1. [18 points: 3 each]
This exercise walks you through the basics of the concepts behind the finite element method that we discussed in class. Consider the strong form of the boundary value problem:

\[-\partial_x (k(x)\partial_x u(x)) = f\]
\[u(0) = u(1) = 0\]

(a) Multiply the strong problem by a test function \(v(x)\), integrate, and apply integration by parts. What assumptions need to be imposed on the test function \(v(x)\) to arrive at the expression:

\[\int_0^1 k(x) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} = \int_0^1 f(x) v(x)\]

(b) Let \(L^2([0,1])\) denote the set of all functions \(h(x)\) such that \(\int_0^1 h^2 \, dx < \infty\). It can be shown that \(L^2([0,1])\) is a vector space; for the rest of this problem you may use that this is true. What assumptions must be made on \(k(x), \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, f(x)\) and, \(v(x)\) so that the integrals in the expression above are defined? Clearly state the assumption, and show how this assumption implies that the integrals are finite. Hint: You may use the fact that if two functions \(h(x)\) and \(g(x)\) satisfy

\[\int_0^1 h^2 \, dx < \infty \quad \int_0^1 g^2 \, dx < \infty\]

Then the product, \(h(x)g(x)\), satisfies

\[\left| \int_0^1 hg \, dx \right| \leq \left( \int_0^1 h^2 \, dx \right) \left( \int_0^1 g^2 \, dx \right)\]

(c) Define the function space

\[V_0 = \left\{ h \in L^2([0,1]) \mid \frac{\partial h}{\partial x} \in L^2([0,1]) \text{ and } h(0) = h(1) = 0 \right\}\]

Suppose that a uniform mesh is constructed on the interval \([0,1]\) having \(N\) internal points (so \(N + 2\) total points counting the endpoints) and define \(V_N \subset V_0\) to be the set of piecewise linear hat functions we discussed in class. \(V_N\) is a subvector space of \(V_0\) of dimension \(N\). Using these spaces write:

i. the weak formulation for the strong form of the boundary value problem.
ii. the corresponding discrete weak problem (the ‘finite element problem’).
(d) Define the expression:

\[ a(u, v) = \int_0^1 k(x) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \]

Show that \( a(u, v) \) defines an inner product on \( V_0 \) if we assume that \( k(x) > 0 \) on the interval \([0, 1]\).

(e) Let \( (u, v)_a = a(u, v) \) denote the inner product (from part (d)) defined on \( V_0 \) the associated inner product norm \( ||h||_a = (h, h)_a^{1/2} = a(h, h)^{1/2} \) is called the energy norm (since it can be shown that this expression is related to a potential energy in a physical sense; this is discussed in the course textbook). Suppose that a uniform mesh is constructed on \([0, 1]\) with \( N = 2 \) internal mesh points and suppose that \( k(x) = 1 \) and \( f(x) = x(1 - x) \) in the strong form of equation (1). Find the best approximation to \( f(x) \) in the subspace \( V_2 \) with respect to the energy norm. Hint: Set up and solve the Gram system.

(f) How is the best approximation of part (e) related to the solution to the weak problem of part of part (c, ii)? Justify your response.

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Solution.

(a) Multiplying equation (1) by \( v(x) \) and integrating gives

\[ \int_0^1 k(x) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} v \bigg|_0^1 = \int_0^1 f(x)v(x) \]

The boundary conditions on the solution \( u(x) \) given in equation (1) are not sufficient to get rid of the boundary term that appears; thus we need to impose the condition \( v(0) = v(1) = 0 \) on the test function \( v(x) \).

(b) Looking at the expression

\[ \int_0^1 k(x) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} = \int_0^1 f(x)v(x) \]

and keeping in mind the given hint we see that assuming \( k(x), \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, f(x) \) and, \( v(x) \) are all functions in \( L^2([0, 1]) \) then all of the integrals will be finite since, using the hint, we have

\[ \left| \int_0^1 k(x) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \right| \leq \left( \int_0^1 k^2 \, dx \right) \left( \int_0^1 \left( \frac{\partial u}{\partial x} \right)^2 \, dx \right) \left( \int_0^1 \left( \frac{\partial v}{\partial x} \right)^2 \, dx \right) \]

Every term on the right hand side of the inequality above is finite since we are assuming that all of these functions are in \( L^2([0, 1]) \) (and that is the definition of being in \( L^2 \)). For the other term we have a similar result since, again using the hint, we get:

\[ \left| \int_0^1 f(x)v(x) \, dx \right| \leq \left( \int_0^1 f^2 \, dx \right) \left( \int_0^1 v^2 \, dx \right) \]

and the result follows from the hypothesis, again, that \( f \) and \( v \) are both in \( L^2([0, 1]) \).

