[0,1]$ which satisfied this and so zero cannot be an eigenvalue of $L$ and hence we can say that $\lambda > 0$ for all eigenvalues $\lambda$ of $L$.

(d) [8 points] The eigenvalues of $L$ are the real numbers $\lambda > 0$ for which there exist nonzero $u \in C^2_m[0,1]$ which are such that $Lu = \lambda u$. When $\lambda > 0$, the general solution to the equivalent differential equation

$$-u''(x) = \lambda u(x)$$

has the form

$$u(x) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x)$$

where $A$ and $B$ are constants. Since

$$u'(x) = A\sqrt{\lambda} \cos(\sqrt{\lambda}x) - B\sqrt{\lambda} \sin(\sqrt{\lambda}x)$$

and thus

$$u'(0) = A\sqrt{\lambda},$$

the boundary condition $u'(0) = 0$ implies that $A = 0$. On the other hand, the boundary condition $u(1) = 0$ implies that

$$u(1) = B \cos(\sqrt{\lambda}) = 0,$$

which can be achieved with nonzero $B$ provided that $\sqrt{\lambda} = (n-1/2)\pi$ for positive integers $n$. We thus have that $L$ has eigenvalues

$$\lambda_n = (n-1/2)^2 \pi^2$$

with corresponding eigenfunctions

$$u_n(x) = B_n \cos(\sqrt{\lambda_n}x) = B_n \cos((n-1/2)\pi x)$$

for nonzero constants $B_n$, for $n = 1, 2, 3, \ldots$.

3. [20 points: 4 points (b), 7 points (a),(c),(d)]

Define the inner product $(u,v)$ to be

$$(u,v) = \int_0^1 u(x)v(x) \, dx$$

and let the norm $\|v\|$ be defined by

$$\|v\| = \sqrt{(v,v)}.$$

Let $N$ be a positive integer and let $\phi_1, \ldots, \phi_N \in C[0,1]$ be such that $\{\phi_1, \ldots, \phi_N\}$ is orthonormal with respect to the inner product $(\cdot, \cdot)$. We wish to approximate a continuous function $f(x)$ with $f_N(x)$

$$f_N(x) = \sum_{n=1}^N \alpha_n \phi_n(x)$$
where
\[ \phi_n(x) = \sqrt{2} \sin(n\pi x), \quad n = 1, 2, \ldots \]
and where \( \alpha_n = (f, \phi_n) \). (Note that \( f_N \) is the best approximation to \( g \) from \( \text{span}\{\phi_1, \ldots, \phi_N\} \) with respect to the norm \( \| \cdot \| \).)

(a) Assume that \( f_N \to f \) as \( N \to \infty \). Show that, since \( \phi_1, \ldots, \phi_N \) are orthonormal,
\[ \|f - f_N\|^2 = \|f\|^2 - \sum_{n=1}^{N} \alpha_n^2. \]

(b) The best approximation to \( f(x) = x(1-x) \) has coefficients \( \alpha_n \) which satisfy
\[ \alpha_n = \frac{2\sqrt{2}}{n^3\pi^3} (1 - (-1)^n). \]
Plot the true function \( f(x) \) and compare it to \( f_N(x) \) for \( N = 5 \). On a separate figure, plot the norm of the error \( \|f - f_N\| \) using the above formula for \( N = 1, 2, \ldots, 100 \) on a log-log scale by using \text{loglog} in \text{MATLAB}.

(c) For \( f(x) = 1 \) (which does not satisfy the same boundary conditions as \( \phi_n(x) \)!), we computed in class that
\[ c_n = \frac{2\sqrt{2}}{(n\pi)^3} \]
for odd \( n \), and \( c_n = 0 \) for even \( n \). Plot the true function \( f(x) \) and compare it to \( f_N(x) \) for \( N = 100 \). On a separate figure, plot the norm of the error \( \|f - f_N\| \) using the above formula for \( N = 1, 2, \ldots, 100 \) on a log-log scale by using \text{loglog} in \text{MATLAB}.

You may have noticed that the rate at which the coefficients \( \alpha_n \to 0 \) determines how fast the error decreases — this is not coincidental!

(d) For \( f(x) = 1 \), the equation \( Lu = f \) has the exact solution \( u(x) = x(1-x)/2 \). Given the result of part (c), the same argument used in part (a) tells us that
\[ \|u - u_N\|^2 = \|u\|^2 - \sum_{n=1}^{N} \frac{c_n^2}{\lambda_n^2}. \]
(You do not need to show this explicitly.) Use this formula to produce a \text{loglog} plot of the error \( \|u - u_N\| \) for \( N = 1, \ldots, 100 \) on the same plot you made in part (b). (Be aware that the error may appear to flatten around \( 10^{-8} \); this is a consequence of the computer’s floating point arithmetic, and is not a concern of ours here. To learn more about this phenomenon, take CAAM 453!!)

