CAAM 336 · DIFFERENTIAL EQUATIONS

Homework 6

Posted Friday 18, November 2016. Due 5pm Monday 28, November 2016.

• A note on proofs (unless otherwise stated by the problem): When asked to prove a statement this means you are to show that the requisite properties (discussed in class) for the item referenced by the statement hold true. When asked to disprove something this means you are to come up with an example that shows the proposed premise is false.

• Unless explicitly stated otherwise in the problem you are free to use MATLAB as you see fit; including for those problems that do not explicitly require it. Please submit any code that you utilize as a printout or, if it is short enough, reference your steps in your writeup directly. (example: I used matlab to compute the inverse to the matrix B and got ... etc)

A total of 100 points is distributed among the following problems

Please write your name and instructor on your homework.

1. [25 points] In class we discussed the spectral method applied to solve an inhomogeneous, time-dependent initial boundary value problem, with Dirichlet boundary conditions, of the general form

\[
\frac{\partial}{\partial t} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) = f(t, x) \tag{1}
\]

\[
\begin{align*}
  u(t, 0) &= a \\
  u(t, 1) &= b \\
  u(0, x) &= h(x)
\end{align*}
\]

In the above model assume that \( f(t, x) = \sin(\pi x) \)

5 pts – Assume \( a = b = 0 \) and that \( h(x) = \sqrt{2}\sin(\pi x) + 2\sqrt{2}\sin(2\pi x) \). Write the system of linear ODE that comes from applying the spectral method to the above problem.

15 pts – Solve the initial boundary value problem of part (a) for the unknown function \( u(t, x) \).

5 pts – Now assume that \( a = 1 \) and \( b = 3 \). Let \( h(x) = h(x) = \sqrt{2}\sin(\pi x) + 2\sqrt{2}\sin(2\pi x) + 2x + 1 \). Use the spectral method to solve the inhomogeneous boundary value problem for \( u(t, x) \).

2. [25 points] This problem considers the spectral method in the context of the homogeneous Neumann boundary conditions. We know from class that the time-independent boundary value problem

\[-\frac{\partial^2}{\partial x^2} u(x) = f(x) \tag{2}
\]

\[
\begin{align*}
  u'(0) &= 0 \\
  u'(1) &= 0
\end{align*}
\]

does not have a unique solution without imposing an additional condition to remove the constant eigenvector. Consider the time-dependent initial boundary value problem

\[
\frac{\partial}{\partial t} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) = f(t, x) \tag{3}
\]

\[
\begin{align*}
  u'(t, 0) &= a \\
  u'(t, 1) &= b \\
  u(0, x) &= h(x)
\end{align*}
\]

With the additional condition that for all times \( t \in [0, 1] \) we have
\[
\int_0^1 u(t, x) \, dx = 0
\]

In the above model assume that \( f(t, x) = x - \frac{1}{2} \) and \( h(x) = \sqrt{8} \cos(3\pi x) - \sqrt{6} \cos(10\pi x) \).

**15 pts** – Assume \( a = b = 0 \) Solve the initial boundary value problem for the unknown function \( u(t, x) \).

**10 pts** – Now assume that \( a = 3 \) and \( b = 1 \). Use the spectral method to solve the inhomogeneous initial boundary value problem for \( u(t, x) \).

**3. [25 points]**

**5 pts** – Consider the function \( u_0(x) = \begin{cases} 
1, & x \in [0, 1/3]; \\
0, & x \in (1/3, 2/3); \\
1, & x \in [2/3, 1]. 
\end{cases} \)

Recall that the eigenvalues of the operator \( L : C^2_N[0, 1] \to C[0, 1] \),
\[
Lu = -u''
\]
are \( \lambda_n = n^2 \pi^2 \) for \( n = 0, 1, \ldots \) with associated (normalized) eigenfunctions \( \psi_0(x) = 1 \) and
\[
\psi_n(x) = \sqrt{2} \cos(n\pi x), \quad n = 1, 2, \ldots .
\]

We wish to write \( u_0(x) \) as a series of the form
\[
u_0(x) = \sum_{n=0}^{\infty} a_n(0) \psi_n(x), \]
where \( a_n(0) = (u_0, \psi_n). \)

Compute these inner products \( a_n(0) = (u_0, \psi_n) \) by hand and simplify as much as possible.

For \( m = 0, 2, 4, 80 \), plot the partial sums
\[
u_{0,m}(x) = \sum_{n=0}^{m} a_n(0) \psi_n(x).
\]

(You may superimpose these on one single, well-labeled plot if you like.)

**10 pts** – Write down a series solution to the homogeneous heat equation
\[
u_t(x, t) = \nu_{xx}(x, t), \quad 0 < x < 1, \quad t \geq 0
\]
with Neumann boundary conditions
\[
u_x(0, t) = \nu_x(1, t) = 0
\]
and initial condition \( u(x, 0) = u_0(x) \).

Create a plot showing the solution at times \( t = 0, 0.002, 0.05, 0.1 \).

You will need to truncate your infinite series to show this plot.

Discuss how the number of terms you use in this infinite series affects the accuracy of your plots.

**5 pts** – Describe the behavior of your solution as \( t \to \infty \).

(To do so, write down a formula for the solution in the limit \( t \to \infty \).)
5 pts – How would you expect the solution to the inhomogeneous heat equation
\[ u_t(x, t) = u_{xx} + 1, \quad 0 < x < 1, \quad t \geq 0 \]
with Neumann boundary conditions
\[ u_x(0, t) = u_x(1, t) = 0 \]
to behave as \( t \to \infty \)?

4. [25 points] We wish to approximate the solution to the heat equation
\[ u_t(x, t) = u_{xx}(x, t) + 100tx, \quad 0 \leq x \leq 1, \quad t \geq 0 \]
with homogeneous Dirichlet boundary conditions
\[ u(0, t) = u(1, t) = 0 \]
and initial condition
\[ u(x, 0) = 0 \]
using the finite element method (method of lines). Let \( N \geq 1, \ h = 1/(N + 1) \), and \( x_k = kh \) for \( k = 0, \ldots, N + 1 \). We shall construct approximations using the hat functions
\[ \phi_k(x) = \begin{cases} \frac{(x - x_{k-1})}{h}, & x \in [x_{k-1}, x_k); \\ \frac{(x_{k+1} - x)}{h}, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise}. \end{cases} \]
The approximate solution shall have the form
\[ u_N(x, t) = \sum_{k=1}^{N} a_k(t) \phi_k(x). \]

10 pts – Write down the system of ordinary differential equations that determines the coefficients \( a_k(t) \), \( k = 1, \ldots, N \). Specify the entries in the mass and stiffness matrices and the load vector.
(You may use results from previous homeworks and class as convenient.)

15 pts – Write a MATLAB code that uses the backward Euler method to solve for the coefficients \( a_k(t) \).
Plot your approximate solution \( u_N(x, t) \) at time \( t = 1 \).
Choose \( N \) and \( \Delta t \) so that your solution appears to be accurate.
Verify this accuracy by superimposing on your plot the computed solution at \( t = 1 \) obtained by using space and time steps that are ten times smaller.