(c) Using the given spaces we have that:

i. The weak formulation is: Find \( u \in V_0 \) such that for all \( v \in V_0 \) we have

\[ a(u, v) = \int_0^1 f \, v \, dx \]

ii. The discrete weak problem is: Find \( u_h \in V_N \) such that for all \( v_h \in V_N \) we have

\[ a(u_h, v_h) = \int_0^1 f \, v_h \, dx \]
GRADERS: do not count points off if the students used the notation \( u \) and \( v \) instead of \( u_h \) and \( v_h \) in the discrete weak problem formulation

(d) To show that \( a(u, v) \) defines an inner product we use the definition of \( a(u, v) \) and have that:

(i) \( a(u, v) = a(v, u) \) since \( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} \)

(ii) \( a(\alpha u + v, w) = \alpha a(u, w) + a(v, w) \) by using the fact that differentiation and integration are linear. Linearity for the other coordinate follows from (ii) and (i) combined.

(iii) \( a(u, u) = \int_0^1 k(x)(\frac{\partial u}{\partial x})^2 dx \geq 0 \) since \( k(x) > 0 \), the square of a function is always non-negative, and the integral of a non-negative function is non-negative. Furthermore if \( a(u, u) = 0 \) then since \( k(x) > 0 \) this implies that \( \frac{\partial u}{\partial x} = 0 \). This means that \( u(x) = C \) is constant, but \( u(0) = 0 \), by \( u \in V_0 \) (e.g. satisfies the boundary conditions of the strong problem) so that \( u = 0 \) follows.

(e) Since \( (\cdot, \cdot)_a \) defines an inner product on \( V_0 \), \( f(x) = x(1 - x) \in V_0 \) and \( V_2 \subset V_0 \) is a subvector space the question of the best approximation to \( f(x) \) in \( V_2 \) is given by the solution to the Gram system:

\[
\begin{bmatrix}
  \langle \phi_1, \phi_1 \rangle_a & \langle \phi_1, \phi_2 \rangle_a \\
  \langle \phi_2, \phi_1 \rangle_a & \langle \phi_2, \phi_2 \rangle_a
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2
\end{bmatrix}
= \begin{bmatrix}
  \langle f, \phi_1 \rangle_a \\
  \langle f, \phi_2 \rangle_a
\end{bmatrix}
\]

Where \( \phi_1 \) is the hat function for the first internal point, and \( \phi_2 \) is the hat function for the second internal point. Note that two internal points in the interval \([0, 1]\) gives that the grid spacing is \( h = 1/3 \). Also we have \( \frac{\partial f}{\partial x} = (1 - x) - x = 1 - 2x \) we have

\[
\langle f, \phi_1 \rangle_a = \frac{1}{h} \int_0^{1/3} 1 - 2x \, dx - \frac{1}{h} \int_{1/3}^{2/3} 1 - 2x \, dx = \frac{2}{3} - 0 = \frac{2}{3}
\]

Likewise

\[
\langle f, \phi_2 \rangle_a = \frac{1}{h} \int_{1/3}^{2/3} 1 - 2x \, dx - \frac{1}{h} \int_{2/3}^{1} 1 - 2x \, dx = 0 - \left( -\frac{2}{3} \right) = \frac{2}{3}
\]

For the matrix we have:

\[
\langle \phi_1, \phi_1 \rangle_a = \frac{1}{h^2} \int_0^{1/3} dx + \frac{1}{h^2} \int_{1/3}^{2/3} dx = \frac{1}{h} + \frac{1}{h} = \frac{2}{h} = 6
\]

Likewise:

\[
\langle \phi_2, \phi_2 \rangle_a = \frac{1}{h^2} \int_{1/3}^{2/3} dx + \frac{1}{h^2} \int_{2/3}^{1} dx = \frac{1}{h} + \frac{1}{h} = \frac{2}{h} = 6
\]

By symmetry we only need to compute the following to finish:

\[
\langle \phi_1, \phi_2 \rangle_a = -\frac{1}{h^2} \int_{1/3}^{2/3} dx = -3
\]