Solution.
(a) [10 points] We have that

\[ \| f - f_N \|^2 = (f - f_N, f - f_N) \]

\[ = \left( f - \sum_{n=1}^{N} \alpha_n \psi_n, f - \sum_{m=1}^{N} \alpha_m \psi_m \right) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) + \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n (\psi_n, f) - \sum_{m=1}^{N} \alpha_m (f, \psi_m) + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

\[ = (f, f) - \sum_{n=1}^{N} \alpha_n^2 - \sum_{m=1}^{N} \alpha_m^2 + \sum_{n=1}^{N} \alpha_n \sum_{m=1}^{N} \alpha_m (\psi_n, \psi_m) \]

where at each equal sign we have used: (1) the definition of the norm \( \| \cdot \| \); (2) the definition of \( g_N \); (3) linearity of the inner product in the second argument; (4) linearity of the inner product in the first argument; (5) the fact that \( (\psi_n, \psi_m) = 0 \) if \( n \neq m \), for \( m, n = 1, 2, \ldots, N \), since \( \{\psi_1, \ldots, \psi_N\} \) is orthonormal with respect to the inner product \( \langle \cdot, \cdot \rangle \); (6) the fact that \( (\psi_n, \psi_n) = 1 \), for \( n = 1, 2, \ldots, N \), since \( \{\psi_1, \ldots, \psi_N\} \) is orthonormal with respect to the inner product \( \langle \cdot, \cdot \rangle \); (7) the fact that \( (f, \psi_n) = (\psi_n, f) = \alpha_n \); (8) algebra; (9) the definition of the norm \( \| \cdot \| \).

(b) [10 points] First calculate the norm of \( f \)

\[ \| f \|^2 = \int_0^1 (f(x))^2 \, dx = \int_0^1 x^2 (1-x)^2 \, dx = \frac{1}{30} \]

Then

\[ \| f - f_N \|^2 = \| f \|^2 - \sum_{n=1}^{N} \alpha_n^2, \]

where

\[ \alpha_n = \frac{2\sqrt{2}}{n^3 \pi^3} \left( 1 - (-1)^n \right). \]

and so

\[ \| f - f_N \|^2 = \frac{1}{30} - \sum_{n=1}^{N} \alpha_n^2. \]

The requested plots are shown below.
Figure 1: Comparison of the true function $f(x)$ and $f_N(x)$ for $N = 5$

Figure 2: Norm of the error for $N = 1, 2 \cdots, 100$ on a log-log scale
Figure 3: Plot for (c).

Figure 4: Plot for (d).
(c) The requested plot for (c) and (d) is shown below.

The code that produced the plot above for (c) and (d) is shown below.

```matlab
n = [1:100]';
cn = (sqrt(2)/pi)*(1+(-1).^(n+1))./(n);
lamm = pi^2*n.^2;
normf2 = 1;
normu2 = 1/120;

figure(1), clf
loglog([1:length(cn)], sqrt(normf2-cumsum(cn.^2)),'r.-')
set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|f-f_N\|$','$\|u-u_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr1

figure(2)
loglog(n, sqrt(normu2-cumsum((cn./lamm).^2)),'b.-')
set(gca,'fontsize',14)
xlabel('$N$','fontsize',16,'interpreter','latex')
legend('$\|u-u_N\|$',3)
set(legend,'interpreter','latex','fontsize',16)
print -depsc2 fourerr2
```

4. [15 points: 5 points each]

Let the symmetric linear operator $L : C^2_M[0, 1] \to C[0, 1]$ be defined by

$$Lv = -v''$$

where

$$C^2_M[0, 1] = \{w \in C^2[0, 1] : w'(0) = w(1) = 0\}.$$

Let $N$ be a positive integer and let $f \in C[0, 1]$ be defined by

$$f(x) = \begin{cases} 1 - 2x & \text{if } x \in [0, \frac{1}{2}] ; \\ 0 & \text{otherwise.} \end{cases}$$

(a) The operator $L$ has eigenvalues $\lambda_n$ with corresponding eigenfunctions

$$\phi_n(x) = \sqrt{2} \cos \left( \frac{2n-1}{2} \pi x \right)$$

for $n = 1, 2, \ldots$. We have that, for $m, n = 1, 2, \ldots$,

$$(\phi_m, \phi_n) = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

Obtain a formula for the eigenvalues $\lambda_n$ for $n = 1, 2, \ldots$.

(b) Compute $f_N$, the best approximation to $f$ from span $\{\phi_1, \ldots, \phi_N\}$ with respect to the norm $\| \cdot \|$.

Plot $f_N$ for $N = 1, 2, 3, 4, 5, 6$. 

(c) Use the spectral method to obtain a series solution to the problem of finding \( \tilde{u} \in C^2[0,1] \) such that
\[
-\tilde{u}''(x) = f(x), \quad 0 < x < 1
\]
and
\[
\tilde{u}'(0) = \tilde{u}(1) = 0.
\]

Solution.