Thus the system is:

\[
\begin{bmatrix}
  6 & -3 \\
  -3 & 6
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2
\end{bmatrix}
= \begin{bmatrix}
  2/3 \\
  2/3
\end{bmatrix}
\]

which has solution

\[
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2
\end{bmatrix}
= \frac{1}{5}
\begin{bmatrix}
  2 & 1 \\
  1 & 2
\end{bmatrix}
\begin{bmatrix}
  2/3 \\
  2/3
\end{bmatrix}
= \begin{bmatrix}
  2/9 \\
  2/9
\end{bmatrix}
\]

Thus the best approximation in \( V_2 \) to \( f(x) = x(1 - x) \) in the energy norm (for the case where \( k(x) = 1 \) is \( \frac{\hat{u}}{a} \phi_1(1) + \frac{\hat{u}}{a} \phi_2(x) \)

(f) The solution, \( u_h \in V_2 \) of the discrete problem of part (e, ii) satisfies: for all \( v_h \in V_2 \)

\[
\langle u_h, v_h \rangle_a = a(u_h, v_h) = \int_0^1 f v_h \, dx
\]
Suppose that \( u \) solves the weak problem (c, i). That is suppose \( u \in V_0 \) such that for all \( v \in V_0 \):

\[
(u, v)_a = a(u, v) = \int_0^1 f v \, dx
\]

is true. It then follows that if \( u \) solves the weak problem (c, i) and \( u_h \) solves the discrete weak problem (c, ii) then since \( V_2 \subset V_0 \) we have that for every \( v_h \in V_2 \)

\[
(u, v_h)_a = a(u, v_h) = \int_0^1 f v_h \, dx = a(u_h, v_h) = (u_h, v_h)_a
\]

Which means that \( (u - u_h, v_h)_a = 0 \) for every \( v_h \in V_2 \) (note: this property is called Galerkin orthogonality and was discussed in class). But since \( (\cdot, \cdot)_a = a(\cdot, \cdot) \) defines an inner product on \( V_0 \) we see that this is precisely the statement that \( u_h \in V_2 \) is the closest approximation to \( f \in V_0 \). Furthermore, we know that this approximation is unique. This shows that the finite element solution \( u_h \), under the right conditions, is nothing more than the best approximation to the function \( f(x) \) with respect to the inner product that comes from the boundary value problem of equation (1).

**Graders:** please give full credit for any response that reasonably links together the idea of the projection theorem / best approximation and the finite element solution.

2. [20 points: 5 each]

Use the finite element method to solve the differential equation

\[-(u'(x)\kappa(x))' = 2x, \quad 0 < x < 1\]

for \( \kappa(x) = 1 + x^2 \), subject to homogeneous Dirichlet boundary conditions,

\[u(0) = u(1) = 0,\]

with the approximation space \( V_N \) given by the piecewise linear hat functions: For \( n \geq 1 \), \( h = 1/(N + 1) \), and \( x_k = kh \) for \( k = 0, \ldots, N + 1 \), we have

\[
\phi_k(x) = \begin{cases}
\frac{(x - x_{k-1})}{h}, & x \in [x_{k-1}, x_k); \\
\frac{(x_{k+1} - x)}{h}, & x \in [x_k, x_{k+1}); \\
0, & \text{otherwise}.
\end{cases}
\]

(a) Write MATLAB code that constructs the stiffness matrix \( A \) for a given value of \( N \), with \( \kappa(x) = 1 + x^2 \).

[You may edit the fem_demo1.m code from the class website. You should compute all necessary integrals (by hand or using a symbolic package) so as to obtain clean formulas that depend on \( h \) and the index of the hat functions involved (e.g., \( a(\phi_j, \phi_j) \) can depend on \( j \)).]

(b) Write MATLAB code that constructs the load vector \( f \) for a given value of \( N \), with \( f(x) = 2x \).

(c) For \( N = 7 \) and \( N = 15 \), produce plots comparing your solution \( u_N \) to the true solution

\[u(x) = \left(\frac{4}{\pi}\right) \tan^{-1}(x) - x.\]

(Note that you can compute \( \tan^{-1}(x) \) as \texttt{atan(x)} in MATLAB.)
(d) Produce a loglog plot showing how the error

$$\max_{x \in [0,1]} |u_N(x) - u(x)|$$

decreases as $N$ increases. (For example, take $N = 8, 16, 32, 64, 128, 256, 512$.) On the same plot, show $N^{-2}$ for the same values of $N$. If your code from parts (a) and (b) is working, your error curve should have the same slope as the $N^{-2}$ curve. (Consult the fem_demo1.m code on the website for a demonstration of the style of plot we intend for part (d); edit this code as you like.)
Solution.

(a) First we compute the energy inner product of the basis functions. Note that
\[
\frac{d\phi_k}{dx}(x) = \begin{cases} 
1/h, & x \in [x_{k-1}, x_k); \\
-1/h, & x \in [x_k, x_{k+1}); \\
0, & \text{otherwise}.
\end{cases}
\]
Thus we have
\[
a(\phi_j, \phi_j) = \int_0^1 (1 + x^2) \left(\frac{d\phi_j}{dx}(x)\right)^2 \, dx
= \int_{x_{j-1}}^{x_j} (1 + x^2) \left(\frac{1}{h}\right)^2 \, dx + \int_{x_j}^{x_{j+1}} (1 + x^2) \left(-\frac{1}{h}\right)^2 \, dx
= \frac{1}{h^2} \int_{x_{j-1}}^{x_{j+1}} (1 + x^2) \, dx 
= \frac{1}{h^2} \left[ x + \frac{x^3}{3} \right]_{x_{j-1}}^{x_{j+1}}
= \frac{2}{h} + \frac{2h}{3} + 2h^2 j,
\]
\[
a(\phi_j, \phi_{j+1}) = \int_0^{x_j} (1 + x^2) \left(\frac{d\phi_j}{dx}(x)\right) \left(\frac{d\phi_{j+1}}{dx}(x)\right) \, dx
= \int_{x_j}^{x_{j+1}} (1 + x^2) \left(-\frac{1}{h}\right) \left(\frac{1}{h}\right) \, dx
= -\frac{1}{h^2} \int_{x_j}^{x_{j+1}} (1 + x^2) \, dx 
= -\frac{1}{h^2} \left[ x + \frac{x^3}{3} \right]_{x_j}^{x_{j+1}}
= -\frac{1}{h} \left( j^2 + j + \frac{1}{3} \right),
\]
and for \(|j - k| > 1\),
\[
a(\phi_j, \phi_k) = 0
\]
since \((d\phi_j(x)/dx)(d\phi_k(x)/dx) = 0\) for all \(x \in [0, 1]\) (except at the nodes \(x_\ell\), where strictly speaking these derivatives are not defined—but these single isolated points do not add anything to the integral). The stiffness matrix is given by
\[
A = \begin{bmatrix}
\vdots & \ddots & \vdots \\
a(\phi_1, \phi_1) & \cdots & a(\phi_1, \phi_n) \\
a(\phi_n, \phi_1) & \cdots & a(\phi_n, \phi_n)
\end{bmatrix}.
\]

(b) Next we compute the entries of the load vector:
\[
(f, \phi_j) = \int_0^1 f(x) \phi_j(x) \, dx
= \int_{x_{j-1}}^{x_j} (2x) \left(\frac{x - x_{j-1}}{h}\right) \, dx + \int_{x_j}^{x_{j+1}} (2x) \left(\frac{x_{j+1} - x}{h}\right) \, dx
= \frac{1}{h} \left[ \frac{2x^3}{3} - x^2 x_{j-1} \right]_{x_{j-1}}^{x_j} + \frac{1}{h} \left[ x^2 x_{j+1} - \frac{2x^3}{3} \right]_{x_j}^{x_{j+1}}
= 2h^2 j.
\]
The load vector is given by
\[
f = \begin{bmatrix} (f, \phi_1) \\
\vdots \\
(f, \phi_n) \end{bmatrix}.
The MATLAB code at the end of this problem shows generates these matrices and produces plots similar to those shown in (b) and (c).

(c) The following plots show the solution (and error) at $N = 7$ (left) and $N = 15$ (right).

(d) The following plot shows the decay of the error as a function of $N$. Notice that the error decays like $1/N^2$. 