(a) [5 points] We can compute that, for \( n = 1, 2, \ldots \),
\[
\phi'_n(x) = -\sqrt{2} \left( \frac{2n - 1}{2} \right) \pi \sin \left( \frac{2n - 1}{2} \pi x \right).
\]
and
\[
\phi''_n(x) = -\sqrt{2} \left( \frac{2n - 1}{2} \right)^2 \pi^2 \cos \left( \frac{2n - 1}{2} \pi x \right).
\]
and so
\[
L\phi_n = -\phi''_n = \left( \frac{2n - 1}{2} \right)^2 \pi^2 \phi_n.
\]
Hence,
\[
\lambda_n = \left( \frac{2n - 1}{2} \right)^2 \pi^2 = (2n - 1)^2 \frac{\pi^2}{4} \quad \text{for } n = 1, 2, \ldots.
\]

(b) [5 points] Since \( \{\phi_1, \ldots, \phi_N\} \) is orthonormal with respect to the inner product \( (\cdot, \cdot) \), the best approximation to \( f \) from \( \text{span} \{\phi_1, \ldots, \phi_N\} \) with respect to the norm \( \| \cdot \| \) is
\[
f_N = \sum_{n=1}^{N} (f, \phi_n) \phi_n.
\]
Now, for \( n = 1, 2, \ldots \),

\[
(f, \phi_n) = \int_0^1 f(x) \phi_n(x) \, dx
\]

\[
= \int_0^{1/2} f(x) \phi_n(x) \, dx + \int_{1/2}^1 f(x) \phi_n(x) \, dx
\]

\[
= \int_0^{1/2} (1 - 2x) \sqrt{2} \cos \left( \frac{2n - 1}{2} \pi x \right) \, dx + \int_{1/2}^1 0 \, dx
\]

\[
= \sqrt{2} \int_0^{1/2} (1 - 2x) \cos \left( \frac{2n - 1}{2} \pi x \right) \, dx + 0
\]

\[
= \sqrt{2} \left( \left[ (1 - 2x) \frac{2}{(2n - 1) \pi} \sin \left( \frac{2n - 1}{2} \pi x \right) \right]_0^{1/2} - \int_0^{1/2} (-2x) \frac{2}{(2n - 1) \pi} \sin \left( \frac{2n - 1}{2} \pi x \right) \, dx \right)
\]

\[
= \sqrt{2} \left( 0 - 0 + \frac{4}{(2n - 1) \pi} \int_0^{1/2} \sin \left( \frac{2n - 1}{2} \pi x \right) \, dx \right)
\]

\[
= \sqrt{2} \left( \frac{4}{(2n - 1) \pi} \left[ - \frac{2}{(2n - 1) \pi} \cos \left( \frac{2n - 1}{2} \pi x \right) \right]_0^{1/2} \right)
\]

\[
= \frac{4\sqrt{2}}{(2n - 1) \pi} \left( - \frac{2}{(2n - 1) \pi} \cos \left( \frac{2n - 1}{4} \pi \right) - \left( - \frac{2}{(2n - 1) \pi} \right) \right)
\]

\[
= \frac{8\sqrt{2}}{(2n - 1)^2 \pi^2} \left( 1 - \cos \left( \frac{2n - 1}{4} \pi \right) \cos \left( \frac{2n - 1}{2} \pi x \right) \right).
\]

Hence,

\[
f_N(x) = \sum_{n=1}^{N} (f, \phi_n) \phi_n(x)
\]

\[
= \sum_{n=1}^{N} (f, \phi_n) \sqrt{2} \cos \left( \frac{2n - 1}{2} \pi x \right)
\]

\[
= \sum_{n=1}^{N} \frac{16}{(2n - 1)^2 \pi^2} \left( 1 - \cos \left( \frac{2n - 1}{4} \pi \right) \right) \cos \left( \frac{2n - 1}{2} \pi x \right).
\]

The requested plot is below.
The above plot was produced using the following MATLAB code.

```matlab
clear
c1c
colors='rgbcmy';
x = linspace(0,1,1000);

figure(1)
c1f
legendStr{1}={'$f(x)$'};
plot(x,-(x-1/2)+(x-1/2).*sign(x-1/2),'k-')
hold on
fk = zeros(size(x));
for k=1:6
    fk = fk + 16*(1-cos(((2*k-1)/4)*pi))./((2*k-1).^2*pi^2)*cos(((2*k-1)/2)*pi*x);
    plot(x,fk,colors(k))
    legendStr{k+1}={'$f_{' num2str(k) '} (x)$'};
end
xlabel('x')
legend(legendStr,'interpreter','latex','location','best');
saveas(figure(1),'hw72c','epsc')
```

(c) [5 points] Now, $\tilde{u}$ is the solution to $L\tilde{u} = f$ and so the spectral method yields the series solution

$$\tilde{u}(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{\lambda_n} \phi_n(x) = \sum_{n=1}^{\infty} \frac{64}{(2n-1)^4 \pi^4} \left(1 - \cos \left(\frac{2n-1}{4} \frac{\pi}{x}\right)\right) \cos \left(\frac{2n-1}{2} \frac{\pi}{x}\right).$$