% demo of the finite element method for the problem
% -d/dx((1+x^2) du/dx) = 2x, 0 < x < 1, u(0) = u(1) = 0.
% which has exact solution $u(x) = (4/pi)\times\arctan(x) - x.$

Nvec = [4 8 16 32 64 128 256 512];  % vector of N values we shall use
maxerr = zeros(size(Nvec));  % vector to hold the max errors for each N

% each pass of the following loop handles a new N value...
for j=1:length(Nvec)
    N = Nvec(j);
    h = 1/(N+1);
    x = [1:N]*h;

    % ...
3. [12 points: 3 each]
A classical problem in quantum mechanics models a particle moving in an infinite square well, subject to an infinite potential at a point. The result is a Schrödinger operator posed on $C^2_D[0,1]$ of the form

$$Lu = -u'' + \delta_{1/2} u,$$

where $\delta_{1/2}$ is a “delta function” centered at the location of the infinite potential, $x = 1/2$. A beautiful theory supports these exotic functions (more properly called distributions). For this problem, you need
only know the following fact: for any function \( g \in C[0, 1] \),
\[
\int_0^1 \delta_{1/2}(x) g(x) \, dx = g(1/2).
\]
The equation \( Lu = f \) has the equivalent weak form
\[
a(u, w) = (f, w) \quad \text{for all } w \in V_N,
\]
where
\[
a(u, w) = \int_0^1 \left( u'(x)w'(x) + \delta_{1/2}(x)u(x)w(x) \right) \, dx.
\]
We wish to use the Galerkin method to approximate solutions to \( Lu = f \) from the finite dimensional subspace \( V_N = \text{span}\{\phi_1, \ldots, \phi_N\} \). Use as basis vectors the eigenfunctions from the problem without the potential at \( x = 1/2 \):
\[
\phi_k(x) = \sqrt{2} \sin(k\pi x).
\]
(a) Compute a general formula for \( a(\phi_j, \phi_k) \).
(b) Write out (by hand) the stiffness matrix for \( N = 5 \).
(c) Write down a general formula for the entries in the load vector, \( (f, \phi_k) \), when \( f(x) = 1 \).
   (You may use any formulas from prior homeworks and the online notes.)
(d) Plot your approximate solutions to \( -u''(x) + \delta_{1/2}(x)u(x) = 1 \) for \( N = 5 \) and \( N = 35 \).

\textbf{Solution.}

(a) Compute
\[
a(\phi_j, \phi_k) = \int_0^1 \left( \phi'_j(x)\phi'_k(x) + \delta_{1/2}(x)\phi_j(x)\phi_k(x) \right) \, dx
\]
\[
= 2k_j^2 \pi^2 \int_0^1 \cos(j\pi x) \cos(k\pi x) \, dx + 2 \int_0^1 \delta_{1/2}(x) \sin(j\pi x) \sin(k\pi x)
\]
\[
= 2k_j^2 \pi^2 \int_0^1 \cos(j\pi x) \cos(k\pi x) \, dx + 2 \sin(j\pi/2) \sin(k\pi/2).
\]
The integral in this last expression is \( 1/2 \) when \( j = k \), and zero otherwise. The second term will be zero if either \( j \) or \( k \) is even (since in that case one of the sine terms must be zero). If both \( j \) and \( k \) are odd, this term will be nonzero, \( \pm 2 \). In general, we can write
\[
a(\phi_j, \phi_k) = \begin{cases} 
  j^2\pi^2 + 2\sin^2(j\pi/2), & \text{if } j = k; \\
  2\sin(j\pi/2)\sin(k\pi/2), & \text{otherwise.}
\end{cases}
\]

[\textbf{GRADERS:} the amount that students simplify \( a(\phi_j, \phi_k) \) will vary. The ultimate solution need not take the precise form that we have given above, but it should be simplified beyond just writing down the definition of \( a(\phi_j, \phi_k) \).]
(b) For $N = 5$ we have
\[
\begin{bmatrix}
\pi^2 + 2 & 0 & -2 & 0 & 2 \\
0 & 4\pi^2 & 0 & 0 & 0 \\
-2 & 0 & 9\pi^2 + 2 & 0 & -2 \\
0 & 0 & 0 & 16\pi^2 & 0 \\
2 & 0 & -2 & 0 & 25\pi^2 + 2 \\
\end{bmatrix}
\]

(c) The entries of the load vector are
\[
(f, \phi_k) = \int_0^1 1 \cdot \sqrt{2} \sin(k\pi x) \, dx = \begin{cases} 
2\sqrt{2}/(n\pi), & \text{if } k \text{ is odd}; \\
0, & \text{if } k \text{ is even},
\end{cases}
\]
as computed in previous examples earlier in the semester.

[GRADERS: students do not need to show work for this formula.]

(d) Approximate solutions for $N = 5$ and $N = 35$ are shown below, followed by the code that produced them.

```
for N = [5 35]
    K = zeros(N); f = zeros(N,1);
    for j=1:N, for k=1:N
        K(j,k) = 2*sin(j*pi/2)*sin(k*pi/2);
    end, end
    K = K + diag([1:N].^2*pi^2);
    for k=1:N
        f(k) = (sqrt(2)/pi)*(1+(-1).^(k+1))./k;
    end
    c = K;
    xx = linspace(0,1,1000);
    uN = zeros(size(xx));
    for k=1:N
        uN = uN + c(k)*sqrt(2)*sin(k*pi*xx);
    end
    figure(N), clf
```

for $N = [5 \ 35]$

K = zeros(N); f = zeros(N,1);

for j=1:N, for k=1:N
    K(j,k) = 2*sin(j*pi/2)*sin(k*pi/2);
end, end
K = K + diag([1:N].^2*pi^2);

for k=1:N
    f(k) = (sqrt(2)/pi)*(1+(-1).^(k+1))./k;
end

c = K;

xx = linspace(0,1,1000);
uN = zeros(size(xx));

for k=1:N
    uN = uN + c(k)*sqrt(2)*sin(k*pi*xx);
end

figure(N), clf
4. [15 points: 3 each]

Let $k(x)$ and $p(x)$ be two positive-valued continuous functions on $[0, 1]$.

(a) Define

$$V = C^1_D[0, 1] = \{ u \in C^1[0, 1] : u(0) = u(1) = 0 \}.$$ 

Derive the weak form of the differential equation

$$-\frac{d}{dx}(k(x)\frac{du}{dx}) + p(x)u = f(x), \quad 0 < x < 1,$$

subject to the boundary conditions

$$u(0) = u(1) = 0;$$

that is, transform this differential equation into a problem of the form:

Find $u \in V$ such that $a(u, v) = (f, v)$ for all $v \in V$,

where $(\cdot, \cdot)$ denotes the usual inner product $(f, g) = \int_0^1 f(x)g(x)\,dx$, and $a(\cdot, \cdot)$ is some bilinear form that you should specify. Verify that $a(u, v)$ is still an inner product.

(b) Let $p(x) = 1$, $k(x) = \epsilon$, and let the source function $f(x) = 1$. Construct the finite element system $A\alpha = b$, where

$$A_{ij} = a(\phi_j, \phi_i), \quad b_i = \int_0^1 f(x)\phi_i(x)$$

using the approximation space $V_N$ given by the piecewise linear hat functions: for $n \geq 1$, $h = 1/(N + 1)$, and $x_k = kh$ for $k = 0, \ldots, N + 1$, we have

$$\phi_k(x) = \begin{cases} 
\frac{(x - x_{k-1})}{h}, & x \in [x_{k-1}, x_k); \\
\frac{(x_{k+1} - x)}{h}, & x \in [x_k, x_{k+1}); \\
0, & \text{otherwise.}
\end{cases}$$

Hint: for this specific choice of $p(x), k(x)$, it may be easier to show that you can express

$$A = \epsilon K + M, \quad K_{ij} = \int_0^1 \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \,dx$$

where the form of $M$ is the so-called mass matrix given by

$$M_{ij} = \int_0^1 \phi_j \phi_i \,dx$$

(c) This specific equation corresponds to the simplest steady-state reaction-diffusion equation, where $u(x)$ is the concentration of some solvent, and the choices of $p(x)$ model local chemical reactions that may occur due to that solvent (multiple chemicals interacting may be modeled using systems of reaction-diffusion equations).
Setup and solve the above system by choosing the hat function basis for the basis functions \( \phi_i, i = 1, 2, \ldots N \). Recall that in order to use the hat function basis you must first have a (uniform) mesh of the interval \([0, 1]\). Let \( N \) be the number of interior points of the mesh, so that there are \( N \) hat functions in your basis, and solve the problem for \( N = 32 \) and \( \epsilon = .1, .25, 1 \). What do you observe about the solution as \( \epsilon \) becomes smaller?

(d) Let the space \( V \) now be defined as

\[
V = \left\{ u \in C^1[0, 1] : u(0) = \frac{du}{dx}(1) = 0 \right\}.
\]

Derive the weak form of the differential equation

\[
-\frac{d}{dx} \left( k(x) \frac{du}{dx} \right) + p(x)u = f(x), \quad 0 < x < 1,
\]

subject to the boundary conditions

\[
u(0) = \frac{du}{dx}(1) = 0;
\]

that is, transform this differential equation into a problem of the form:

Find \( u \in V \) such that \( a(u, v) = (f, v) \) for all \( v \in V \),

where \((\cdot, \cdot)\) denotes the usual inner product \((f, g) = \int_0^1 f(x)g(x) \, dx\), and \( a(\cdot, \cdot) \) is some bilinear form that you should specify.

(e) Show that the form \( a(u, v) \) from part (d) is an inner product for \( u, v \in V \).

Solution.

(a) Multiply the differential equation with some function \( v \) from the space \( V \) and integrate from \( x = 0 \) to \( x = 1 \) to obtain

\[
\int_0^1 \left( -\frac{d}{dx} \left( k(x) \frac{du}{dx} \right) v(x) + p(x)u(x)v(x) \right) \, dx = \int_0^1 f(x)v(x) \, dx.
\]

Break the integral on the left into pieces to obtain

\[
\int_0^1 \left( -\frac{d}{dx} \left( k(x) \frac{du}{dx} \right) \right) v(x) \, dx + \int_0^1 \left( p(x)u(x) \right) v(x) \, dx = \int_0^1 f(x)v(x) \, dx.
\]

Integrate the first integral by parts to obtain

\[
\left[ k(x) \frac{du}{dx}(x)v(x) \right]_0^1 + \int_0^1 k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) \, dx + \int_0^1 \left( p(x)u(x) \right) v(x) \, dx = \int_0^1 f(x)v(x) \, dx.
\]

The boundary terms vanish due to the fact that \( v(0) = v(1) = 0 \) if \( v \in V = C^1_D[0, 1] \). We consolidate the integrals on the left to arrive at the weak problem:

Find \( u \in V \) such that \( a(u, v) = (f, v) \) for all \( v \in V \),

where

\[
a(u, v) = \int_0^1 \left( k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) + p(x)u(x)v(x) \right) \, dx.
\]

To show that the form \( a(u, v) \) in part (a) is an inner product, we must verify the three basic properties:
• **Symmetry** is apparent by inspection:

\[
a(u, v) = \int_0^1 \left( k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) + p(x)u(x)v(x) \right) dx
\]

\[
= \int_0^1 \left( k(x) \frac{dv}{dx}(x) \frac{du}{dx}(x) + p(x)u(x)v(x) \right) dx = a(v, u).
\]

• **Linearity** follows from the linearity of differentiation and integration:

\[
a(\alpha u + \beta v, w) = \int_0^1 \left( k(x) \frac{d(\alpha u(x) + \beta v(x))}{dx}(x) \frac{dv}{dx}(x) + p(x)(\alpha u(x) + \beta v(x))w(x) \right) dx
\]

\[
= \int_0^1 \left( k(x) \left( \alpha \frac{du}{dx}(x) + \beta \frac{dv}{dx}(x) \right) \frac{dv}{dx}(x) + p(x)(\alpha u(x) + \beta v(x))w(x) \right) dx
\]

\[
= \alpha \int_0^1 \left( k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) + p(x)u(x)w(x) \right) dx
\]

\[
+ \beta \int_0^1 \left( k(x) \frac{dv}{dx}(x) \frac{dv}{dx}(x) + p(x)v(x)w(x) \right) dx
\]

\[
= \alpha a(u, w) + \beta a(v, w).
\]

• **Positivity** requires that \(a(u, u) \geq 0\) and \(a(u, u) = 0\) only when \(u = 0\). Note that

\[
a(u, u) = \int_0^1 \left( k(x) \frac{du}{dx}(x) \frac{du}{dx}(x) + p(x)u(x)u(x) \right) dx
\]

\[
= \int_0^1 \left( k(x) \left( \frac{du}{dx}(x) \right)^2 + p(x)(u(x))^2 \right) dx.
\]

Since \(k(x)\) and \(p(x)\) are both positive for all \(x \in [0, 1]\), each integrand is non-negative, and hence \(a(u, u) \geq 0\). To have \(a(u, u) = 0\), we must have \(u(x) = 0\) for all \(x \in [0, 1]\), and \(du(x)/dx = 0\) for all \(x \in [0, 1]\), which is only possible if \(u(x) = 0\) for all \(x \in [0, 1]\), i.e., \(u = 0\).

(b) If \(p(x) = 1\) and \(k(x) = \epsilon\), our formulation reduces down to \(a(u, v) = (f, v)\) where

\[
a(u, v) = \int_0^1 u(x)v(x)dx + \epsilon \int_0^1 \frac{du}{dx} \frac{dv}{dx} dx.
\]

Then, if \(A_{ij} = a(\phi_j, \phi_i)\), we have

\[
A_{ij} = \int_0^1 \phi_j(x)\phi_i(x)dx + \epsilon \int_0^1 \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} dx = M_{ij} + \epsilon K_{ij}
\]

where \(M_{ij} = \int_0^1 \phi_j(x)\phi_i(x)dx\) is the Gram matrix for hat functions using the \(L^2\) inner product.

The entries of \(M\) and \(K\) are known to be

\[
M_{ij} = \begin{cases} 
M_{i,i} = 2h/3 \\
M_{i+1,i} = h/6 \\
M_{ij} = 0, \quad |i-j| > 1
\end{cases}
\]

\[
K_{ij} = \begin{cases} 
K_{i,i} = 2/h \\
K_{i+1,i} = -1/h \\
K_{ij} = 0, \quad |i-j| > 1
\end{cases}
\]

from class and from previous homeworks.

(c) The code to generate the figures for this problem are given below.

```
iter = 1;
C = hsv(3);
```
for ep = [1 .25 .1];
  N = 32;
  h = 1/(N+1);
  x = [1:N]*h;

  M = (2/3)*diag(ones(N,1)) + (1/6)*diag(ones(N-1,1),1) + (1/6)*diag(ones(N-1,1),-1);
  M = h*M;

  K = 2*diag(ones(N,1)) - diag(ones(N-1,1),1) - diag(ones(N-1,1),-1);
  K = (1/h)*K;

  A = M + ep*K;
  b = h*ones(N,1);
  c = A\b;

  xx = linspace(0,1,500);  \% finely spaced points between 0 and 1.
  hold on
  \% plot the approximation solution
  uN = zeros(size(xx));
  for j=1:N
    uN = uN + c(j)*hat(xx,j,N);
  end
  plot(xx,uN,'color',C(iter,:),'linewidth',2)
  iter = iter + 1;
end

legend('\epsilon = 1','\epsilon = .25','\epsilon = .1')
set(gca,'fontsize',14)
xlabel('x','fontsize',15)
ylabel('Solution u(x)','fontsize',15)
print(gcf,'-depsc','../ep1.eps')

As \epsilon decreases, the temperature in the bar increases. It is difficult to see with \epsilon \geq .1, but as \epsilon gets small, the solution actually develops additional characteristics called boundary layers, where the solution becomes very steep near the boundaries. Graders: please give credit just for noting the temperature increases, as the boundary layer phenomena was not visible for the range of \epsilon specified in the problem.

(d) The process is very similar to part (a). Multiply the differential equation with some function v from the space V and integrate from x = 0 to x = 1 to obtain

$$
\int_0^1 \left( - \frac{d}{dx} \left( k(x) \frac{du}{dx}(x) \right) v(x) + p(x)u(x)v(x) \right) dx = \int_0^1 f(x)v(x) dx.
$$

Break the integral on the left into pieces to obtain

$$
\int_0^1 \left( - \frac{d}{dx} \left( k(x) \frac{du}{dx}(x) \right) \right) v(x) dx + \int_0^1 \left( p(x)u(x) \right) v(x) dx = \int_0^1 f(x)v(x) dx.
$$
Integrate the first integral by parts to obtain

\[- \left[ \kappa(x) \frac{du}{dx}(x)v(x) \right]_0^1 + \int_0^1 k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) \, dx + \int_0^1 \left( p(x)u(x) \right)v(x) \, dx = \int_0^1 f(x)v(x) \, dx.\]

The first term disappears because of the boundary conditions \(v(0) = 0\) and \(du(1)/dx = 0\). We consolidate the integrals on the left to arrive at the weak problem:

Find \(u \in V\) such that \(a(u, v) = (f, v)\) for all \(v \in V\),

where

\[a(u, v) = \int_0^1 \left( k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) + p(x)u(x)v(x) \right) \, dx.\]

(e) The proof for (e) is identical to the proof for (a). Graders: please give full credit if the student notices this.